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APPROXIMATIONS OF DYNAMIC NASH GAMES WITH GENERAL STATE AND ACTION SPACES AND ERGODIC COSTS FOR THE PLAYERS

Abstract. The purpose of this paper is to prove existence of an ε-equilibrium point in a dynamic Nash game with Borel state space and long-run time average cost criteria for the players. The idea of the proof is first to convert the initial game with ergodic costs to an "equivalent" game endowed with discounted costs for some appropriately chosen value of the discount factor, and then to approximate the discounted Nash game obtained in the first step with a countable state space game for which existence of a Nash equilibrium can be established. From the results of Whitt we know that if for any $\varepsilon > 0$ the approximation scheme is selected in an appropriate way, then Nash equilibrium strategies for the approximating game are also ε-equilibrium strategies for the discounted game constructed in the first step. It is then shown that these strategies constitute an ε-equilibrium point for the initial game with ergodic costs as well. The idea of canonical triples, introduced by Dynkin and Yushkevich in the control setting, is adapted here to the game situation.

1. Introduction. We are considering a two-person Markov game over an infinite time horizon. The state space E of the process $\{x_t\}_{t=0}^{\infty}$ controlled by the players is taken to be a Borel space E equipped with the Borel σ-algebra $\mathcal{E}$. The action spaces U_1 and U_2 of player 1 and 2, respectively, are compact subsets of some metric spaces. Let $\mathcal{U}_i$ denote the Borel σ-algebra on U_i, and let $\mathcal{P}(U_i)$ denote the set of all probability measures on $(U_i,\mathcal{U}_i)$,

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$i = 1, 2$. By $h_t = (x_0, u_0^1, u_0^2, x_1, \ldots, u_{t-1}^1, u_{t-2}^2, x_t)$ we denote the state and controls history up to time t, where u_s^i stands for the ith player's action taken at time s, $0 \le s \le t$, $i = 1, 2$. The set of all histories h_t will be denoted by $H_t := (E \times U_1 \times U_2)^t \times E$, $t \ge 0$. For each $x \in E$, $B \in \mathcal{E}$ and $u^i \in U_i$, $i = 1, 2$, we denote by $p(x, B, u^1, u^2)$ the time-homogeneous one-step transition probability of the controlled process, from the state x to the set B under the actions u^1 and u^2 taken by the players. Let also $c^i(x, u^1, u^2)$ be an immediate cost incurred by player $i = 1, 2$ if the controlled process is in the state x and the actions u^1 and u^2 are applied by the players. The following assumptions will hold throughout the paper:

A1. $p(x, B, u^1, u^2)$ is uniformly continuous in x and continuous in u^1, u^2 uniformly in B.

A2. $c^i(x, u^1, u^2)$ is bounded and uniformly continuous in x and continuous in u^1, u^2 for $i = 1, 2$.

A3. There is a Borel measure ν on $(E, \mathcal{E})$ satisfying

(i) $0 < \nu(E) < 1$,

(ii) $\nu(B) \le p(x, B, u^1, u^2)$ for every $x \in E$, $B \in \mathcal{E}$ and $u^i \in U_i$, $i = 1, 2$.

Definition 1.1. (i) By an *admissible strategy* of player i we mean a sequence $\pi^i = (\mu_0^i, \mu_1^i, \mu_2^i, \ldots)$, where for each $t \ge 0$, $\mu_t^i(du^i|h_t)$ is a regular stochastic kernel on $(U_i, \mathcal{U}_i)$ with respect to the history h_t, $i = 1, 2$ (see [1], p. 134). The set of admissible strategies of player i will be denoted by Π_i, $i = 1, 2$.

(ii) A sequence $\mu^{i,\infty} = (\mu^i, \mu^i, \ldots)$ is called a *stationary strategy* for player i if μ^i is a Borel measurable mapping from E to $\mathcal{P}(U_i)$, $i = 1, 2$. We shall denote the sets of stationary strategies for the players by M_1 and M_2.

Now, for any initial state of the controlled process, $x_0 = x \in E$, and for any pair of admissible strategies (π^1, π^2) we denote by $(\Omega, \mathcal{F}, P_x(\pi^1, \pi^2))$ the canonical probability space for the process $\{(x_t, u_t^1, u_t^2)\}_{t=0}^\infty$, with the actions u_t^i chosen from U_i by player i at time t according to the probability law μ_t^i, $t \ge 0$. (See for example Bertsekas and Shreve [1] for the definition of the canonical probability space introduced above.) The expectation operator with respect to the measure $P_x(\pi^1, \pi^2)$ will be denoted by $E_x^{\pi^1,\pi^2}$.

In this paper we shall be mainly concerned with ergodic cost functionals for both players. Thus, for every initial state x, and for every pair $(\pi^1, \pi^2) \in \Pi_1 \times \Pi_2$ we define the cost functional of player i as

$$J^i(x, \pi^1, \pi^2) := \limsup_{t\to\infty} t^{-1} E_x^{\pi^1,\pi^2} \sum_{s=0}^{t-1} c^i(x_s, u_s^1, u_s^2) \tag{1.1}$$

for $i = 1, 2$.

We shall also need an auxiliary notion of α-discounted cost functional for player i,

$$J_\alpha^i(x,\pi^1,\pi^2) := \lim_{t\to\infty} E_x^{\pi^1,\pi^2}\sum_{s=0}^{t-1}\alpha^s c^i(x_s,u_s^1,u_s^2) \tag{1.2}$$

for $\alpha\in[0,1)$, $x\in E$ and $\pi^i\in\Pi_i$, $i=1,2$.

Definition 1.2. (i) For each $\varepsilon>0$ and $\alpha\in[0,1)$ the pair $(\pi^1_{\varepsilon\alpha},\pi^2_{\varepsilon\alpha})$ of admissible strategies is called an *$\varepsilon\alpha$-Discounted Nash Equilibrium Point* (*$\varepsilon\alpha$-DNEP*) if for all $x\in E$, $\pi^1\in\Pi_1$ and $\pi^2\in\Pi_2$, the following inequalities hold:

$$\begin{aligned} J_\alpha^1(x,\pi^1_{\varepsilon\alpha},\pi^2_{\varepsilon\alpha})-\varepsilon &\le J_\alpha^1(x,\pi^1,\pi^2_{\varepsilon\alpha}),\\ J_\alpha^2(x,\pi^1_{\varepsilon\alpha},\pi^2_{\varepsilon\alpha})-\varepsilon &\le J_\alpha^2(x,\pi^1_{\varepsilon\alpha},\pi^2). \end{aligned}\tag{1.3}$$

(ii) For each $\varepsilon>0$ the pair $(\pi^1_\varepsilon,\pi^2_\varepsilon)$ of admissible strategies is called an *ε-Ergodic Nash Equilibrium Point* (*ε-ENEP*) if for all $x\in E$, $\pi^1\in\Pi_1$ and $\pi^2\in\Pi_2$, the following inequalities hold:

$$\begin{aligned} J^1(x,\pi^1_\varepsilon,\pi^2_\varepsilon)-\varepsilon &\le J^1(x,\pi^1,\pi^2_\varepsilon),\\ J^2(x,\pi^1_\varepsilon,\pi^2_\varepsilon)-\varepsilon &\le J^2(x,\pi^1_\varepsilon,\pi^2). \end{aligned}\tag{1.4}$$

The following theorem is due to Whitt ([3], Theorem 5.1).

Theorem 1.1. *Assume that* A1 *and* A2 *are satisfied. Then for any $\varepsilon>0$ and $\alpha\in[0,1)$ there exists an $\varepsilon\alpha$-DNEP in the set $M_1\times M_2$.*

Whitt obtained the above result by means of approximating the initial discounted game on a general state space with an appropriate sequence of simpler games defined on countable state spaces, and demonstrating that if the approximation parameter n is large enough than the α-discounted Nash equilibrium strategies for the approximating game also constitute an $\varepsilon\alpha$-DNEP for the initial game with discounted costs. See [3] for details.

In Section 2 we shall use Theorem 1.1 and some ideas from Dynkin and Yushkevich ([2], Chapter 7) to prove the existence of an ε-ENEP for the ergodic game considered in this paper.

2. Existence results. We start with defining, for each pair of measures $(\mu^1,\mu^2)\in\mathcal{P}(U_1)\times\mathcal{P}(U_2)$, operators acting on the space $U(E)$ of real-valued, bounded and universally measurable functions on E:

$$T^i_{\mu^1,\mu^2}r(x) := \int_{U_1}\int_{U_2}\Big[c^i(x,u^1,u^2)+\int_E r(y)\,p(x,dy,u^1,u^2)\Big]\,\mu^1(du^1)\,\mu^2(du^2),$$

$$S^1_{\mu^1,\mu^2} r(x) := \inf_{\mu\in\mathcal{P}(U_1)} T^1_{\mu,\mu^2} r(x), \qquad S^2_{\mu^1,\mu^2} r(x) := \inf_{\mu\in\mathcal{P}(U_2)} T^2_{\mu^1,\mu} r(x),$$

$$\Lambda_{\mu^1,\mu^2} r(x) := \int_{U_1}\int_{U_2}\int_{E} r(y)\, p(x,dy,u^1,u^2)\,\mu^1(du^1)\,\mu^2(du^2),$$

for all $r\in U(E)$ and $x\in E$.

Definition 2.1. For each $\varepsilon>0$, $\mu^i\in\mathcal{P}(U_i)$ and $f^i, r^i\in U(E)$, $i=1,2$, the following system of conditions is called an *ε-canonical system*:

$$\begin{aligned} f^1(x) &= \inf_{\mu\in\mathcal{P}(U_1)} \Lambda_{\mu,\mu^2} f^1(x), && \forall x\in E,\\ f^2(x) &= \inf_{\mu\in\mathcal{P}(U_2)} \Lambda_{\mu^1,\mu} f^2(x), && \forall x\in E,\\ f^i(x) &= \Lambda_{\mu^1,\mu^2} f^i(x), && \forall x\in E,\ i=1,2,\\ S^i_{\mu^1,\mu^2} r^i(x) &\ge r^i(x)+f^i(x)\\ &\ge T^i_{\mu^1,\mu^2} r^i(x)-\varepsilon, && \forall x\in E,\ i=1,2. \end{aligned} \tag{2.1}$$

The quadruples $((r^i,f^i,\mu^1,\mu^2),\ i=1,2)$ satisfying (2.1) are called *ε-canonical quadruples.*

Lemma 2.1. *For each $\varepsilon>0$, if $((r^i,f^i,\mu^1,\mu^2),\ i=1,2)$ are any ε-canonical quadruples then the pair of stationary strategies $(\mu^{1,\infty},\mu^{2,\infty})$ constitutes an ε-ENEP.*

Proof. Let us first define, for $x\in E$, $\nu^i\in\mathcal{P}(U_i)$, $i=1,2$, and a finite integer N,

$$W^{iN}_{r^i}(x,\nu^1,\nu^2) := J^{iN}(x,\nu^{1,\infty},\nu^{2,\infty}) + E^{\nu^{1,\infty},\nu^{2,\infty}}_x r^i(x_N),$$

where

$$J^{iN}(x,\nu^{1,\infty},\nu^{2,\infty}) := E^{\nu^{1,\infty},\nu^{2,\infty}}_x \sum_{s=0}^{N-1} c^i(x_s,u^1_s,u^2_s), \qquad i=1,2.$$

Using induction on N we shall show that

$$\begin{aligned} W^{1N}_{r^1}(x,\mu,\mu^2) &\ge r^1(x)+Nf^1(x) \ge W^{1N}_{r^1}(x,\mu^1,\mu^2)-N\varepsilon,\\ &\qquad\qquad \forall x\in E,\ \mu\in\mathcal{P}(U_1),\\ W^{2N}_{r^2}(x,\mu^1,\mu) &\ge r^2(x)+Nf^2(x) \ge W^{2N}_{r^2}(x,\mu^1,\mu^2)-N\varepsilon,\\ &\qquad\qquad \forall x\in E,\ \mu\in\mathcal{P}(U_2). \end{aligned} \tag{2.2}$$

For $N=1$ inequalities (2.2) follow by assumption. Suppose then that (2.2) is valid for $N=k>1$. From the definition of $W^{1N}_{r^1}$ we obtain

$$
\begin{aligned}
&W_{r^1}^{1(k+1)}(x,\mu,\mu^2)\\
&\quad = \int_{U_1}\int_{U_2}\Big[c^1(x,u^1,u^2)+\int_E W_{r^1}^{1k}(y,\mu,\mu^2)\,p(x,dy,u^1,u^2)\Big]\,\mu^2(du^2)\,\mu(du^1)\\
&\quad \ge \int_{U_1}\int_{U_2}\Big[c^1(x,u^1,u^2)+\int_E (r^1(y)+kf^1(y))\,p(x,dy,u^1,u^2)\Big]\,\mu^2(du^2)\,\mu(du^1)\\
&\quad \ge S^1_{\mu^1,\mu^2}(r^1(x)+kf^1(x))\ge r^1(x)+(k+1)f^1(x),
\end{aligned}
$$

for all $x\in E$ and $\mu\in\mathcal{P}(U_1)$. Similar reasoning for $i=2$ leads to

$$W_{r^2}^{2(k+1)}(x,\mu^1,\mu)\ge r^2(x)+(k+1)f^2(x),$$

for all $x\in E$ and $\mu\in\mathcal{P}(U_2)$.

On the other hand, we have

$$
\begin{aligned}
&W_{r^i}^{i(k+1)}(x,\mu^1,\mu^2)\\
&\qquad \le \int_{U_1}\int_{U_2}\Big[c^1(x,u^1,u^2)\\
&\qquad\quad + \int_E (r^i(y)+kf^i(y))\,p(x,dy,u^1,u^2)\Big]\,\mu^2(du^2)\,\mu^1(du^1)+k\varepsilon\\
&\qquad = T^i_{\mu^1,\mu^2}(r^i(x)+kf^i(x))+k\varepsilon\le r^i(x)+(k+1)f^i(x)+(k+1)\varepsilon
\end{aligned}
$$

for all $x\in E$, $i=1,2$.

To end the proof it is now sufficient to divide (2.2) by N, and let the N go to ∞, also using boundedness of r^i, $i=1,2$. ∎

We precede the statement of Lemma 2.2 below with the following definitions:

$$
\begin{aligned}
f^{1,\alpha}_{\mu^{1,\infty}\mu^{2,\infty}} &:= \inf_{\nu^{1,\infty}\in M_1} J^1_\alpha(x,\nu^{1,\infty},\mu^{2,\infty}),\\
f^{2,\alpha}_{\mu^{1,\infty}\mu^{2,\infty}} &:= \inf_{\nu^{2,\infty}\in M_2} J^2_\alpha(x,\mu^{1,\infty},\nu^{2,\infty}),
\end{aligned}
\tag{2.3}
$$

for all $x\in E$, $(\mu^{1,\infty},\mu^{2,\infty})\in M_1\times M_2$ and $\alpha\in[0,1)$.

Remark 2.1. It follows from Corollary 9.4.1 of [1], or Theorem B on p. 85 of [2], that the functions $f^{i,\alpha}_{\mu^{1,\infty}\mu^{2,\infty}}$, $i=1,2$, defined in (2.3) are universally measurable.

LEMMA 2.2. *Let the pair $(\mu^{1,\infty}_{\varepsilon\alpha},\mu^{2,\infty}_{\varepsilon\alpha})$ be an $\varepsilon\alpha$-DNEP for $\varepsilon>0$ and $\alpha\in[0,1)$. Then*

$$f^{i,\alpha}_{\mu^{1,\infty}_{\varepsilon\alpha}\mu^{2,\infty}_{\varepsilon\alpha}}(x)+\varepsilon\ge T^i_{\mu^1_{\varepsilon\alpha},\mu^2_{\varepsilon\alpha}}(\alpha f^{i,\alpha}_{\mu^{1,\infty}_{\varepsilon\alpha}\mu^{2,\infty}_{\varepsilon\alpha}})(x)\tag{2.4}$$

for all $x\in E$ and $i=1,2$.

Proof. From (2.3) and the additivity of the cost functionals J^i_α we get

$$\begin{aligned}
&f^{i,\alpha}_{\mu^{1,\infty}_{\varepsilon\alpha}\,\mu^{2,\infty}_{\varepsilon\alpha}}(x)+\varepsilon \ge J^i_\alpha(x,\mu^{1,\infty}_{\varepsilon\alpha},\mu^{2,\infty}_{\varepsilon\alpha})\\
&=\int\limits_{U_1}\int\limits_{U_2}\Big[c^i(x,u^1,u^2)\\
&\quad+\alpha\int\limits_E J^i_\alpha(y,\mu^{1,\infty}_{\varepsilon\alpha},\mu^{2,\infty}_{\varepsilon\alpha})p(x,dy,u^1,u^2)\Big]\,\mu^1_{\varepsilon\alpha}(du^1)\,\mu^2_{\varepsilon\alpha}(du^2)\\
&\ge\int\limits_{U_1}\int\limits_{U_2}\Big[c^i(x,u^1,u^2)+\alpha\int\limits_E f^{i,\alpha}_{\mu^{1,\infty}_{\varepsilon\alpha}\,\mu^{2,\infty}_{\varepsilon\alpha}}(y)p(x,dy,u^1,u^2)\Big]\,\mu^1_{\varepsilon\alpha}(du^1)\,\mu^2_{\varepsilon\alpha}(du^2)
\end{aligned}$$

for $i=1,2$. ∎

Let us now set $\beta=1-\nu(E)$ and define new transition probabilities

$$\overline{p}(x,B,u^1,u^2):=(1/\beta)(p(x,B,u^1,u^2)-\nu(E)) \tag{2.5}$$

for any $x\in E$, $B\in\mathcal{E}$, $u^1\in U_1$ and $u^2\in U_2$.

We shall refer to the game with dynamics given in terms of the transition probabilities p as the *game* $N(p)$, whereas the game with the modified dynamics given in (2.5) will be referred to as the *game* $N(\overline{p})$. The following modifications of the operators $T^i_{\mu^1,\mu^2}$, $S^i_{\mu^1,\mu^2}$, and Λ_{μ^1,μ^2} will be needed in the remaining part of the paper:

$$\overline{T}^i_{\mu^1,\mu^2}r(x):=\int\limits_{U_1}\int\limits_{U_2}\Big[c^i(x,u^1,u^2)+\beta\int\limits_E r(y)\bar{p}(x,dy,u^1,u^2)\Big]\,\mu^1(du^1)\,\mu^2(du^2),$$

$$\overline{S}^1_{\mu^1,\mu^2}r(x):=\inf_{\mu\in\mathcal{P}(U_1)}\overline{T}^1_{\mu,\mu^2}r(x),\qquad \overline{S}^2_{\mu^1,\mu^2}r(x):=\inf_{\mu\in\mathcal{P}(U_2)}\overline{T}^2_{\mu^1,\mu}r(x),$$

$$\overline{\Lambda}_{\mu^1,\mu^2}r(x):=\int\limits_{U_1}\int\limits_{U_2}\int\limits_E r(y)\overline{p}(x,dy,u^1,u^2)\,\mu^1(du^1)\,\mu^2(du^2),$$

for all $r\in U(E)$ and $x\in E$.

Theorem 2.1 and Corollary 2.1 below are the main results in the paper.

THEOREM 2.1. *Assume* A1–A2. *Then, for each* $\varepsilon>0$, *if a pair* $(\mu^{1,\infty}_{\varepsilon\alpha},\mu^{2,\infty}_{\varepsilon\alpha})$ *of stationary strategies is an $\varepsilon\beta$-DNEP in the game $N(\overline{p})$, then it is also an ε-ENEP in the game $N(p)$.*

Proof. For $i=1,2$ and for each $r\in U(E)$ we have

$$\begin{aligned}
\overline{T}^i_{\mu^1,\mu^2}(r)&=T^i_{\mu^1,\mu^2}(r)-\nu(r),\\
\overline{S}^i_{\mu^1,\mu^2}(r)&=S^i_{\mu^1,\mu^2}(r)-\nu(r),\\
\overline{\Lambda}_{\mu^1,\mu^2}(r)&=(\Lambda_{\mu^1,\mu^2}(r)-\nu(r))/\beta,
\end{aligned}\tag{2.6}$$

where $\nu(r):=\int_E r(x)\,\nu(dx)$. Next, let the functions $\overline{f}^{i,\alpha}_{\mu^{1,\infty}\mu^{2,\infty}}$ be defined as in (2.3) for the game $N(\overline{p})$, $i=1,2$. We know that $\overline{f}^{i,\beta}_{\mu^{1,\infty}\mu^{2,\infty}}$ is a fixed point of $\overline{S}^i_{\mu^1,\mu^2}$ in $U(E)$, $i=1,2$ (see e.g. [2], Section 7.10). From this, and from Lemma 2.2 it then follows that

$$\overline{S}^i_{\mu^1_{\varepsilon\beta},\mu^2_{\varepsilon\beta}}\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}=\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}\ge\overline{T}^i_{\mu^1_{\varepsilon\beta},\mu^2_{\varepsilon\beta}}(\beta\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}})-\varepsilon, \tag{2.7}$$

for $i=1,2$. From (2.6) and (2.7) we thus obtain

$$\begin{aligned}S^i_{\mu^1_{\varepsilon\beta},\mu^2_{\varepsilon\beta}}\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}-\nu(\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}})&=\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}\\&\ge T^i_{\mu^1_{\varepsilon\beta},\mu^2_{\varepsilon\beta}}\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}-\nu(\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}})-\varepsilon\end{aligned}$$

for $i=1,2$, and hence

$$\begin{aligned}S^i_{\mu^1_{\varepsilon\beta},\mu^2_{\varepsilon\beta}}\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}&=\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}+\nu(\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}})\\&\ge T^i_{\mu^1_{\varepsilon\beta},\mu^2_{\varepsilon\beta}}\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}-\varepsilon\end{aligned} \tag{2.8}$$

for $i=1,2$. From (2.8) we conclude that the quadruples

$$((\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}},\nu(\overline{f}^{i,\beta}_{\mu^{1,\infty}_{\varepsilon\beta}\mu^{2,\infty}_{\varepsilon\beta}}),\mu^{1,\infty}_{\varepsilon\beta},\mu^{2,\infty}_{\varepsilon\beta}),\ i=1,2)$$

are ε-canonical quadruples. This, together with Lemma 2.1, proves the theorem. ∎

COROLLARY 2.1. *Assume* A1–A3. *Then for each* $\varepsilon>0$ *there exists an* ε*-ENEP for the game* $N(p)$.

Proof. The corollary is a direct consequence of Theorems 1.1 and 2.1. ∎

3. Concluding remarks. We remark that the existence and uniqueness of a Nash equilibrium point for the ergodic game considered in this paper is still an open and challenging problem.

Also, a further study is required in order to examine extendibility of the results presented above to the games endowed with trajectory-wise ergodic cost functionals of the type

$$\limsup_{t\to\infty} t^{-1}\sum_{s=0}^{t-1} c^i(x_s,u^1_s,u^2_s)$$

for $i=1,2$.

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