SINGULARITIES AND DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 33
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

SYMMETRIES OF CONTROL SYSTEMS

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Abstract. Symmetries of the control systems of the form $\mathbf{u}_t = \mathbf{f}(t, \mathbf{u}, \mathbf{v})$, $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$ are studied. Some general results concerning point symmetries are obtained. Examples are provided.

Introduction. Technically, the control systems are underdetermined systems of differential equations. These are not familiar objects for symmetry analysis, probably because their full symmetry algebras are presumed to be unresonably large. In [2] the first- and second-order generalized symmetries of the underdetermined "system" $u_x = (v_{xx})^2$, where u and v are scalar functions of x, were studied. The resulting Lie algebra of second-order symmetries is the noncompact real form of the exceptional Lie algebra G_2. Later Kersten, [1], obtained the description of the general higher-order symmetry algebra for this equation. Moreover, he gave the elegant and short derivation of the full Lie algebra of generalized symmetries for general "scalar system"

$$u_x = f(u, v, v_x, v_{xx}, \ldots, v_{x^k}), \tag{1}$$

x, u and v being scalars. In short, any $n+2$-order generalized symmetry may be obtained from an arbitrarily chosen function $H(x, u, v, \ldots, v_{x^n})$ by explicit procedure, provided n is sufficiently greater than k. References [4]–[8] deal mostly with the setting of a problem (there is a choice: whether to consider v-type variables as functional parameters or as unknown functions on a par with u-type ones; we choose the latter).

As became lately known to the author, Proposition 1 was obtained independently by Krishenko [3]. He also obtained some necessary conditions for a control system to admit a decomposition in terms of the system's symmetry algebra.

We are mainly concerned here with the system of the form

$$\mathbf{u}_t = \mathbf{f}(x, \mathbf{u}, \mathbf{v}), \tag{2}$$

which is the general form of a control system and also with a more general system

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1991 *Mathematics Subject Classification*: Primary 58F37; Secondary 34H05.

The work was partially supported by NSF grant N2F300.

The paper is in final form and no version of it will be published elsewhere.

$$\mathbf{u}_t = \mathbf{f}(x, \mathbf{u}, \mathbf{v}, \mathbf{v}_t, \dots, \mathbf{v}_{t^k}), \tag{3}$$

where $t \in \mathbb{R}$, $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^m$.

General discussion

1. *Higher symmetries.* The symmetry equation for (3) is of the form

$$D_t\mathbf{A} - \mathbf{f}_{\mathbf{u}}\mathbf{A} - \sum_{s=0}^{k} \mathbf{f}_{\mathbf{v}_{t^s}}\mathbf{B}|_{\{\mathbf{u}_t=\mathbf{f}(t,\mathbf{u},\mathbf{v},\mathbf{v}_t,\dots,\mathbf{v}_{t^k})\}} = 0, \tag{4}$$

where subscripts stand for partial derivatives, $(\mathbf{A}, \mathbf{B})$ is a symmetry and D_t denotes the total derivative with respect to t. To be precise,

$$D_t = \partial_t + \sum_{s=0}^{\infty}\left(\sum_{i=1}^{n} u^i_{t^{s+1}}\frac{\partial}{\partial u^i_{t^s}} + \sum_{j=1}^{m} v^j_{t^{s+1}}\frac{\partial}{\partial v^j_{t^s}}\right)$$

is the scalar operator acting on the n-vector $\mathbf{A} = \mathbf{A}(t, \mathbf{u}, \mathbf{v}, \mathbf{u}_t, \mathbf{v}_t, \dots, \mathbf{u}_{t^k}, \mathbf{v}_{t^k})$. Note that in (4), $\mathbf{B} = \mathbf{B}(t, \mathbf{u}, \mathbf{v}, \mathbf{u}_t, \mathbf{v}_t, \dots, \mathbf{u}_{t^k}, \mathbf{v}_{t^k})$ is an m-vector, while $\mathbf{f}_{\mathbf{u}}$ and $\mathbf{f}_{\mathbf{v}}$ are $n \times n$ and $n \times m$ matrices with entries $f^i_{u^j}$ and $f^i_{v^j}$ respectively.

When restricted to (2), D_t becomes

$$\overline{D}_t = \partial_t + \sum_{i=1}^{n} A^j\frac{\partial}{\partial u^i} + \sum_{s=0}^{\infty}\sum_{j=1}^{m} v^j_{t^{s+1}}\frac{\partial}{\partial v^j_{t^s}}. \tag{5}$$

Besides, $\mathbf{A}$ and $\mathbf{B}$ restricted to (2) depend on $t, \mathbf{u}, \mathbf{v}, \mathbf{v}_t, \dots, \mathbf{v}_{t^k}$ only, that is, do not depend on any derivatives of $\mathbf{u}$.

To simplify notations we shall write $\mathbf{v}_s$ instead of $\mathbf{v}_{t^s}$. Substituting (5) into (4) we get

$$\partial_t\mathbf{A} + \sum_{i=1}^{n}\frac{\partial\mathbf{A}}{\partial u^i}f^i + \sum_{s=0}^{k}\sum_{j=1}^{m} v^j_{s+1}\frac{\partial\mathbf{A}}{\partial v^j_s} - \mathbf{f}_{\mathbf{u}}\mathbf{A} - \mathbf{f}_{\mathbf{v}}\mathbf{B} = 0. \tag{6}$$

The maximal order derivatives entering (6) are v^j_{s+j}. They enter it linearly; their contribution to (6) is

$$\sum_{j=1}^{m}\frac{\partial\mathbf{A}}{\partial v^j_s}.$$

There are no other summands to cancel them, and it follows that

$$\forall j: \qquad \frac{\partial\mathbf{A}}{\partial v^j_s} = 0.$$

In other words, if $\mathbf{B}$ depends on derivatives of $\mathbf{v}$ of orders up to k, then $\mathbf{A}$ depends on v^j_s, $s \le k-1$.

2. *Point symmetries.* Let us first consider point symmetries of (2). As is well known, in that case

$$\begin{aligned} \mathbf{A} &= \mathbf{S}(t, \mathbf{u}, \mathbf{v}) + \alpha(t, \mathbf{u}, \mathbf{v})\mathbf{u}_t, \\ \mathbf{B} &= \mathbf{T}(t, \mathbf{u}, \mathbf{v}) + \alpha(t, \mathbf{u}, \mathbf{v})\mathbf{v}_t, \end{aligned} \tag{7}$$

which corresponds to diffeomorphisms of the $(t, \mathbf{u}, \mathbf{v})$ space (the space of dependent and independent variables) with infinitesimal generators

$$-\alpha\frac{\partial}{\partial t} + \sum_{i=1}^{n} S^i \frac{\partial}{\partial u^i} + \sum_{j=1}^{m} T^i \frac{\partial}{\partial v^j}.$$

Here α is a scalar function. The symmetry (7) restricted to (2) becomes

$$\begin{aligned} \mathbf{A} &= \mathbf{S}(t, \mathbf{u}, \mathbf{v}) + \alpha(t, \mathbf{u}, \mathbf{v})\mathbf{f}, \\ \mathbf{B} &= \mathbf{T}(t, \mathbf{u}, \mathbf{v}) + \alpha(t, \mathbf{u}, \mathbf{v})\mathbf{v}_t, \end{aligned} \tag{8}$$

in accordance with the previous conclusion. Substituting (8) into (6) we subsequently observe that maximal order derivatives in (6) are components of $\mathbf{v}_1$, entering linearly:

$$\partial_t(\mathbf{S} + \alpha\mathbf{f}) + \sum_{i=1}^{n} \frac{\partial(\mathbf{S} + \alpha\mathbf{f})}{\partial u^i} f^i + \sum_{j=1}^{m} v_1^j \frac{\partial(\mathbf{S} + \alpha\mathbf{f})}{\partial v^j} - \mathbf{f}_{\mathbf{u}}(\mathbf{S} + \alpha\mathbf{f}) - \mathbf{f}_{\mathbf{v}}(\mathbf{T} + \alpha\mathbf{v}_1) = 0. \tag{9}$$

Therefore the coefficient by $\mathbf{v}_1$ equals zero:

$$\sum_{j=1}^{m} \frac{\partial(\mathbf{S} + \alpha\mathbf{f})}{\partial v^j} - \mathbf{f}_{\mathbf{v}}\alpha = 0,$$

that is, $(\mathbf{S} + \alpha\mathbf{f})_{\mathbf{v}} - \alpha\mathbf{f}_{\mathbf{v}} = 0$ or, furthermore,

$$\mathbf{S}_{\mathbf{v}} + \alpha_{\mathbf{v}}\mathbf{f} = 0. \tag{10}$$

In this notation $\alpha_{\mathbf{v}}$ is an $n \times 1$ matrix and $\mathbf{f}$ is a $1 \times n$ matrix.

PROPOSITION 1. *If* $\alpha_{\mathbf{v}} \neq 0$ *then* $\operatorname{rank} \mathbf{f}_{\mathbf{v}} \leq 1$.

Proof. On components, the relation (10) means

$$S^i_{v^j} = -\alpha_{v^j} f^i.$$

The compatibility conditions $S^i_{v^j v^k} = S^i_{v^k v^j}$ yield relations $f^i_{v^j}\alpha_{v^k} = f^i_{v^k}\alpha_{v^i}$ or

$$\forall i; \forall j, k: \quad \begin{vmatrix} f^i_{v^j} & f^i_{v^k} \\ \alpha_{v^j} & \alpha_{v^k} \end{vmatrix} = 0. \tag{11}$$

If $\alpha_{\mathbf{v}} \neq 0$, this means that f^i and α as functions of $\{v^j\}$, $j = 1, \ldots, m$, are functionally dependent for all i. Thus the equation (11) shows that $\operatorname{rank} \mathbf{f}_{\mathbf{v}} \leq 1$, that is, *de facto*, there is no more than one independent control parameter for the system (2) in case of $\alpha_{\mathbf{v}} \neq 0$. ∎

2.1. $\operatorname{rank} \mathbf{f}_{\mathbf{v}} \leq 1$. Of course, the absence of control parameters is a situation of no interest in the present context. In the reasonable case of $\operatorname{rank} \mathbf{f}_{\mathbf{v}} = 1$ one can choose $\alpha(t, \mathbf{u}, \mathbf{v})$ as a new variable which will be the sole control parameter. Thus $\mathbf{f} = \mathbf{\Phi}(t, \mathbf{u}, \alpha(t, \mathbf{u}, \mathbf{v}))$ or simply $\mathbf{f} = \mathbf{\Phi}(t, \mathbf{u}, \alpha)$ in accordance with (11). So the situation $\alpha_{\mathbf{v}} \neq 0$ makes sense only for $m = 1$. Now (7) takes the form

$$\begin{aligned} \mathbf{A} &= \mathbf{S}(t, \mathbf{u}, v) + \alpha(t, \mathbf{u}, v)\mathbf{f}(t, \mathbf{u}, v), \\ \mathbf{B} &= T(t, \mathbf{u}, v) + \alpha(t, \mathbf{u}, v)v_1. \end{aligned}$$

Here B, T and v are scalars. The symmetry equation becomes

$$\begin{cases} \mathbf{S}_v + \alpha_v \mathbf{f} = 0, \\ \partial_t(\mathbf{S} + \alpha\mathbf{f}) + (\mathbf{S} + \alpha\mathbf{f})_{\mathbf{u}}\mathbf{f} - \mathbf{f}_{\mathbf{u}}(\mathbf{S} + \alpha\mathbf{f}) - \mathbf{f}_v T = 0, \end{cases}$$

or

$$\begin{cases} \mathbf{A}_v = \alpha\mathbf{f}_v, \\ \partial_t\mathbf{A} + \mathbf{A}_{\mathbf{u}}\mathbf{f} - \mathbf{f}_{\mathbf{u}}\mathbf{A} = T\mathbf{f}_v. \end{cases} \tag{12}$$

To obtain a symmetry, get $\mathbf{A}$ using the former equation. The T is a kind of eigenvalue (if there are any) in the latter equation. See also Examples 1 and 2 below.

2.2. $\operatorname{rank}\mathbf{f}_{\mathbf{v}} > 1$. In another case, if $\alpha_{\mathbf{v}} = 0$ then $\mathbf{S}_{\mathbf{v}} = 0$ and in place of (7) we get

$$\begin{aligned} \mathbf{A} &= \mathbf{S}(t, \mathbf{u}) + \alpha(t, \mathbf{u})\mathbf{f}, \\ \mathbf{B} &= \mathbf{T}(t, \mathbf{u}, v) + \alpha(t, \mathbf{u})v_t. \end{aligned} \tag{13}$$

and in place of (9) we get

$$\partial_t(\mathbf{S} + \alpha\mathbf{f}) + \sum_{i=1}^{n} \frac{\partial(\mathbf{S} + \alpha\mathbf{f})}{\partial u^i} f^i - \mathbf{f}_{\mathbf{u}}(\mathbf{S} + \alpha\mathbf{f}) - \mathbf{f}_{\mathbf{v}}\mathbf{T} = 0 \tag{14}$$

or

$$\partial_t(\mathbf{S} + \alpha\mathbf{f}) + (\mathbf{S} + \alpha\mathbf{f})_{\mathbf{u}}\mathbf{f} - \mathbf{f}_{\mathbf{u}}(\mathbf{S} + \alpha\mathbf{f}) - \mathbf{f}_{\mathbf{v}}\mathbf{T} = 0.$$

After differentiation this takes the form

$$\mathbf{S}_t + \mathbf{S}_{\mathbf{u}}\mathbf{f} - \mathbf{f}_{\mathbf{u}}\mathbf{S} + [(\alpha\mathbf{f})_t + (\alpha_{\mathbf{u}}\mathbf{f})\mathbf{f}] = \mathbf{f}_{\mathbf{v}}\mathbf{T}. \tag{15}$$

In case $m = n$ or, rather, $\operatorname{rank}\mathbf{f}_{\mathbf{v}} = n$ the solution of (15) is readily obtained.

PROPOSITION 2. *In case $m = n$ point symmetries correspond to arbitrary transformations of* $\mathbf{u}$ *variables.*

Proof. Indeed, for arbitrary $n + 1$ functions α, S^i, $i = 1, \dots, n$, of t, $\mathbf{u}$ we get

$$\mathbf{T} = \mathbf{f}_{\mathbf{v}}^{-1}\{\mathbf{S}_t + \mathbf{S}_{\mathbf{u}}\mathbf{f} - \mathbf{f}_{\mathbf{u}}\mathbf{S} + [(\alpha\mathbf{f})_t + (\alpha_{\mathbf{u}}\mathbf{f})\mathbf{f}]\}, \tag{16}$$

since the matrix $\mathbf{f}_{\mathbf{v}}^{-1}$ is nondegenerate in this situation. Since any symmetry produces (infinitesimally) a transformation $\mathbf{u}_\tau = \mathbf{S} + \alpha\mathbf{f}$ compatible with (2), this proves the statement. The formulas (16) and (13) give the full description of point symmetries in case of $m = n$. ∎

The last remark concerns the case $1 < m < n$. As follows from (15),

$$\mathbf{S}_t + \mathbf{S}_{\mathbf{u}}\mathbf{f} - \mathbf{f}_{\mathbf{u}}\mathbf{S} + [(\alpha\mathbf{f})_t + (\alpha_{\mathbf{u}}\mathbf{f})\mathbf{f}] \in \operatorname{Im}\mathbf{f}_{\mathbf{v}}.$$

The dimension of the latter equals m, and this is a first rough obstruction to the existence of a symmetry. Yet there are situations where the maximal algebra is attained: see Example 3 below.

EXAMPLE

The case $m = 1$

1. As follows from Proposition 1, only in case of $m = 1$ the dependence of α on v is possible. Yet often enough α is independent of v even in this case. Consider the control system

$$\begin{cases} u^1_t = g(t, u^1, u^2), \\ u^2_t = h(t, u^1, u^2) + v. \end{cases}$$

Its point symmetries are the solutions of (12). Here

$$\mathbf{A} = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}, \qquad \mathbf{A_u} = \begin{pmatrix} A^1_{u^1} & A^1_{u^2} \\ A^2_{u^1} & A^2_{u^2} \end{pmatrix},$$

$$\mathbf{f}_v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{f_u} = \begin{pmatrix} g_{u^1} & g_{u^2} \\ h_{u^1} & h_{u^2} \end{pmatrix}.$$

Thus

$$\begin{cases} A^1_v = 0, \quad A^2_v = \alpha, \\ A^1_t + A^1_{u^1} g + A^1_{u^2}(h + v) - A^1 g_{u^1} - A^2 g_{u^2} = 0, \\ A^2_t + A^2_{u^1} g + A^2_{u^2}(h + v) - A^1 h_{u^1} - A^2 h_{u^2} = T. \end{cases}$$

Differentiating the third equation with respect to v and taking the first one into account we obtain $A^1_{u^2} = A^2_v g_{u^1} = \alpha g_{u^1}$ (the last equality follows from the second equation of the system). Since A^1 does not depend on v, this is also true for α.

Now $A^1 = A^1(t, u^1, u^2)$ is an arbitrary function, while A^2, α and T are obtained immediately from the latter system.

However, in the following example α does depend on v.

2. Consider the system

$$\begin{cases} u^1_t = v u^2, \\ u^2_t = v u^1. \end{cases}$$

Here

$$\mathbf{f}_v = \begin{pmatrix} u^2 \\ u^1 \end{pmatrix}, \qquad \mathbf{f_u} = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}.$$

We take $\alpha = v$. Then

$$\begin{cases} A^1_v = v u^2, \quad A^2_v = v u^1, \\ A^1_t + v(A^1_u u^2 + A^1_{u^2} u^1) - v A^2 = T u^2, \\ A^2_t + v(A^2_{u^1} u^2 + A^2_{u^2} u^1) - v A^1 = T u^1. \end{cases}$$

It follows from the first two equations that

$$A^1 = \tfrac{1}{2} v^2 u^2 + p(t, u^1, u^2), \qquad A^2 = \tfrac{1}{2} v^2 u^1 + q(t, u^1, u^2),$$

for some p, q. To satisfy the remaining equations it is sufficient to choose p and q in such a way that $pu^1 - qu^2 = 0$ (then $T = 0$). For instance, there is the following symmetry:

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2}(v^2 u^2 + t u^1 (u^2)^2 \\ \frac{1}{2}(v^2 u^1 + t (u^2)^3 \end{pmatrix}, \qquad B = v v_1.$$

The case $1 < m < n$

3. Let us consider an example of a linear system of the form $\mathbf{u}_t = P(t)\mathbf{u} + Q(t)\mathbf{v}$, where P and Q some proper-sized matrices. Multiplying it by $\exp(-\int P(t)\,dt)$ we obtain $\mathbf{w}_t = \mathcal{Q}\mathbf{v}$ for $\mathbf{w} = \exp(-\int P(t)dt)\mathbf{u}$ and $\mathcal{Q} = \exp(-\int P(t)dt)Q$. If $\operatorname{rank} \mathcal{Q} = m$, then by an invertible transformation on u^i's the simplest general form of such a system may be obtained: $\mathbf{U}_t = \mathbf{V}$, where $\mathbf{U} = (U^1, \dots, U^n)$ and $\mathbf{V} = (V^1, \dots, V^m, 0, \dots, 0)$.

The symmetry equation (4) for the latter system is as follows:

$$D_t\mathbf{A} - \mathbf{B}|_{\{\mathbf{U}_t=\mathbf{V}\}} = 0.$$

For point symmetries (13) we get

$$D_t\mathbf{S} + [\alpha_t\mathbf{V} + (\alpha_{\mathbf{U}}\mathbf{V})\mathbf{V}] = \mathbf{V}_{\mathbf{V}}\mathbf{T}$$

On components it means that

$$\begin{cases} T^i = D_tS^i + \alpha_tV^i + \Big(\sum\limits_{j=0}^{n} \alpha_{U^j}V^j\Big)V^i, \ 1 \le i \le m, \\ D_tS^i = 0, \ m < i \le n. \end{cases}$$

Thus, S^i, $i > m$, are arbitrary constants, $\alpha = \alpha(t,\mathbf{U})$ and $S^i(t,\mathbf{U})$, $0 \le i \le m$, are arbitrary functions, while $T^i(t,\mathbf{U},\mathbf{V})$ are defined by (17).

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