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ACCELERATION PROPERTIES OF THE HYBRID PROCEDURE FOR SOLVING LINEAR SYSTEMS

Abstract. The aim of this paper is to discuss the acceleration properties of the hybrid procedure for solving a system of linear equations. These properties are studied in a general case and in two particular cases which are illustrated by numerical examples.

1. The hybrid procedure. Let us consider the system of linear equations

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ and $x, b \in \mathbb{R}^m$. We denote by $\widetilde{x}$ the solution of (1).

Let $G = Z^T Z$ be a symmetric positive definite matrix. The G-inner product and the corresponding G-norm are respectively defined by $(x,y)_G = (x, Gy)$ and $\|x\|_G = \sqrt{(x,x)_G}$. The corresponding G-matrix norm is given by

$$\|A\|_G = \sup_{x\neq 0} \frac{\|Ax\|_G}{\|x\|_G} = \sqrt{\varrho((ZAZ^{-1})^T ZAZ^{-1})}.$$

We shall also use the notation $x \perp_G y$ if $(x,y)_G = 0$. For simplicity, the subscript G will be suppressed when unnecessary.

Let us now assume that two iterative methods for solving the system (1) are used simultaneously. Their iterates are denoted respectively by x'_n and x''_n and the corresponding residual vectors by $r'_n = b - Ax'_n$ and $r''_n = b - Ax''_n$.

The hybrid procedure defined in [1] consists of constructing a new iterate x_n and a new residual $r_n = b - Ax_n$ by

$$x_n = \alpha_n x'_n + (1-\alpha_n) x''_n, \qquad r_n = \alpha_n r'_n + (1-\alpha_n) r''_n, \tag{2}$$

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with
$$\alpha_n = -\frac{(r'_n - r''_n, r''_n)}{(r'_n - r''_n, r'_n - r''_n)}.$$
From the definition of r_n, we see that
$$\|r_n\| = \min_\alpha \|\alpha r'_n + (1-\alpha) r''_n\|.$$
We have
$$(r_n, r'_n) = (r_n, r''_n) = (r_n, r_n)$$
and, setting $p_n = r'_n - r''_n$, (2) can be written as
$$r_n = r''_n - \frac{(p_n, r''_n)}{(p_n, p_n)} p_n, \qquad r_n = r'_n - \frac{(p_n, r'_n)}{(p_n, p_n)} p_n. \tag{3}$$
It is easy to check that
$$(r_n, r_n) = \frac{(r'_n, r'_n)(r''_n, r''_n) - (r'_n, r''_n)^2}{(r'_n - r''_n, r'_n - r''_n)} \tag{4}$$
$$= (r''_n, r''_n) - \frac{(p_n, r''_n)^2}{(p_n, p_n)} \tag{5}$$
$$= (r'_n, r'_n) - \frac{(p_n, r'_n)^2}{(p_n, p_n)}. \tag{6}$$

2. Properties of the hybrid procedure. We now study the acceleration properties of the hybrid procedure.

2.1. *Asymptotic behavior of the hybrid procedure*. Let θ_n be the angle between Zr'_n and Zr''_n. Using the relation $(r'_n, r''_n) = \|r'_n\|\|r''_n\| \cos\theta_n$ we have
$$\alpha_n = -\frac{\|r'_n\|\|r''_n\| \cos\theta_n - \|r''_n\|^2}{\|r'_n\|^2 - 2\|r'_n\|\|r''_n\| \cos\theta_n + \|r''_n\|^2}$$
and
$$\|r_n\|^2 = \frac{\|r'_n\|^2 \|r''_n\|^2 (1 - \cos^2\theta_n)}{\|r'_n\|^2 - 2\|r'_n\|\|r''_n\| \cos\theta_n + \|r''_n\|^2}.$$
Setting $\varrho_n = \|r'_n\|/\|r''_n\|$ we obtain
$$\alpha_n = -\frac{\varrho_n \cos\theta_n - 1}{\varrho_n^2 - 2\varrho_n \cos\theta_n + 1},$$
$$\frac{\|r_n\|^2}{\|r'_n\|^2} = \frac{1 - \cos^2\theta_n}{\varrho_n^2 - 2\varrho_n\cos\theta_n + 1} \tag{7}$$
$$= 1 - \frac{(\varrho_n - \cos\theta_n)^2}{(\varrho_n - \cos\theta_n)^2 + \sin^2\theta_n} \tag{8}$$
$$= \frac{\sin^2\theta_n}{(\varrho_n - \cos\theta_n)^2 + \sin^2\theta_n} \tag{9}$$

$$= \frac{\sin^2 \theta_n}{\varrho_n^2 - 2\varrho_n \cos\theta_n + 1}. \tag{10}$$

From these relations, we immediately obtain

THEOREM 2.1. *Suppose that the limit* $\lim_{n\to\infty} \theta_n = \theta$ *exists.*

1. *If* $\lim_{n\to\infty} \varrho_n = 0$ *then* $\lim_{n\to\infty} \alpha_n = 1$.
2. *If* $\lim_{n\to\infty} \varrho_n = 1$ *and* $\theta \neq 0, \pi$ *then* $\lim_{n\to\infty} \alpha_n = 1/2$.
3. *If* $\lim_{n\to\infty} \varrho_n = \infty$ *then* $\lim_{n\to\infty} \alpha_n = 0$.

This theorem shows that the hybrid procedure asymptotically selects the best method among the two.

Let us now consider the convergence behavior of $\|r_n\|/\|r'_n\|$. From (10), we immediately have

THEOREM 2.2. *If the limits* $\lim_{n\to\infty} \varrho_n = \varrho$ *and* $\lim_{n\to\infty} \theta_n = \theta$ *exist and* $\varrho^2 - 2\varrho\cos\theta + 1 \neq 0$, *then*

$$\lim_{n\to\infty} \frac{\|r_n\|^2}{\|r'_n\|^2} = \frac{\sin^2\theta}{\varrho^2 - 2\varrho\cos\theta + 1} \leq 1.$$

Remark 1. Obviously if $\varrho \leq 1$, we also have $\lim_{n\to\infty} \|r_n\|^2/\|r''_n\|^2 \leq 1$. Thus $\lim_{n\to\infty} \|r_n\|/\min(\|r'_n\|, \|r''_n\|)$ exists and is not greater than 1.

Similar results can be obtained by considering the ratio $\|r_n\|^2/\|r''_n\|^2$.

It must also be noticed that $\|r_n\|^2/\|r'_n\|^2$ tends to 1 if and only if $\varrho = \cos\theta$. This result comes out directly from (8) and we also get

THEOREM 2.3. *A necessary and sufficient condition for the existence of an* N *such that*

$$0 \leq \|r_n\|^2/\|r'_n\|^2 < 1 \quad \textit{for all } n \geq N$$

is that $(r'_n - r''_n, r'_n) \neq 0$ *for all* $n \geq N$.

Proof. Suppose that $(r'_n - r''_n, r'_n) \neq 0$ for all $n \geq N$. Thus we have

$$\varrho_n - \cos\theta_n = \frac{\|r'_n\|}{\|r''_n\|} - \frac{(r'_n, r''_n)}{\|r'_n\|\|r''_n\|} = \frac{(r'_n, r'_n - r''_n)}{\|r'_n\|\|r''_n\|} \neq 0$$

and, from (8), it follows that $\|r_n\|^2/\|r'_n\|^2 < 1$. The reverse implication is proved similarly. ∎

Let us now study some cases where (r_n) converges to zero faster than (r'_n) and (r''_n). From (9), we have

THEOREM 2.4. *If there are* ϱ *and* N *such that* $0 \leq \varrho_n \leq \varrho < 1$ *for all* $n \geq N$, *then a necessary and sufficient condition for*

$$\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$$

to hold is that (θ_n) *tends to* 0 *or* π.

Proof. First let us prove the sufficiency. Suppose that (θ_n) tends to 0 or π. Thus, since $\varrho_n \le \varrho < 1$, from (9) we have $\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$.

To prove the necessity, suppose that $\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$. The condition $\varrho_n \le \varrho < 1$ implies that $\sin\theta_n$ tends to 0, which ends the proof. ■

Remark 2. Since $\varrho_n < 1$ we have $\|r'_n\| < \|r''_n\|$ for all $n \ge N$ and so

$$\lim_{n\to\infty} \|r_n\|/\min(\|r'_n\|, \|r''_n\|) = 0.$$

Let us now study the case where (ϱ_n) tends to 1. From (10), we first have

Theorem 2.5. *If* $\lim_{n\to\infty} \varrho_n = 1$, *then a sufficient condition for*

$$\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$$

to hold is that (θ_n) *tends to* π.

Remark 3. Since $\lim_{n\to\infty} \varrho_n = 1$, it follows that

$$\lim_{n\to\infty} \|r_n\|/\min(\|r'_n\|, \|r''_n\|) = 0.$$

Another result in the case where (ϱ_n) tends to 1 is given by

Theorem 2.6. *If* $\|r'_n\|/\|r''_n\| = 1 + a_n$ *with* $\lim_{n\to\infty} a_n = 0$, *then a sufficient condition for*

$$\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$$

to hold is that $\theta_n = o(a_n)$.

Proof. We have

$$\cos\theta_n = 1 - \theta_n^2/2 + \mathcal{O}(\theta_n^4), \qquad \sin\theta_n = \theta_n + \mathcal{O}(\theta_n^3).$$

Replacing in formula (10), we have

$$\begin{aligned}
\frac{\|r_n\|}{\|r'_n\|} &= \frac{\sin^2\theta_n}{\varrho_n^2 - 2\varrho_n\cos\theta_n + 1} \\
&= \frac{(\theta_n(1+\mathcal{O}(\theta_n^2)))^2}{(1+a_n)^2 - 2(1+a_n)(1-\theta_n^2/2+\mathcal{O}(\theta_n^4)) + 1} \\
&= \frac{\theta_n^2(1+\mathcal{O}(\theta_n^2))}{a_n^2 + \theta_n^2 + a_n\theta_n^2 + (1+a_n)\mathcal{O}(\theta_n^4)} \\
&= \frac{1+\mathcal{O}(\theta_n^2)}{(a_n/\theta_n)^2 + 1 + a_n + (1+a_n)\mathcal{O}(\theta_n^2)}
\end{aligned}$$

and the result follows. ■

Remark 4. Since $\lim_{n\to\infty} \varrho_n = 1$, the conclusion of Remark 3 still holds.

Another presentation consists in considering the angle ϑ_n between Zr'_n and Zp_n. From (6) we have $\|r_n\|^2 = \|r'_n\|^2 \sin^2 \vartheta_n$. Directly from this equation we obtain

THEOREM 2.7. *If there exists* $\vartheta \neq \pi/2$ *such that* $\lim_{n\to\infty} \vartheta_n = \vartheta$ *then* $\lim_{n\to\infty} \|r_n\|/\|r'_n\| = |\sin\vartheta| < 1$.

Also, we have

THEOREM 2.8. $\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$ *if and only if* (ϑ_n) *tends to* 0 *or* π.

These results are simpler than the preceding ones, in particular those of Theorems 2.2–2.4.

Remark 5. Similarly, if we denote by φ_n the angle between Zr''_n and Zp_n, we have $\|r_n\|^2 = \|r''_n\|^2 \sin^2 \varphi_n$. Obviously $\theta_n = \varphi_n - \vartheta_n$.

2.2. *Geometric behavior of the hybrid procedure*. A sphere in $\mathbb{R}^m$ with respect to the G-norm will be denoted by

$$\Upsilon_G(q, r) = \{y \in \mathbb{R}^m : \|y - q\|_G = r\}.$$

We have the following properties:

PROPERTY 1. $r_n \in \Upsilon_G(r'_n/2, \|r'_n\|_G/2) \cap \Upsilon_G(r''_n/2, \|r''_n\|_G/2)$.

Proof. By definition, we have $(r_n, r_n) = (r_n, r'_n) = (r_n, r''_n)$. Computing $\|r_n - r'_n/2\|^2$ we get

$$\|r_n - r'_n/2\|^2 = \|r_n\|^2 - (r_n, r'_n) + \tfrac{1}{4}\|r'_n\|^2 = \tfrac{1}{4}\|r'_n\|^2.$$

In the same way, we can prove that $\|r_n - r''_n/2\|^2 = \frac{1}{4}\|r''_n\|^2$ and the result follows. ∎

Let us denote by $e_n = \widetilde{x} - x_n$, $e'_n = \widetilde{x} - x'_n$, $e''_n = \widetilde{x} - x''_n$ the error vectors corresponding respectively to x_n, x'_n, x''_n. Using the relation $r_n = Ae_n$ and the preceding property we have

PROPERTY 2.

$$e_n \in \Upsilon_{A^TGA}(e'_n/2, \|e'_n\|_{A^TGA}/2) \cap \Upsilon_{A^TGA}(e''_n/2, \|e''_n\|_{A^TGA}/2).$$

The hybrid procedure is a projection method because there exists a matrix $\wp_n \in \mathbb{R}^{m\times m}$ such that

$$r_n = \wp_n r'_n = \wp_n r''_n \quad \text{with} \quad \wp_n = I - \frac{p_n p_n^T G}{p_n^T G p_n}.$$

It is easy to see that $\wp_n^2 = \wp_n$ and $(G\wp_n)^T = G\wp_n$. So $\wp_n$ is a G-orthogonal projection. By definition of $\wp_n$ we get

$$\begin{aligned} \wp_n v &= v \quad \text{if } v \perp_G p_n, \\ \wp_n v &= 0 \quad \text{if } v \in \operatorname{span}\{p_n\}. \end{aligned}$$

The above results can be considered as a generalization of the results given in [8].

3. Applications. It seems quite difficult to obtain more theoretical results on the convergence of the hybrid procedure in the general case. So, (r'_n) being an arbitrary sequence of residual vectors, we shall assume that we are in one of the following particular cases:

(i) $r''_n = Br_{n-1}$,
(ii) $r''_n = Br'_n$.

Such a situation arises, for example, if we consider a splitting of the matrix A,

$$A = M - N,$$

and if x''_n is obtained from y (equal to x_{n-1} or x'_n) by

$$x''_n = M^{-1}Ny + M^{-1}b.$$

In this case the associated residual has the form

$$\begin{aligned} r''_n &= b - Ax''_n = b - A(M^{-1}Ny + M^{-1}b) \\ &= b - (M-N)(M^{-1}Ny + M^{-1}b) = NM^{-1}(b - Ay). \end{aligned}$$

Thus we have $B = NM^{-1}$ with $y = x_{n-1}$ (case (i)) and $y = x'_{n-1}$ (case (ii)). It must be noticed that $B = I - AM^{-1}$. This situation also holds if $B = I - AC$ with C an arbitrary matrix. In this case, we have

$$x_n = \alpha_n x'_n + (1-\alpha_n)(y + C(b - Ay)).$$

(We are indebted to one of the referees for this remark.)

3.1. *Case* (i). Let r_n be computed by the hybrid procedure from $r''_n = Br_{n-1}$ and r'_n. We have

$$r_n = \alpha_n r'_n + (1-\alpha_n)Br_{n-1}$$

and we get

LEMMA 3.1. *Let $r_0 = r'_0$. Then, for all $n \ge 1$,*

$$H1(n) \qquad r_n = \sum_{i=0}^{n} a_i^{(n)} B^{n-i} r'_i$$

with

$$H2(n) \qquad \sum_{i=0}^{n} a_i^{(n)} = 1.$$

Proof. $a_0^{(0)} = 1$ and so $H1(0)$ and $H2(0)$ are true. Suppose that $H1(n-1)$ and $H2(n-1)$ hold. From the definition of r_n and from $H1(n-1)$, we get

$$r_n = \alpha_n r'_n + (1-\alpha_n) B r_{n-1} = \alpha_n r'_n + (1-\alpha_n) B \sum_{i=0}^{n-1} a_i^{(n-1)} B^{n-1-i} r'_i$$
$$= \alpha_n r'_n + \sum_{i=0}^{n-1} (1-\alpha_n) a_i^{(n-1)} B^{n-i} r'_i = \sum_{i=0}^{n} a_i^{(n)} B^{n-i} r'_i,$$

where the $a_i^{(n)}$'s are given by

$$a_i^{(n)} = (1-\alpha_n) a_i^{(n-1)}, \quad i = 0, \ldots, n-1,$$
$$a_n^{(n)} = \alpha_n.$$

Thus $H1(n)$ is true with $H2(n)$ obviously satisfied. ∎

Remark 6. When r'_n is computed by a polynomial method of the form $r'_n = P_n(B) r'_0$ then $r_n = Q_n(B) r'_0$ with Q_n given by $Q_n(t) = \alpha_n P_n(t) + (1-\alpha_n) t Q_{n-1}(t)$.

Let us now prove other results. We have

THEOREM 3.2. *Let γ be an eigenvector of B. If $r_n = c_n \gamma + a_n$ with*

$$\lim_{n\to\infty} (\gamma, r'_n)/\|r'_n\| = \|\gamma\| \quad \textit{and} \quad \lim_{n\to\infty} \|a_n\|/c_n = 0$$

then $\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$.

Proof. Let θ_n be the angle between ZBr_{n-1} and Zr'_n. We have

$$(Br_{n-1}, r'_n)^2 = c_{n-1}^2 \lambda^2 (\gamma, r'_n)^2 + (Ba_{n-1}, r'_n)^2 + 2c_{n-1}\lambda(\gamma, r'_n)(Ba_{n-1}, r'_n),$$
$$\|Br_{n-1}\|^2 = c_{n-1}^2 \lambda^2 \|\gamma\|^2 + \|Ba_{n-1}\|^2 + 2c_{n-1}\lambda(\gamma, Ba_{n-1}),$$

where λ is the eigenvalue of B corresponding to γ. Thus

$$\lim_{n\to\infty} \cos^2\theta_n = \lim_{n\to\infty} \frac{(Br_{n-1}, r'_n)^2}{\|Br_{n-1}\|^2 \|r'_n\|^2}$$
$$= \lim_{n\to\infty} \left[\frac{\lambda^2 \dfrac{(\gamma, r'_n)^2}{\|r'_n\|^2} + \dfrac{(Ba_{n-1}, r'_n)^2}{c_{n-1}^2 \|r'_n\|^2} + 2\lambda \dfrac{(\gamma, r'_n)(Ba_{n-1}, r'_n)}{c_{n-1}\|r'_n\|^2}}{\lambda^2 \|\gamma\|^2 + \dfrac{\|Ba_{n-1}\|^2}{c_{n-1}^2} + 2\lambda \dfrac{(\gamma, a_{n-1})}{c_{n-1}}} \right]$$
$$= 1$$

and the result follows from Theorem 2.4. ∎

From the minimization property of r_n we have

$$\|r_n\| \le \|Br_{n-1}\| \le \|B\| \|r_{n-1}\|$$

and thus $\|r_n\| \le \|r_{n-1}\|$ if $\|B\| \le 1$.

In particular, consider a splitting $A = M - N$ of the matrix A. Premultiplying the system (1) from the right by M^{-1} we get a new system of the form

$$A^{(M)}x = b^{(M)}$$

with

$$B^{(M)} = M^{-1}N, \quad A^{(M)} = I - B^{(M)}, \quad b^{(M)} = M^{-1}b.$$

Applying the method described above to this new system, we get

$$\|r_n\| \leq \|B^{(M)}\|\|r_{n-1}\|.$$

Thus, a good choice of $B^{(M)}$ is equivalent to a good choice of the preconditioner M from the right-hand side.

When $B = I$, the method is called the *Minimal Residual Smoothing* (MRS) algorithm. It was introduced in [6, 7] and applied to some well known methods. For more details, see [2, 9–12].

We now apply it to an error-minimization method [4]. Set $e'_n = \widetilde{x} - x'_n$ and $e_n = \widetilde{x} - x_n$. Let φ be any norm in $\mathbb{R}^m$. For any $x \in \mathbb{R}^m$ we denote by $z(x)$ a vector such that

$$(z(x), x) = \varphi(x).$$

This is called a *decomposition* of the norm φ. Such decompositions were introduced by Gastinel [3] for the case of the l_1-norm.

Let x'_0 be a given vector. The *Transformed Norm Decomposition Method* (TNDE) [4] is defined by

$$r'_0 = b - Ax'_0, \qquad p'_0 = A^T z_0,$$

and for $n = 0, 1, \ldots,$

$$x'_{n+1} = x'_n - \beta_n p'_n, \qquad r'_{n+1} = r'_n + \beta_n A p'_n,$$
$$p'_{n+1} = A^T z_{n+1} + \sum_{i=0}^{n} \gamma_{n+1}^{(i)} p'_i,$$

where z_i is such that $(z_i, r_i) = \varphi(r_i)$. The coefficients β_n and $\gamma_{n+1}^{(i)}$ are given by

$$\beta_n = \frac{(p'_n, e'_n)}{(p'_n, p'_n)} = -\frac{\varphi(r'_n)}{(p'_n, p'_n)}$$

and

$$\gamma_{n+1}^{(i)} = -\frac{(p'_i, A^T z_{n+1})}{(p'_i, p'_i)}, \qquad i = 0, \ldots, n.$$

The sequence $(e'_n = \widetilde{x} - x'_n)$ has the following properties:

1. $e'_n \in V_n = e'_0 + \operatorname{span}\{p'_0, \ldots, p'_n\}$,
2. $V_{n-1} \subset V_n$,
3. $\|e'_n\| = \min_{e \in V_n} \|e\|$,
4. $\|e'_n\| \leq \|e'_{n-1}\|$.

If the MRS is applied to the TNDE, that is, to the sequence (r'_n) defined above with $r_0 = r'_0$, then we have

THEOREM 3.3. *If* $\alpha_n \in]0, 1[$ *then* $\|e_n\| \le \|e_{n-1}\|$.

Proof. We have

$$e_{n-1} = \alpha_{n-1} e'_{n-1} + (1 - \alpha_{n-1}) e_{n-2}.$$

It follows that $e_{n-1} = \sum_{i=1}^{n-1} a_i^{(n-1)} e'_i \in V_{n-1}$ for all n. Thus, from property 3,

$$\|e'_{n-1}\| \le \|e_{n-1}\|.$$

Using property 4, we get

$$\begin{aligned}\|e_n\| &\le \alpha_n \|e'_n\| + (1-\alpha_n)\|e_{n-1}\| \le \alpha_n \|e'_{n-1}\| + (1-\alpha_n)\|e_{n-1}\| \\ &\le \alpha_n \|e_{n-1}\| + (1-\alpha_n)\|e_{n-1}\| = \|e_{n-1}\|\end{aligned}$$

and the result follows. ■

3.2. *Case* (ii). Suppose now that r_n is given by

$$r_n = \alpha_n r'_n + (1-\alpha_n) B r'_n$$

with $B = I - AM^{-1}$. Then r_n can be written as

$$r_n = r'_n - \frac{(AM^{-1} r'_n, r'_n)}{(AM^{-1} r'_n, AM^{-1} r'_n)} AM^{-1} r'_n.$$

Remark 7. If $r'_n = r_{n-1}$ and $M = I$, then the hybrid procedure is identical to the Minimal Residual Method.

DEFINITION 1. Consider two vector sequences $(u_n), (v_n) \in \mathbb{R}^m$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. We say that (u_n) *converges with the same speed as* (v_n) if there exists N such that for all $n \ge N$ there are $M_n \in \mathbb{R}^{m\times m}$ and $a_n \in \mathbb{R}^m$ with $\|a_n\| \le \varepsilon$ such that

- $v_{n+1} = M_n v_n$,
- $u_{n+1} = M_n u_n + a_n$.

LEMMA 3.4. *Suppose that there exists* N *such that for all* $n \ge N$, *there is* $M_n \in \mathbb{R}^{m\times m}$ *such that* $r'_{n+1} = M_n r'_n$ *and* $AM^{-1} M_n = M_n AM^{-1}$. *If* $\lim_{n\to\infty} \alpha_n = \alpha$ *exists and if there is* K *such that* $\|M_n\| < K$ *for all* n, *then* (r_n) *converges with the same speed as* (r'_n).

Proof. If (α_n) converges, then there is a sequence (ε_n) with $\lim_{n\to\infty} \varepsilon_n = 0$ such that $\alpha_{n+1} = \alpha_n - \varepsilon_n$ for all n. Setting $a_n = \varepsilon_n AM^{-1} M_n r'_n$, we get from the definition

$$\begin{aligned}r_{n+1} &= r'_{n+1} - (1-\alpha_{n+1}) AM^{-1} r'_{n+1} = M_n r'_n - (1 - \alpha_n + \varepsilon_n) AM^{-1} M_n r'_n \\ &= M_n (r'_n - (1-\alpha_n) AM^{-1} r'_n) + \varepsilon_n AM^{-1} M_n r'_n = M_n r_n + a_n.\end{aligned}$$

Obviously $\lim_{n\to\infty} a_n = 0$ and the result follows. ■

We now assume that $r'_n = c_n\gamma + a_n$, where $c_n \in \mathbb{R}$, $a_n \in \mathbb{R}^m$, and that γ is an eigenvector of B. In this case we get

LEMMA 3.5. *Let γ be an eigenvector of B. If $r'_n = c_n\gamma + a_n$, then there are $K \in \mathbb{R}$ and $M_n \in \mathbb{R}^m$ such that for all n, $\|M_n\| \le K$ and $r_n = M_n a_n$.*

Proof. We know that

$$r_n = \wp_n r'_n = c_n \wp_n \gamma + \wp_n a_n$$

with

$$\wp_n = I - p_n p_n^T G/(p_n^T G p_n),$$

where $p_n = AM^{-1} r'_n$. Premultiplying p_n by $\wp_n$ we get

$$0 = \wp_n p_n = (1-\lambda) c_n \wp_n \gamma + \wp_n AM^{-1} a_n,$$

where λ is the eigenvalue of B corresponding to γ. Thus, since A is assumed to be regular,

$$c_n \wp_n \gamma = -\frac{1}{1-\lambda} \wp_n AM^{-1} a_n.$$

Setting

$$M_n = \wp_n \left(I - \frac{1}{1-\lambda} AM^{-1} \right),$$

we get $r_n = M_n a_n$. The matrix $\wp_n$ is a G-orthogonal projection and thus $\|\wp_n\|_G = 1$. It follows that

$$\|M_n\| \le 1 + \frac{1}{|1-\lambda|} \|AM^{-1}\|$$

which ends the proof. ∎

Remark 8. As a consequence of Lemma 3.5 we have $\|r_n\| = \mathcal{O}(\|a_n\|)$.

From Theorem 2.8, we easily get

THEOREM 3.6. *Let γ be an eigenvector of B with the corresponding eigenvalue λ. If $r'_n = c_n\gamma + a_n$ with $\lim_{n\to\infty} \|a_n\|/c_n = 0$ then*

$$\lim_{n\to\infty} \alpha_n = -\frac{\lambda}{1-\lambda} \quad \text{and} \quad \lim_{n\to\infty} \frac{\|r_n\|}{\|r'_n\|} = 0.$$

Proof. We have

$$r'_n = c_n\gamma + a_n, \quad Br'_n = \lambda c_n \gamma + Ba_n, \quad AM^{-1} r'_n = (1-\lambda) c_n \gamma + AM^{-1} a_n,$$

and

$$\begin{aligned}
\lim_{n\to\infty} \alpha_n &= -\frac{(Br'_n, AM^{-1}r'_n)}{(AM^{-1}r'_n, AM^{-1}r'_n)} \\
&= \lim_{n\to\infty} \left[-\frac{\lambda(1-\lambda)c_n^2(\gamma,\gamma) + c_n\lambda(\gamma, AM^{-1}a_n)}{(1-\lambda)^2c_n^2(\gamma,\gamma) + 2(1-\lambda)c_n(\gamma, AM^{-1}a_n) + (AM^{-1}a_n, AM^{-1}a_n)} \right. \\
&\qquad \left. + \frac{(1-\lambda)c_n(Ba_n,\gamma) + (Ba_n, AM^{-1}a_n)}{(1-\lambda)^2c_n^2(\gamma,\gamma) + 2(1-\lambda)c_n(\gamma, AM^{-1}a_n) + (AM^{-1}a_n, AM^{-1}a_n)} \right] \\
&= \lim_{n\to\infty} \left[-\frac{\lambda(1-\lambda)(\gamma,\gamma) + \lambda\dfrac{(\gamma, AM^{-1}a_n)}{c_n}}{(1-\lambda)^2(\gamma,\gamma) + 2(1-\lambda)\dfrac{(\gamma, AM^{-1}a_n)}{c_n} + \dfrac{(AM^{-1}a_n, AM^{-1}a_n)}{c_n^2}} \right. \\
&\qquad \left. + \frac{(1-\lambda)\dfrac{(Ba_n,\gamma)}{c_n} + \dfrac{(Ba_n, AM^{-1}a_n)}{c_n^2}}{(1-\lambda)^2(\gamma,\gamma) + 2(1-\lambda)\dfrac{(\gamma, AM^{-1}a_n)}{c_n} + \dfrac{(AM^{-1}a_n, AM^{-1}a_n)}{c_n^2}} \right] \\
&= -\frac{\lambda}{1-\lambda}.
\end{aligned}$$

Let θ_n be the angle between Zr'_n and $ZAM^{-1}r'_n$. Replacing r'_n and $AM^{-1}r'_n$ by their expressions above, we also get

$$\begin{aligned}
\lim_{n\to\infty} \cos^2\theta_n &= \lim_{n\to\infty} \frac{(r'_n, AM^{-1}r'_n)^2}{\|r'_n\|^2\|AM^{-1}r'_n\|^2} \\
&= \lim_{n\to\infty} \left[\frac{1}{c_n^2(\gamma,\gamma) + 2c_n(\gamma, a_n) + (a_n, a_n)} \right. \\
&\qquad \left. \times \frac{[(1-\lambda)c_n^2(\gamma,\gamma) + c_n(\gamma, AM^{-1}a_n) + (1-\lambda)c_n(a_n,\gamma) + (a_n, AM^{-1}a_n)]^2}{(1-\lambda)^2c_n^2(\gamma,\gamma) + 2(1-\lambda)c_n(\gamma, AM^{-1}a_n) + (AM^{-1}a_n, AM^{-1}a_n)} \right] \\
&= \lim_{n\to\infty} \left[\frac{1}{(\gamma,\gamma) + 2\dfrac{(\gamma,a_n)}{c_n} + \dfrac{(a_n,a_n)}{c_n^2}} \right. \\
&\qquad \left. \times \frac{\left[(1-\lambda)(\gamma,\gamma) + \dfrac{(\gamma, AM^{-1}a_n)}{c_n} + (1-\lambda)\dfrac{(a_n,\gamma)}{c_n} + \dfrac{(a_n, AM^{-1}a_n)}{c_n^2}\right]^2}{(1-\lambda)^2(\gamma,\gamma) + 2(1-\lambda)\dfrac{(\gamma, AM^{-1}a_n)}{c_n} + \dfrac{(AM^{-1}a_n, AM^{-1}a_n)}{c_n^2}} \right] \\
&= 1
\end{aligned}$$

and the result follows by Theorem 2.8. ■

The conditions of Lemma 3.5 and Theorem 3.6 seem difficult to check in practice. We now give an example where these results can be applied.

EXAMPLE. Let $\{\lambda_i\}_{i=1}^m$ be the eigenvalues of $B = I - A$ with the corresponding eigenvectors $\{\gamma_i\}_{i=1}^m$. Suppose that $|\lambda_1| \geq \ldots \geq |\lambda_m|$ and that the eigenvectors form a basis of $\mathbb{R}^m$. Let r'_n be such that $r'_n = Br'_{n-1}$ and let r_n be obtained by the hybrid procedure from r'_n and r'_{n+1}. Let $r'_0 = \sum_{i=1}^m d_i\gamma_i$. Thus

$$r'_n = \sum_{i=1}^m d_i\lambda_i^n\gamma_i = d_1\lambda_1^n\gamma_1 + \sum_{i=2}^m d_i\lambda_i^n\gamma_i.$$

Setting

$$c_n = d_1\lambda_1^n, \qquad a_n = \sum_{i=2}^m d_i\lambda_i^n\gamma_i,$$

we get from Remark 8 and Theorem 3.6

THEOREM 3.7. *If* $r'_n = Br'_{n-1}$, $r_0 = r'_0$ *and if* r_n *is obtained by the hybrid procedure from* r'_n *and* r'_{n+1}, *then* $\|r_n\| = \mathcal{O}(|\lambda_2|^n)$. *Moreover*, *if* $|\lambda_2| < |\lambda_1|$, *then* $\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$.

Remark 9. This theorem holds even if $|\lambda_2| < 1 < |\lambda_1|$.

Remark 10. Since (α_n) converges, Lemma 3.4 shows that (r_n) converges with the same speed as (r'_n). In this case, the iterations will be stopped when $|\alpha_n + \lambda_1/(1-\lambda_1)| \leq \varepsilon$, where ε is an arbitrary threshold. Of course the value of λ_1 is usually unknown and this test cannot be used in practice. Thus the iterations will be stopped when $|\alpha_n - \alpha_{n-1}| \leq \varepsilon$. However, it must be noticed that, due to a possible stagnation of the method, this test does not guarantee that the recurrence is close to the limit.

4. Numerical examples. In all the examples we take $G = I$, $M = I$, $B = I - A$ and $x_0 = 0$. The right-hand side is computed in order that the solution be $\widetilde{x} = [1,\ldots,1]^T$. Each figure shows $\log\|r'_n\|$ and $\log\|r_n\|$ as a function of the number n of iterations and the lowest curve always corresponds to the hybrid procedure.

Let $\{\lambda_i\}_{i=1}^m$ be the set of eigenvalues of B. The elements of the matrix $A \in \mathbb{R}^{100\times100}$ were randomly chosen in $[0,1]$. The values of $\|B\|$, $|\lambda_i|$ ($i = 1,\ldots,100$) were computed with Matlab with a precision of 10^{-20}.

4.1. *Case* (i). Let r'_n be obtained by the norm decomposition method of Gastinel [3] with $\varphi_1(r) = \sum_{i=1}^m |r_i|$. This method is as follows: for $n = 0,1,\ldots,$

$$x'_{n+1} = x'_n - \alpha'_n A^T z_n, \qquad r'_{n+1} = r'_n + \alpha'_n AA^T z_n,$$

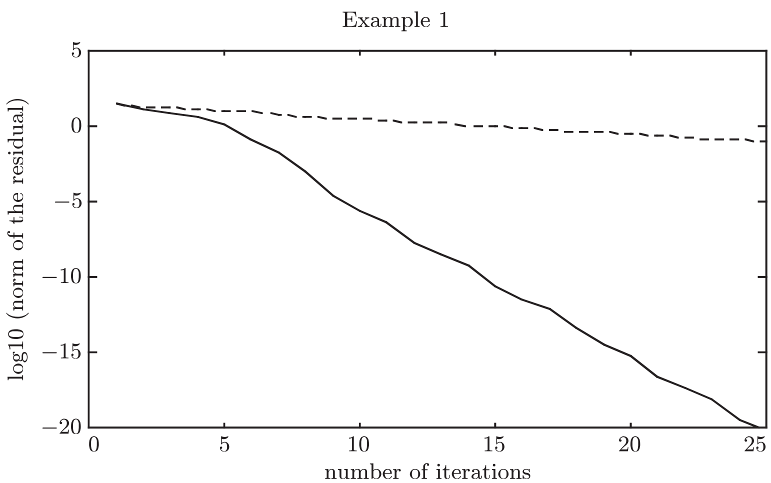
where $z_n = \mathrm{sgn}(r'_n)$. Thus, $(z_n, r'_n) = \varphi_1(r'_n)$ and

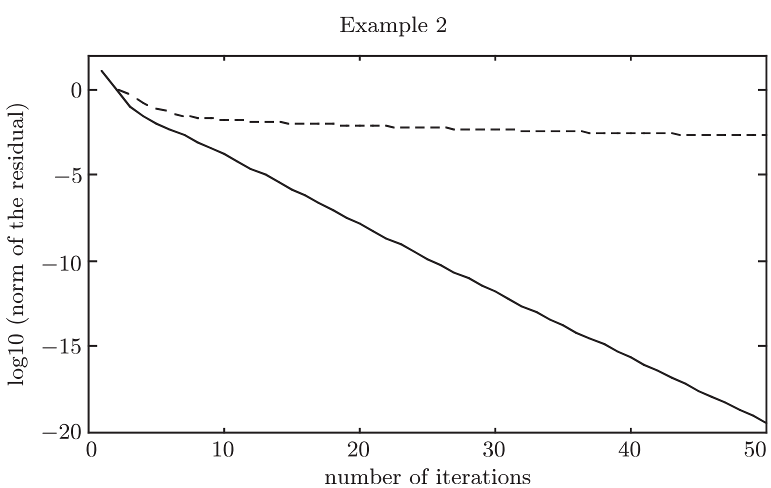
$$\alpha'_n = -\frac{\varphi_1(r'_n)}{(A^T z_n, A^T z_n)}.$$

Let r_n be computed by the hybrid procedure from r'_n and Br_{n-1}.

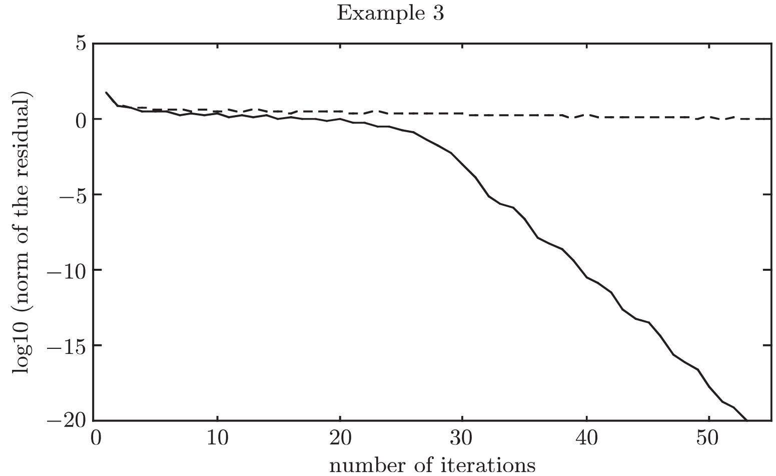
	Example 1	Example 2	Example 3
$\lVert B \rVert$	0.998605	0.663839	1.485374
$\lvert\lambda_1\rvert$	0.030418	0.661562	0.078104
$\lvert\lambda_2\rvert$	0.030372	0.040093	0.046249

For each example $|\lambda_2| < |\lambda_1|$ and thus condition 2 of Theorem 3.2 is satisfied. We did not check condition 1 but the numerical results show that, in this case, the convergence of Gastinel's method has been accelerated.





Example 3



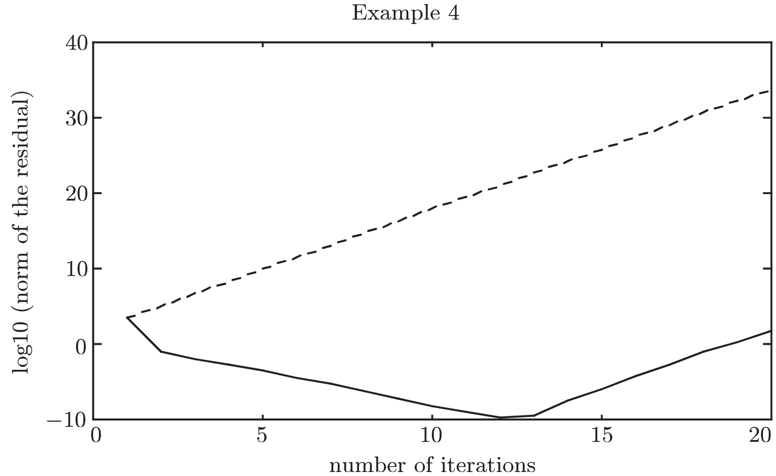
4.2. *Case* (ii). Let r'_n be such that $r'_n = Br'_{n-1}$ and let r_n be computed by the hybrid procedure from r'_n and r'_{n+1}.

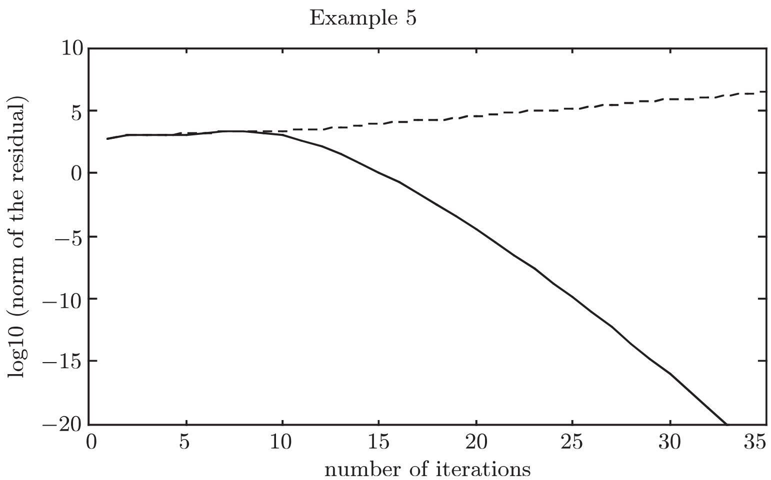
	Example 4	Example 5	Example 6
$\|B\|$	6.296298	3.273282	6.457731
$\|\lambda_1\|$	6.274695	1.158723	0.822448
$\|\lambda_2\|$	0.380272	0.099341	0.195185

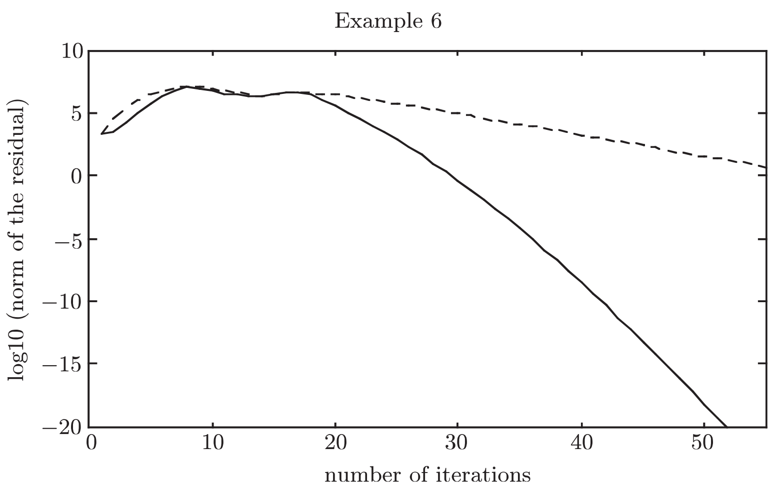
Let N be the index such that $|\alpha_N + \lambda_1/(1-\lambda_1)| \le 10^{-20}$. We get

	Example 4	Example 5	Example 6
N	12	> 35	> 55
$\log \|r'_N\|$	20.992904		
$\log \|r_N\|$	-9.771103		

Example 4







For each example we have $|\lambda_2| < |\lambda_1|$. Thus the conditions of Theorem 3.7 are satisfied and we get $\lim_{n\to\infty} \|r_n\|/\|r'_n\| = 0$ even if $\lim_{n\to\infty} \|r'_n\| = \infty$ (see Examples 4 and 5). For Example 1 we get, at iteration 12, $|\alpha_{12} + \lambda_1/(1-\lambda_1)| \leq 10^{-20}$. Moreover, we also have $|\alpha_n + \lambda_1/(1-\lambda_1)| \leq 10^{-20}$ for $n \in [12, 20]$, and thus we see that (r_n) converges with the same speed as (r'_n). We can also remark that, since the sequence (r'_n) diverges, so does (r_n) (from iteration 12) and thus it is better to stop the iterations at $n = 12$.

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