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# TRANSFER OF ESTIMATES FROM CONVEX TO STRONGLY PSEUDOCONVEX DOMAINS IN $\mathbb{C}^N$

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**Abstract.** In this article, estimates of the hyperbolic and Carathéodory distances in domains $G \subset\subset \mathbb{C}^n$, $n \geq 1$, are obtained. They are equally valid for the Kobayashi distance.

**1. Introduction.** In Section 2, general definitions and notions are given, upper and lower estimates of the hyperbolic distance $C_{E_r}(z, w),\ z,\ w \in E_r$ in the disc $E_r \subset \mathbb{C}$ with radius $r$ are obtained, which show that these estimates depend mainly on mutual ratios of the distances $d(z,\ \partial E_r),\ d(w,\ \partial E_r)$ of $z,\ w$ from the boundary $\partial E_r$ of $E_r$ and their distance apart $|z - w|$.

These results are used in Section 3 to obtain estimates in the Euclidean ball $B_r \subset \mathbb{C}^n$, $n > 1$, using the idea that: if $z,\ w \in B_r$ then the plane section $D$ of $B_r$ by the 1-dimensional analytic complex plane through $z,\ w$ is a metric plane in the terminology of Carathéodory [2] or a geodesic in the terminology of Vesentini [7]. Sufficient and necessary conditions for the boundedness of $C_{B_r}(z,\ w)$ are obtained. An example shows that the necessary and sufficient condition for $E_r \subset \mathbb{C}$ is only sufficient but not necessary in $B_r \subset \mathbb{C}^n$, $n > 1$.

In Section 4, it is shown that it is possible to use chains of balls to get estimates of $C_G(z,\ w)$ in a domain $G \subset\subset \mathbb{C}^n$, under certain conditions; which are easily applied if $G$ is convex. The estimates is then transfered from convex domains to strongly pseudoconvex domains by means of local biholomorphic maps [1, page 132]. A sufficient condition (but not necessary) for the boundedness of $C_G(z,\ w),\ z,\ w \in G$ is obtained. An example is given.

Finally, in section 5, the continuous extension of biholomorphic maps between strongly pseudoconvex domains with $C^2$ boundaries is proved [cf. 8].

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## 2. Basic notions

DEFINITIONS:

1- Let $G \subset\subset \mathbb{C}^n$ be a domain $G$ is called *smooth* if for every $z_o \in \partial G$ (the boundary of $G$) there exist a neighbourhood $U$ of $z_o$ and a real valued function $f \in C^2(U)$ such that

$$G \cap U = \{z \in U : \ f(z) < 0\} \quad \text{and} \quad df \neq 0 \ in \ G \cap U.$$

2- In definition (1), the 1-dimensional real inward normal to $\partial G$ at $z_o$, will be denoted by $N_{z_o}$.

3- If $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, then $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

4- If in definition (1), $z' \in G$, then $d(z', \ \partial G)$ denotes the distance of $z'$ from $\partial G$. Obvious there is a point $z'_o \in \partial G$ such that $z' \in N_{z'_o}$, $z'_o$ will be called the *projection of* $z'$ *on* $\partial G$.

5- Let $G \subset \mathbb{C}^n$ be a domain, $U \subset \mathbb{C}^n$, then $\partial G \cap U$ will be denoted by $\beta_U$.

6- If $z, \ w \in \mathbb{C}^n$, then the 1-dimensional complex analytic plane through $z, \ w$ will be denoted by $P(z, \ w)$.

7- Let $B_r \subset \mathbb{C}^n$ be the ball $|z| < r$, $z \in \mathbb{C}^n$. Let $G \subset\subset \mathbb{C}^n$ be a smooth domain and $z_o \in \partial G$, then $B_r$ when placed tangential to $\partial G$ at $z_o$, with $N_{z_o}$ lying on a diameter will be denoted by $B_r^{(z_o)}$. If $B_r^{(z_o)} \subset G$ for all $z_o \in \partial G$, then $B_r$ is called an *admissible ball to G*. If $G \subset\subset \mathbb{C}^n$ is smooth, then there exist always admissible balls $B_r$.
If $z_o$ has a neighbourhood $U$ such that $B_r^{(w)} \subset G$ for all $w \in \beta_U$, then $B_r$ is called an *admissible ball to G* at $z_o$.

8- Let $G \subset\subset \mathbb{C}^n$, $z, \ w \in G$; the Carathéodory and Kobayashi Distances in $G$ will be denoted by $C_G(z, \ w)$ and $K_G(z, \ w)$ respectively.

PROPOSITION 2.1. *Let $D, \ D_1 \subset \mathbb{C}^n$ be domains, $D' \subset\subset D$, $D'_1 \subset\subset D_1$ and $\phi$ a biholomorphic map of $D$ onto $D_1$, $\phi(D') = D'_1$, $z^{(j)} \in D'$, $j = 1, \ 2$ and $\phi(z^{(j)}) = w^{(j)} \in D'_1$, $j = 1, \ 2$, $\ell_1 = |z^{(1)} - z^{(2)}|$, $\ell_2 = |w^{(1)} - w^{(2)}|$. If the line segments*

$$L_1 = \overrightarrow{z^{(1)} z^{(2)}}, \ L_2 = \overrightarrow{w^{(1)} w^{(2)}}, \ L_1 \subset D', \ L_2 \subset D'_1$$

*then, there exist $0 < \alpha < \beta < +\infty$ such that $\alpha < (\ell_1/\ell_2) < \beta$.*

Proof. Let $v = z^{(2)} - z^{(1)} = (v_1, \cdots, v_n)$, $\gamma = \phi(L_1)$, $\phi = (\phi_1, \cdots, \phi_n)$, $\ell'_2 =$ length of the curve $\gamma$, $(d\phi_j, \ v) = \sum_{\mu=1}^{n} \frac{\partial \phi_j}{\partial z_\mu} v_\mu$. Since $\phi$ is regular in $D$ and $v \neq 0$, $|d\phi_j|_{D'} \neq 0$, $j = 1, \cdots, n$ and $(d\phi_j, \ v)$, $j = 1, \cdots, n$, do not vanish simultaneously. Let $M = \max_{D'}\{|d\phi_j|, \ j = 1, \ 2, \cdots, n\}$. Now, $|(d\phi_j, v)| \leq |d\phi_j| \ . \ |v| \leq M\ell_1$, thus

$$\ell_2 \leq \ell'_2 = \int_0^1 \sqrt{\sum_{j=1}^{n} |(d\phi_j, v)|^2 \ dt} \ \leq \sqrt{n} M \ell_1 = k_1 \ell_1$$

Similarly, there exist $k_2 > 0$ such that $\ell_1 \leq k_2 \ell_2$, thus, $1/k_1 \ \leq \ \ell_1/\ell_2 \ \leq \ k_2$.

PROPOSITION 2.2. *In Proposition 2.1, let $H \subset D'$, $H_1 \subset D'_1$ be smooth hypersurfaces such that $H_1 = \phi(H)$. Furthermore, let $z \in D'$, $w = \phi(z)$, $n_1 = d(z, \ H)$, $n_2 = d(w, \ H_1)$,* *then* $1/k_1 \ \leq \ (n_1/n_2) \ \leq \ k_2$ *provided that the projection of* $z$ *on* $H$ *is* $\in H$ *and projection of* $w$ *on* $H_1$ *is* $\in H_1$.

Proof. Let $z_o \in H$ and $w_o \in H_1$ be such that $n_1 = |z_o - z|, \ n_2 = |w_o - w|$. Let $\phi(z_o) = w'_o \in H_1, \ \phi^{-1}(w_o) = z'_o \in H, \ \ell_1 = |z - z'_o|, \ \ell_2 = |w - w_o|$. From Proposition 2.1, we have

$$\frac{1}{k_1} \ \leq \ \frac{n_1}{\ell_2} \ \leq \ \frac{n_1}{n_2} \ \ \textit{and} \ \ k_2 \ \geq \ \frac{\ell_1}{n_2} \ \geq \ \frac{n_1}{n_2}.$$

PROPOSITION 2.3. *Let* $E_r \subset \mathbb{C}$ *be the disc* $|z| < r$, *and* $z, \ w \in E_r$, *then*

$$C_{E_r}(z, \ w) = \log\left[\sqrt{1 + \frac{r^2}{(2r - r_1)(2r - r_2)} \cdot \frac{R^2}{r_1 r_2}} + \frac{r}{\sqrt{(2r - r_1)(2r - r_2)}} \frac{R}{\sqrt{r_1 r_2}}\ \right],$$

*where* $R = |z - w|, \ r_1 = d(z, \ \partial E_r), \ r_2 = d(w, \ \partial E_r)$.

COROLLARY 2.1. *Obviously*,

$$\frac{1}{4} \ < \ \frac{r^2}{(2r - r_1)(2r - r_2)} \ \leq \ 1.$$

*Thus*

$$\log\left[\sqrt{1 + \frac{1}{4}\frac{R^2}{r_1 r_2}} + \frac{1}{2}\frac{R}{\sqrt{r_1 r_2}}\right] \ < \ C_{E_r}(z, \ w) \ \leq \ \log\left[\sqrt{1 + \frac{R^2}{r_1 r_2}} + \frac{R}{\sqrt{r_1 r_2}}\right],$$

*which are inequalities independent on* $r$, *they depend only on ratios* $(R/r_j), \ j = 1, \ 2$

COROLLARY 2.2. *Let* $\{z_\nu\}_{\nu=1}^{\infty}, \ \{w_\nu\}_{\nu=1}^{\infty} \subset E_r$. *The necessary and sufficient conditions for* $\{C_{E_r}(z_\nu, \ w_\nu)\}$ *to be bounded, is that there exists*

$$0 \ \leq \ M \ < \ +\infty \quad \textit{such that} \quad \frac{R_\nu^2}{r_\nu r'_\nu} \ \leq \ M \quad \textit{for all} \ \nu,$$

*where* $R_\nu = |z_\nu - w_\nu|, \ r_\nu = d(z_\nu, \ \partial E_r), \ r'_\nu = d(w_\nu, \ \partial E_r)$.

The condition $(R_\nu^2/r_\nu r'_\nu) \leq M$ is equivalent to $(R_\nu/r_\nu) \leq M_1 < +\infty$ and $(R_\nu/r'_\nu) \leq M_2 < +\infty$, for all $\nu$.

## 3. The Euclidean ball

THEOREM 3.1. *Let* $B_r \subset \mathbb{C}^n, \ n > 1$, *be the ball* $|z| < r, \ z \in \mathbb{C}^n$. *If* $z, \ w \in B_r$, *then* $C_{B_r}(z, \ w) = C_D(z, \ w)$, *where* $D$ *is the disc* $P(z, \ w) \cap B_r$ *(see definition 6)*.

Proof. Let $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$. Obvious if $z' = (z'_1, 0, \cdots, 0), \ z'' = (z''_1, 0, \cdots, 0)$, $|z'_1| < r$, $|z''_1| < r$ and $D'$ be the disc $|z_1| < r, \ z_j = 0, \ j = 2, \cdots, n$. Then

$$C_{B_r}(z', \ z'') = C_{D'}(z'_1, \ z''_1)$$

Now, there exists [3] an automorphism of $B_r$, which maps $D$ conformally on $D'$, which proves the theorem.

From Proposition 2.3, Corollaries 2.1 and 2.2 we get:

Corollary 3.1.

$$C_{B_r}(z,\ w) = \log\left(\sqrt{1+\frac{\rho^2}{(2\rho-\rho_1)(2\rho-\rho_2)}\cdot\frac{R^2}{\rho_1\rho_2}}+\frac{\rho}{\sqrt{(2\rho-\rho_1)(2\rho-\rho_2)}}\cdot\frac{R}{\sqrt{\rho_1\rho_2}}\right)$$

*where* $\rho =$ *radius of* $D$, $\rho_1 = d(z,\ \partial D),\ \rho_2 = d(w,\ \partial D)$ *and* $\frac{1}{4} < \frac{\rho^2}{(2\rho-\rho_1)(2\rho-\rho_2)} \leq 1$. *Thus*

$$\log\left(\sqrt{1+\frac{1}{4}\frac{R^2}{\rho_1\rho_2}}+\frac{1}{2}\frac{R}{\sqrt{\rho_1\rho_2}}\right) < C_{B_r}(z,w) \leq \log\left(\sqrt{1+\frac{R^2}{\rho_1\rho_2}}+\frac{R}{\sqrt{\rho_1\rho_2}}\right)$$

Corollary 3.2. *Let* $\{z_\nu\}_{\nu=1}^{\infty},\ \{w_\nu\}_{\nu=1}^{\infty} \subset B_r$; *the necessary and sufficient condition for* $\{C_{B_r}(z_\nu,\ w_\nu)\}$ *to be bounded is that there exists*

$$0 \leq M < +\infty \quad \textit{such that} \quad \frac{R_\nu^2}{\rho_\nu \rho'_\nu} \leq M$$

*where* $R_\nu = |z_\nu - w_\nu|,\ \rho_\nu = d(z_\nu,\ \partial D_\nu),\ \rho'_\nu = d(w_\nu,\ \partial D_\nu)$, *or equivalently* $(R_\nu/\rho_\nu) \leq M' < +\infty,\ \rho_\nu \leq \rho'_\nu$, *where* $D_\nu = P(z_\nu,\ w_\nu) \cap B_r$.

(Notice that $\rho_\nu + R_\nu \geq \rho'_\nu,\ \rho'_\nu + R_\nu \geq \rho_\nu$).

Corollary 3.3. *From Corollary* 3.1, *it follows that*

$$C_{B_r}(z,\ \nu) = \log\left(\sqrt{1+\frac{\rho}{(2r-r_1)(2r-r_2)}\cdot\frac{R^2}{r_1 r_2}}+\frac{\rho}{\sqrt{(2r-r_1)(2r-r_2)}}\cdot\frac{R}{\sqrt{r_1 r_2}}\right)$$

*where* $r_1 = d(z,\ B_r),\ r_2 = d(w,\ \partial B_r)$. *It is obvious that*

$$0 \ < \ \frac{\rho^2}{(2r-r_1)(2r-r_2)} \ \leq \ 1.$$

*Thus*,

$$C_{B_r}(z,\ w) \ \leq \ \log\left(\sqrt{1+\frac{R^2}{r_1 r_2}}+\frac{R}{\sqrt{r_1 r_2}}\right).$$

Corollary 3.4. *In Corollary* 3.2, *let* $r_\nu = d(z_\nu,\ \partial B_r),\ r'_\nu = d(w_\nu,\ \partial B_r)$.

From Corollary 3.3 we get:

For $\{C_{B_r}(z_\nu,\ w_\nu)\}$ to be bounded it is sufficient that there exists $0 \leq M < +\infty$ such that$(R_\nu^2/r_\nu r'_\nu) \ \leq \ M$ (or equivalently $(R_\nu/r_\nu) < M' < +\infty,\ r_\nu \leq r'_\nu$)

Corollary 3.5. *In Corollary* 3.4, *the condition* $\frac{R^2\nu}{r_\nu r'_\nu} < M$ *is sufficient but not necessary as is illustrated by the following example*:

Example. Let $B \subset \mathbb{C}^2$ be the unit ball $z_1\overline{z}_1 + z_2\overline{z}_2 < 1$ and $z^{(\nu)} = (\frac{1}{\nu},\ b_\nu),\ w^{(\nu)} = (\frac{1}{\nu}e^{\frac{\pi}{6}},\ b_\nu),\ b_\nu^2 = 1 - \frac{4}{\nu^2},\ \nu \geq 2$.

If $D_\nu = P(z^{(\nu)},\ w^{(\nu)}) \cap B$, then the radius of $D_\nu = \frac{2}{\nu}$. Therefore

$$\rho_\nu = \frac{1}{\nu}, \quad \rho'_\nu = \frac{1}{\nu}, \quad R_\nu = \frac{1}{\nu}$$

Thus, $\frac{R_\nu^2}{\rho_\nu\rho'_\nu} = 1$ for all $\nu$, hence $\{C_B(z^{(\nu)},\ w^{(\nu)})\}$ is bounded.

In fact, from Corollary 3.1:

$$C_B(z^{(\nu)},\ w^{(\nu)}) \equiv \log(\frac{\sqrt{13}+2}{3}), \quad for\ all\ \nu.$$

While $r_\nu = r'_\nu < \frac{3}{\nu^2}$, thus,

$$\frac{R_\nu^2}{r_\nu r'_\nu} > \frac{\nu^2}{9} \;\to\; +\infty$$

as $\nu \to \infty$. Thus, the condition $\frac{R_\nu^2}{r_\nu r'_\nu} \le M < +\infty$ is not necessary for $C_B(z^{(\nu)},\ w^{(\nu)})$ to be bounded.

PROPOSITION 3.1. *Let $z,\ w \in B_r$, $z,\ w \in N_{z_o}$, $z_o \in \partial B_r$, $|z - z_o| = r_1$, $|w - z_o| = r_2$ and $r_1 < r_2 < r$. Then*

$$C_{B_r}(z,\ w) = -\frac{1}{2}\log r_1 + \frac{1}{2} log\ r_2 + \Psi(z,\ w)$$

*where $|\Psi(z,\ w)| \le k < +\infty$*

Proof. This is because $N_{z_o}$ is a geodesic in $B_r$

THEOREM 3.2. *Let $G \subset \mathbb{C}^n$ be a strongly pseudoconvex domain and $z_o \in \partial G$. If $z,\ w \in N_{z_o}$, $r_1 = |z - z_o|$, $r_2 = |w - z_o|$, $r_1 < r_2 < r$, where $B_r$ is an admissible ball to $G$, then*

$$C_G(z,\ w) = -\frac{1}{2}\log r_1 + \frac{1}{2}\log r_2 + \Psi(z,\ w)$$

*where $|\Psi(z, w)| < k < +\infty$.*

Proof. From Proposition 3.1:

$$C_G(z,\ w) \le C_{B_r}(z,\ w) = -\frac{1}{2}\log r_1 + \frac{1}{2}\log r_2 + \Psi(z,\ w)$$

$$|\Psi(z,\ w)| \le k_1 < +\infty, \tag{3.1}$$

In [4], it is proved that if $A \in G$ is fixed and $\xi \in G$, then

$$C_G(A,\ \xi) = -\frac{1}{2}\log r' + \phi(\xi)$$

where

$$|\phi(\xi)| \ \le\ k' \ <\ +\infty \quad \textit{and} \quad r' = d(\xi, \partial G)$$

Thus,

$$\begin{aligned} C_G(z,\ w) \ &\ge\ C_G(A,\ z) - C_G(A,\ w) \\ &= -\frac{1}{2}\log r_1 + \frac{1}{2}\log r_2 + (\phi(z) - \phi(w)). \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), the result follows.

## 4. Domains in $\mathbb{C}^n$

THEOREM 4.1. *Let $G \subset \mathbb{C}^n$ be a smooth domain and $B_r$, $r > 0$ be an admissible ball to $G$. Let $\{z_\nu\}$, $\{w_\nu\} \subset G$ such that:*

(i) *If $L_\nu$ is the line joining $z_\nu$ to $w_\nu$, then $L_\nu \subset G$.*
(ii) *If $\xi \in L_\nu$, then $d(\xi,\ \partial G) \le r$.*
(iii) *Let $\lambda_\nu = d(L_\nu,\ \partial G)$, and $\ell_\nu =$ length of $L_\nu$. If $(\ell_\nu/\lambda_\nu) \le k < +\infty$ for all $\nu$.*

*Then $\{C_G(z_\nu,\ w_\nu)\}$ will be bounded ($\le\ 2\ (k+1)\ \log(\frac{\sqrt{6}+\sqrt{2}}{2})$).*

Proof. Let $m = [k] + 1$, $[k] =$ integral part of $k$.

We divide $L_\nu$ into $2m$ equal parts, each of length $\leq \lambda_\nu/2$ by the points $z_\nu = x_o,\ x_1, \cdots, x_{2m} = w_\nu$.

Thus,

$$R'_j = |x_{j+1} - x_j| \ \leq \lambda_\nu \ / \ 2, \qquad j = 0, \cdots, 2m-1,$$

$$r'_j = d(x_j,\ \partial G) \ \geq \ \lambda_\nu \ \ , \ \ r''_j = d(x_{j+1},\ \partial B_j) \ \geq \ \frac{\lambda\ \nu}{2},$$

(Since $(r''_j + R'_j \ \geq \ r'_j)$, where $B_j = B_r^{(x_j)}$. Thus

$$(R'^2_j / r'_j r''_j) \ \leq \ \frac{1}{2}$$

and thus from Corollary 3.3 we get

$$C_G(x_j, x_{j+1}) \ \leq \ C_{B_j}(x_j, x_{j+1}) \ \leq \ \log \frac{\sqrt{6}+\sqrt{2}}{2},$$

which proves the theorem.

Theorem 4.2. *Let $G \subset \mathbb{C}^n$ be a smooth convex domain $\{z_\nu\}_{\nu=1}^{\infty},\ \{w_\nu\}_{\nu=1}^{\infty} \subset G$, $r_\nu = d(z_\nu,\ \partial G)$, $r'_\nu = d(w_\nu,\ \partial G)$, $\lim_{\nu\to\infty}\ z_\nu = z_o = \lim_{\nu\to\infty}\ w_\nu$, $R_\nu = |z_\nu - w_\nu|$, if $r_\nu \leq r'_\nu$, $R_\nu \leq k r_\nu$, $0 \leq k < +\infty$ (or equivalently $\frac{R_\nu^2}{r_\nu r'_\nu} \leq M < +\infty$) then,*

$$\{C_G(z_\nu,\ w_\nu)\}$$

*is bounded by $k'$.*

The condition $r_\nu \leq r'_\nu$ is not a restriction since $C_G(z_\nu,\ w_\nu)$ is symmetric in $z_\nu$ and $w_\nu$.

Proof. Let $L_\nu$ and $r_\nu$ be as in Theorem 4.1. Since $G$ is convex, $r_\nu = d(L_\nu,\ \partial G)$. Obvious there exists $\nu_o$ such that $d(\xi,\ \partial G) < r$ for $\xi \in L_\nu$, $\nu \geq \nu_o$. Thus all the conditions of Theorem 4.1 are satisfied.

Corollary 4.1. *Theorem 4.2 remains valid if $B_r$ is an admissible ball to $G$ at $z_o$ and $k'$ depends on $k$ and $r$.*

Theorem 4.3. *Let $G \subset\subset \mathbb{C}^n$ be a smooth strongly pseudoconvex, $z_o \in \partial G$, $\{z_\nu\}_{\nu=1}^{\infty}$, $\{w_\nu\}_{\nu=1}^{\infty} \subset G$ be two sequences converging to $z_o$ if $r_\nu = d(z_\nu,\ \partial G)$, $r'_\nu = d(w_\nu,\ \partial G)$, $R_\nu = |z_\nu - w_\nu|$. If $\frac{R_\nu^2}{r_\nu r'_\nu} \leq M < +\infty$ (or equivalently $r_\nu \leq r'_\nu, (R_\nu/r_\nu) \leq M_1 < +\infty$) then*

$$\{C_G(z_\nu,\ w_\nu)\}$$

*is bounded by $k_1 < +\infty$.*

Proof. There exist [1. page 132 ] neighbourhoods $U$ and $U'$ of $z_o$, $U' \subset\subset U$ and a biholomorphic map $\phi :\ U \to W \subset \mathbb{C}^n$ such that $\phi(U' \cap G) = D$ and $D$ is strictly convex.

Let $\phi(z_\nu) = z'_\nu$, $\phi(w_\nu) = w'_\nu$, $d(z'_\nu,\ \partial D) = \rho_\nu$, $d(w'_\nu,\ \partial D) = \rho'_\nu$, $R'_\nu = |z'_\nu - w'_\nu|$, then from Propositions 2.1, 2.2 there exists $0 < k < +\infty$ such that $\frac{R'^2_\nu}{\rho_\nu \rho'_\nu} \leq kM$. From Theorem 4.2, $\{C_D(z'_\nu,\ w'_\nu)\}$ will be bounded. Since $C_G(z_\nu,\ w_\nu) \leq C_D(z'_\nu,\ w'_\nu)$. The result follows.

We notice that $k_1$ depends only on $M_1$ and $z_o$.

As proved in Section 3, the condition $\frac{R_\nu^2}{r_\nu r'_\nu} \leq M < +\infty$, is not necessary.

THEOREM 4.4. *If in Theorem 4.2, the condition* $(R_\nu^2/r_\nu r_\nu') < M$ *is replaced by* $R_\nu \le \lambda r_\nu$ *and* $(R_\nu/r_\nu') \to \infty$ *as* $\nu \to +\infty$, *then*

$$C_G(z_\nu,\ w_\nu) = -\frac{1}{2}\log r_\nu' + \frac{1}{2}\log r_\nu + \phi(z_\nu,\ w_\mu),$$

*and*

$$|\phi(z_\nu,\ w_\nu)|\ \le\ k\ <\ +\infty$$

Proof. Let $A \in G$ be fixed. As in Theorem 3.2

$$C_G(z_\nu,\ w_\nu)\ \ge\ C_G(A,\ w_\nu) - C_G(A,\ z_\nu)$$

$$= -\frac{1}{2}\log r_\nu' + \frac{1}{2}\log r_\nu + k_\nu, \tag{4.1}$$

where $|k_\nu| \le k$. Let $w_\nu'$ be the projection of $w_\nu$ on $\partial G$ and $w_\nu'' \in N_{w_\nu'}$ such that $r_\nu'' = |w_\nu' - w_\nu''| = R_\nu + r_\nu > R_\nu$. Let $R_\nu' = |z_\nu - w_\nu''|$, then $R_\nu' \le 2R_\nu$. Thus, $\frac{R_\nu'^2}{r_\nu r_\nu''} \le 4\lambda$.

Thus, from Theorem 4.3, $\{C_G(w_\nu'',\ z_\nu)\}$ is bounded $< M$. Therefore,

$$C_G(z_\nu,\ w_\nu)\ \le\ C_G(z_\nu,\ w_\nu'') + C_G(w_\nu,\ w_\nu'')$$

$$\le\ M - \frac{1}{2}\log r_\nu' + \frac{1}{2}\log(r_\nu' + R_\nu) + k_\nu' \quad \text{(from Theorem 3.2)}$$

$$\le\ M - \frac{1}{2}\log r_\nu' + \frac{1}{2}\log 2R_\nu + k_\nu'', \quad \le\ M_\nu - \frac{1}{2}\log r_\nu' + \frac{1}{2}\log r_\nu, \tag{4.2}$$

where $M_\nu < k_1$ for all $\nu$.
From (4.1) and (4.2), we get the result.

THEOREM 4.5. *In Theorem* 4.4, *if Condition* $R_\nu \le \lambda r_\nu$ *is replaced by* $(R_\nu/r_\nu) \to +\infty$ *then, there exists a constant* $k$ *such that*

$$C_G(z_\nu, w_\nu)\ \le\ -\frac{1}{2}\log r_\nu - \frac{1}{2}\log r_\nu' + \log(R_\nu + r_\nu) + k$$

*for all* $\nu$.

Proof. Let $z_\nu'$ be the projection of $z_\nu$ on $\partial G$, $z_\nu'' \in N_{z_\nu'}$ such that $|z_\nu' - z_\nu''| = r_\nu'' = r_\nu + R_\nu$.

Then $z_\nu''$, $w_\nu$ satisfy conditions of Theorem 4.3

$$C_G(z_\nu'',\ w_\nu) = -\frac{1}{2}\log r_\nu' + \frac{1}{2}\log(r_\nu + R_\nu) + k_\nu$$

Also, from Theorem 3.2

$$C_G(z_\nu,\ z_\nu'') = -\frac{1}{2}\log r_\nu + \frac{1}{2}\log(r_\nu + R_\nu) + k_\nu'$$

where $k_\nu$ and $k_\nu'$ are bounded.

By the triangle axiom

$$C_G(z_\nu,\ w_\nu)\ \le\ -\frac{1}{2}\log r_\nu - \frac{1}{2}\log r_\nu' + \log(r_\nu + R_\nu) + k.$$

COROLLARY 4.2. *If $r_\nu \geq r'_\nu$ then $1 \leq \frac{r_\nu + R_\nu}{r'_\nu + R_\nu} \leq 2$, (since $r'_\nu + R_\nu \geq r_\nu$). Thus in Theorem 4.5, there exist $0 \leq k' < +\infty$ such that*

$$C_G(z_\nu, w_\nu) \;\leq\; -\frac{1}{2}\log r_\nu - \frac{1}{2}\log r'_\nu + \frac{1}{2}\log(r_\nu + R_\nu)$$

$$+\frac{1}{2}\log(r'_\nu + R_\nu) + k' \tag{4.3}$$

*which is the formula obtained before in [6].*

COROLLARY 4.3. *From Theorems 3.2, 4.3, 4.4, 4.5 and Corollary 4.2, we see that inequality (4.3) is valid for any two sequences $\{z_\nu\}$, $\{w_\nu\} \subset G$ converging to a point $z_o \in \partial G$ where $G$ is a smooth strongly pseudoconvex domain, ($k'$ depends on $z_o$).*

THEOREM 4.6 (a necessary condition). *Let $G \subset \mathbb{C}^n$ be a smooth strongly pseudoconvex domain. Let $\{z_\nu\}_{\nu=1}^\infty$, $\{w_\nu\}_{\nu=1}^\infty \subset G$, for $\{C_G(z_\nu,\ w_\nu)\}$ to be bounded, it is necessary that $0 < \ell_1 \leq \frac{r_\nu}{r'_\nu} \leq \ell_2 < +\infty$ where $r_\nu = d(z_\nu,\ \partial G)$, $r'_\nu = d(w_\nu,\ \partial G)$.*

Proof. Let $A \in G$ be a fixed point. Then [4]

$$C_G(A,\ z_\nu) = -\frac{1}{2}\log r_\nu + k(z_\nu),$$

where $|k(z)| \leq k_1 < +\infty$, for all $z \in G$.

Since,

$$C_G(z_\nu, w_\nu) \;\geq\; |C_G(A,\ z_\nu) - C_G(A,\ w_\nu)|$$

we get the result.

EXAMPLE. We give an example to show that in any smooth strongly pseudoconvex domain $G \subset \mathbb{C}^n$, there exist sequences $\{z_\nu\}$, $\{w_\nu\} \subset G$ converging to a point $z_o \in \partial G$, $\lim_{\nu\to\infty} \frac{R_\nu}{r_\nu} = \lim_{\nu\to\infty} \frac{R_\nu}{r'_\nu} = +\infty$ and in spite of this $\{C_G(z_\nu,\ w_\nu)\}$ is bounded; i.e., the condition $\frac{R_\nu^2}{r_\nu r'_\nu} < k < +\infty$ is not necessary for the boundedness of $\{C_G(z_\nu,\ w_\nu)\}$.

We use the idea of the example given in Section 3. Let $B = B_r \subset \mathbb{C}^n$ be an admissible ball to $G$, $\zeta_o \in \partial G$ and $\{\zeta_\nu\}_{\nu=1}^\infty \subset \partial G$ converging to $\zeta_o$.

Let $z_\nu,\ w_\nu \in B$, $z_\nu = \{\frac{r}{\nu},\ b_\nu,\ 0, \cdots, 0\}$, $w_\nu = \{\frac{r}{\nu}e^{i\frac{\pi}{6}},\ b_\nu,\ 0, \cdots, 0\}$, $\nu = 2,\ 3, \cdots, b_\nu^2 = r^2 - \frac{4r^2}{\nu^2}$.

As in section 3, $C_B(z_\nu, w_\nu) = \log(\frac{\sqrt{13}+2}{3})$ for all $\nu$.

Let $z'_\nu$ be the projection of $z_\nu$ onto $\partial B$. Let $B_{z'_\nu}$ be the ball $B$ placed tangential to $\partial G$ at $\zeta_\nu$ with the point $z'_\nu$ coincident with $\zeta_\nu$ such that the diameter of $B$ through $z'_\nu$, $z_\nu$ lie on $N_{\zeta_\nu}$. Let $z_\nu$, $w_\nu$ coincide with $z''_\nu$, $w'_\nu \in G$. Obvious $z''_\nu \to \zeta_o$ and $w'_\nu \to \zeta_o$. Now

$$|z_\nu - z'_\nu| = |\zeta_\nu - z''_\nu| = d(z_\nu,\ \partial B) = d(z''_\nu,\ \partial G) = r_\nu,$$

and

$$R_\nu = |z_\nu - w_\nu| = |z''_\nu - w_\nu|.$$

As proved in the example in section 3, $\frac{R_\nu}{r_\nu} \to +\infty$.

It is obvious that

$$C_G(z''_\nu,\ w'_\nu) \;\leq\; C_B(z_\nu,\ w_\nu) = \log(\frac{\sqrt{13}+2}{3})$$

Let $r'_\nu = d(w'_\nu,\ \partial G)$. Since $C_G(z''_\nu,\ w'_\nu)$ is bounded, then from Theorem 4.5, we get

$$0\ <\ \ell_1\ \leq\ \frac{r_\nu}{r'_\nu}\ \leq\ \ell_2\ <\ +\infty$$

Since $(R_\nu/r_\nu) \to \infty$, then

$$(R_\nu\ /\ r'_\nu) \to +\infty$$

## 5. Continuous (topological) extensions of biholomorphic maps of strongly pseudoconvex domains

Definition 5.1.

Let $G \subset\subset \mathbb{C}^n$ be a smooth domain:

(i) Let $z,\ w \in G$, $r_1 = d(z,\ \partial G)$, $r_2 = d(w,\ \partial G)$. We define

$$\Psi_G(z,\ w) = -\frac{1}{2}\log r_1 - \frac{1}{2}\log r_2 \quad ,$$

and

$$T_G(z,\ w) = \Psi_G(z,\ w) - C_G(z,\ w)$$

(ii) Let $S = \{z_\nu\}_{\nu=1}^{\infty} \subset G$. $S$ is called a *boundary sequence* if $S$ has no limiting point in $G$. Furthermore, if $\lim_{\nu\to\infty}\ z_\nu = z_o \in \partial G$, $S$ is called a *simple* boundary sequence.

Now, let $G$ be strongly pseudoconvex, $\{z_\nu\}_{\nu=1}^{\infty}$, $\{w_\nu\}_{\nu=1}^{\infty} \subset G$ and $\lim_{\nu\to\infty} z_\nu = \lim_{\nu\to\infty} w_\nu = z_o \in \partial G$, then from Corollary 4.3, we get

$$\lim_{\nu\to\infty}\ T_G\ (z_\nu,\ w_\nu) = +\infty \tag{5.1}$$

In [5], it is proved that if $z_\nu \to z_o \in \partial G$ and $w_\nu \to w_o \in \partial G$, $z_o \neq w_o$, there exists $\nu_o > 0$, such that

$$C_G(z_\nu,\ w_\nu) = -\frac{1}{2}\log r_\nu - \frac{1}{2}\log r'_\nu + \theta(z_\nu,\ w_\nu),$$

where $r_\nu = d(z_\nu,\ \partial G)$, $r'_\nu = d(w_\nu,\ \partial G)$ and $|\theta(z_\nu,\ w_\nu)| \leq k < +\infty$ for all $\nu \geq \nu_o$. Thus,

$$|\ T_G(z_\nu,\ w_\nu)\ |\ \leq\ k' \quad for\ all\ \nu \tag{5.2.}$$

From (5.1) and (5.2), we see that if:

(i) $\{z_\nu\} \subset G$ is simple boundary sequence $\to z_o \in \partial G$,

(ii) $\{w_\nu\} \subset G$ is a boundary sequence then,

$$\lim_{\nu\to\infty} T_G(z_\nu,\ w_\nu) = +\infty,$$

if and only if $\{w_\nu\} \subset G$ is a simple boundary sequence $\to z_o$.

Now, let $\phi$ be a biholomorphic map of $G$ onto another smooth strongly pseudoconvex domain $G_1$, $\phi(z) = x \in G_1$, $r = d(z,\ \partial G)$, $\rho = d(x,\ \partial G_1)$, $A \in G$ be a fixed point and $\phi(A) = A'$. In [4], it is proved that

$$C_G(A,\ z) = -\frac{1}{2}\log r + k(z), \quad |k(z)|\ <\ k_1, \quad for\ all\ z\ \in\ G$$

$$C_{G_1}(A',\ x) = -\frac{1}{2}\log \rho + k'(x), \quad |k'(x)|\ <\ k'_1, \quad for\ all\ x\ \in\ G_1$$

Since

$$C_G(A,\ z) = C_{G_1}(A',\ x),$$

there exist $0 < \ell_1 < \ell_2 < +\infty$ such that $\ell_1 \leq (r/\rho) \leq \ell_2$ for all $z \in G,\ x \in G_1$.

Thus, if $w \in G$, $\phi(w) = y$

$$|\Psi_{G_1}(x,\ y) - \Psi_G(z,\ w)| \ < \ k_3 \qquad \textit{for all } z,\ w \in G,$$

Thus,

$$|\ [\Psi_{G_1}(x,\ y) - C_{G_1}(x,\ y)] - [\Psi_G(z,\ w) - C_G(z,\ w)]\ | \ < \ k_3$$

i.e.,

$$|T_{G_1}(x,\ y) - T_G(z,\ w)| \ < \ k_3 \tag{5.3.}$$

Now, let $\{z_\nu\} \subset G$ be any simple boundary sequence $\to z_o \in \partial G$, such that $\{x_\nu = \phi(z_\nu)\}_{\nu=1}^{\infty} \subset G_1$ be also a simple boundary sequence $\to x_o \in \partial G_1$

Furthermore, let $\{w_\nu\}_{\nu=1}^{\infty} \subset G$ be any simple boundary sequence $\to z_o \in \partial G$. Thus, from (5.1)

$$\lim_{\nu\to\infty} T_G(z_\nu,\ w_\nu) = +\infty,$$

From (5.3) if $y_\nu = \phi(w_\nu)$, then

$$\lim_{\nu\to\infty} T_{G_1}(x_\nu,\ y_\nu) = +\infty$$

Since $\{x_\nu\} \subset G_1$ is a simple boundary sequence $\to x_o$, then $\{y_\nu\}$ will be a simple boundary sequence $\to x_o$.

Doing the same thing with $\phi^{-1}$, we see that if $\{y_\nu\}_{\nu=1}^{\infty} \subset G_1$ is any boundary sequence converging to $x_o$, then $\{w_\nu = \phi^{-1}(y_\nu)\}_{\nu=1}^{\infty}$ will be also a simple boundary sequence converging to $z_o$. Thus if we define $\phi(z_o) = w_o$, we get the following theorem.

THEOREM 5.1. *Any biholomorphic map of a strongly pseudoconvex domain $G \subset \mathbb{C}^n$ with a $C^2$ boundary onto a strongly pseudoconvex domain $G_1 \subset \mathbb{C}^n$ with a $C^2$ boundary, has a topological extension to be boundary.*

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