APPLICATIONES MATHEMATICAE
24,3 (1997), pp. 243–250

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EFFECTIVE COMPUTATION OF THE FIRST LYAPUNOV QUANTITIES FOR A PLANAR DIFFERENTIAL EQUATION

Abstract. We take advantage of the complex structure to compute in a short way and without using any computer algebra system the Lyapunov quantities V_3 and V_5 for a general smooth planar system.

1. Introduction. Consider the differential equation $(\dot{x}, \dot{y}) = (f(x,y), g(x,y))$, $(x,y) \in \mathbb{R}^2$, in the plane where f and g are analytic functions satisfying $f(0,0) = g(0,0) = 0$. It is well known that when the origin is a non-hyperbolic critical point of focus type the study of its stability can be reduced to the computation of the so called Lyapunov quantities, $V_{2k+1}, k = 1, 2, \ldots$; see [ALGM] for more details. By making a linear change of coordinates and a rescaling of the time variable if necessary, the planar differential equation can be written as

$$\dot{z} = F(z, \overline{z}) = iz + \sum_{k=2}^{\infty} F_k(z, \overline{z}), \tag{1}$$

where $z = x + iy = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$, and F_k is a complex homogeneous polynomial of degree k.

In this paper we make some modifications in the standard techniques explained in [ALGM] to obtain the Lyapunov quantities. These modifications simplify their effective computation. The main idea is to keep the complex structure of (1) during all the process.

In Section 2 we give some preliminary results and in Section 3 we prove:

1991 *Mathematics Subject Classification*: 34C25, 58F14.

Key words and phrases: stability, weak focus, Lyapunov quantities.

Research of the first author partially supported by a DGICYT grant, number PB93-0860.

Theorem A. *Consider the differential equation* (1). *Set*

$$F_2(z,\overline{z}) = Az^2 + Bz\overline{z} + C\overline{z}^2,$$
$$F_3(z,\overline{z}) = Dz^3 + Ez^2\overline{z} + Fz\overline{z}^2 + G\overline{z}^3,$$
$$F_4(z,\overline{z}) = Hz^4 + Iz^3\overline{z} + Jz^2\overline{z}^2 + Kz\overline{z}^3 + L\overline{z}^4,$$
$$F_5(z,\overline{z}) = Mz^5 + Nz^4\overline{z} + Oz^3\overline{z}^2 + Pz^2\overline{z}^3 + Qz\overline{z}^4 + R\overline{z}^5.$$

Then the first Lyapunov quantities of (1) *are*:

(i) $V_3 = 2\pi[\mathrm{Re}(E) - \mathrm{Im}(AB)]$,

(ii) $$V_5 = \frac{\pi}{3}[6\,\mathrm{Re}(O) + \mathrm{Im}(3E^2 - 6DF + 6A\overline{I} - 12BI - 6B\overline{J} - 8CH - 2C\overline{K}) + \mathrm{Re}(-8C\overline{C}E + 4AC\overline{F} + 6A\overline{B}F + 6B\overline{C}F - 12B^2D - 4ACD - 6A\overline{B}\overline{D} + 10B\overline{C}\overline{D} + 4A\overline{C}G + 2BC\overline{G}) + \mathrm{Im}(6A\overline{B}^2C + 3A^2B^2 - 4A^2\overline{B}C + 4\overline{B}^3C)].$$

The above result already appears in [CGMM, FLLL, G, GW, HW], but the proof that we present is shorter and does not use any computer algebra system.

2. Preliminary results. We briefly recall the definition of the Lyapunov constants.

In the (r,θ)-polar coordinates $z\overline{z} = r^2$, $\theta = \arctan\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$, (1) is converted into

$$\frac{dr}{d\theta} = \frac{\mathrm{Re}[\overline{z}F(z,\overline{z})]/r}{\mathrm{Im}[\overline{z}F(z,\overline{z})]/r^2}\bigg|_{z=re^{i\theta}},$$

or equivalently, for r small enough,

$$\frac{dr}{d\theta} = \frac{\sum_{k=2}^{\infty} r^k\,\mathrm{Re}(S_k(\theta))}{1 + \sum_{k=2}^{\infty} r^{k-1}\,\mathrm{Im}(S_k(\theta))} = \sum_{k=2}^{\infty} R_k(\theta) r^k, \tag{2}$$

where $S_k(\theta) = \overline{z}F_k(z,\overline{z})|_{z=e^{i\theta}} = e^{-i\theta}F_k(e^{i\theta}, e^{-i\theta})$, $R_2(\theta) = \mathrm{Re}(S_2(\theta))$ and

$$R_k(\theta) = \mathrm{Re}(S_k(\theta)) - \sum_{j=1}^{k-2} R_{k-j}(\theta)\,\mathrm{Im}(S_{j+1}(\theta)) \quad \text{for } k \ge 3. \tag{3}$$

Denote by $r(\theta,s)$ the solution of (2) which takes the value s at $\theta = 0$. Consider

$$r(\theta,s) - s = \sum_{k=2}^{\infty} u_k(\theta)s^k, \quad \text{where } u_k(0) = 0 \text{ for } k \ge 2. \tag{4}$$

Then the stability of the origin of (1) is given by the sign of the first non-zero value $u_k(2\pi)$. It is well known that the corresponding k is odd (see [ALGM, p. 243]).

Assume that $u_k(2\pi)=0$ for $k=1,\dots,2m$ and $u_{2m+1}(2\pi)\neq 0$. Then the mth *Lyapunov quantity* is defined by $V_{2m+1}=u_{2m+1}(2\pi)$.

The next result is inspired by [AL] and it allows us to compute the first values $u_k(2\pi)$. In the sequel, we use the notation $\widetilde{f}=\widetilde{f}(\theta)=(f)^{\sim}(\theta)=\int_0^\theta f(s)\,ds$.

PROPOSITION 1. *Given* (2), *the functions* $u_i(\theta)$, $i=2,3,4,5$, *involved in its solution* (4) *are*

$$\begin{aligned}
u_2(\theta)&=\widetilde{R}_2,\\
u_3(\theta)&=(\widetilde{R}_2)^2+\widetilde{R}_3,\\
u_4(\theta)&=(\widetilde{R}_2)^3+2\widetilde{R}_2\widetilde{R}_3+\widetilde{\widetilde{R}_2R_3}+\widetilde{R}_4,\\
u_5(\theta)&=(\widetilde{R}_2)^4+3(\widetilde{R}_2)^2\widetilde{R}_3+\widetilde{(\widetilde{R}_2)^2R_3}+2\widetilde{R}_2\widetilde{\widetilde{R}_2R_3}\\
&\qquad+\tfrac{3}{2}(\widetilde{R}_3)^2+2\widetilde{R}_2\widetilde{R}_4+2\widetilde{R_4\widetilde{R}_2}+\widetilde{R}_5.
\end{aligned}$$

Proof. Direct substitution gives

$$\sum_{k=2}^{\infty}R_k(\theta)[r(\theta,s)]^k=\sum_{k=2}^{\infty}u_k'(\theta)s^k.$$

By using the expression for a power series raised to some power (see [GR], for instance), whenever $k\geq 2$, we have

$$u_k'(\theta)=\sum_{m=2}^{k}R_m(\theta)\Big[\sum_{M}\left(\binom{m}{a_1\dots a_{k-1}}\right)u_2^{a_2}(\theta)u_3^{a_3}(\theta)\dots u_{k-1}^{a_{k-1}}(\theta)\Big],$$

where $M=\{(a_1,\dots,a_{k-1})\in\mathbb{N}^{k-1}: a_1+\dots+a_{k-1}=m,\ a_1+\dots+(k-1)a_{k-1}=k\}$. Then the proof follows from judicious integration. As an example we prove the expression for $u_4(\theta)$. By using the previous formula we have

$$u_4(\theta)=\int_0^\theta(R_2(\Psi)(2u_3(\Psi)+u_2^2(\Psi))+R_3(\Psi)3u_2(\Psi)+R_4(\Psi))\,d\Psi.$$

We obtain the desired result from the last expression, by substituting the values of $u_2(\Psi)$ and $u_3(\Psi)$ and integrating, as follows:

$$\begin{aligned}
u_4(\theta)&=\int_0^\theta[R_2(\Psi)(2\widetilde{R}_3(\Psi)+3(\widetilde{R}_2(\Psi))^2)+R_3(\Psi)3\widetilde{R}_2(\Psi)+R_4(\Psi)]\,d\Psi\\
&=(\widetilde{R}_2(\theta))^3+2\int_0^\theta[\widetilde{R}_3(\Psi)R_2(\Psi)+\widetilde{R}_2(\Psi)R_3(\Psi)]\,d\Psi\\
&\qquad+\int_0^\theta\widetilde{R}_2(\Psi)R_3(\Psi)\,d\Psi+\widetilde{R}_4(\theta)\\
&=(\widetilde{R}_2)^3+2\widetilde{R}_2\widetilde{R}_3+\widetilde{\widetilde{R}_2R_3}+\widetilde{R}_4. \quad\blacksquare
\end{aligned}$$

Corollary 2. *The first Lyapunov quantities of* (1) *are*

$$V_3 = \widetilde{R}_3(2\pi),$$
$$V_5 = (R_3\widetilde{(\widetilde{R}_2)^2} + 2\widetilde{R_4\widetilde{R}_2} + \widetilde{R}_5)(2\pi),$$

where the functions $R_i(\theta)$ *are defined by*

$$\begin{aligned} R_2 &= \operatorname{Re} S_2,\\ R_3 &= \operatorname{Re} S_3 - \operatorname{Re} S_2 \operatorname{Im} S_2,\\ R_4 &= \operatorname{Re} S_4 - \operatorname{Re} S_3 \operatorname{Im} S_2 + \operatorname{Re} S_2 (\operatorname{Im} S_2)^2 - \operatorname{Re} S_2 \operatorname{Im} S_3,\\ R_5 &= \operatorname{Re} S_5 - \operatorname{Re} S_4 \operatorname{Im} S_2 - \operatorname{Re} S_2 \operatorname{Im} S_4 + 2 \operatorname{Re} S_2 \operatorname{Im} S_2 \operatorname{Im} S_3\\ &\quad - \operatorname{Re} S_3 \operatorname{Im} S_3 + \operatorname{Re} S_3 (\operatorname{Im} S_2)^2 - \operatorname{Re} S_2 (\operatorname{Im} S_2)^3, \end{aligned}$$

and $S_k(\theta) = e^{-i\theta} F_k(e^{i\theta}, e^{-i\theta})$.

Proof. From the fact that $u_2(2\pi) = 0$ and using Proposition 1, we have $\widetilde{R}_2(2\pi) = 0$. Hence, the result on V_3 follows by using Proposition 1. Assuming that $V_3 = 0$, we get $u_4(2\pi) = 0$ and from Proposition 1, again, we get the desired result on V_5. On the other hand, the expression of R_k when $k = 2$, 3, 4 and 5 follows directly from (3). As an example we prove the expression for R_4. From (3) we have

$$\begin{aligned} R_4 &= \operatorname{Re} S_4 - \sum_{j=1}^{2} R_{4-j}(\theta) \operatorname{Im} S_{j+1}(\theta)\\ &= \operatorname{Re} S_4 - (\operatorname{Re} S_3 - \operatorname{Re} S_2 \operatorname{Im} S_2) \operatorname{Im} S_2 - \operatorname{Re} S_2 \operatorname{Im} S_3, \end{aligned}$$

which gives the expected value of R_4. ■

We now recall the following formulas that will be frequently used in the sequel:

$$\begin{aligned} 2 \operatorname{Re}\alpha \operatorname{Re}\beta &= \operatorname{Re}[\alpha\beta + \overline{\alpha}\beta],\\ 2 \operatorname{Im}\alpha \operatorname{Im}\beta &= \operatorname{Re}[-\alpha\beta + \overline{\alpha}\beta],\\ 2 \operatorname{Re}\alpha \operatorname{Im}\beta &= \operatorname{Im}[\alpha\beta + \overline{\alpha}\beta], \qquad \alpha, \beta \in \mathbb{C}. \end{aligned} \tag{5}$$

3. Proof of Theorem A. Firstly we will express the Lyapunov quantities of (1) in terms of the trigonometric polynomials S_k.

Proposition 3. *The first two Lyapunov quantities of system* (1) *are*

$$V_3 = \operatorname{Re} \int_0^{2\pi} S_3(\Psi)\, d\Psi - \tfrac{1}{2} \operatorname{Im} \int_0^{2\pi} S_2^2(\Psi)\, d\Psi,$$

$$V_5 = \operatorname{Re} \int_0^{2\pi} S_5(\Psi)\, d\Psi$$

$$- \operatorname{Im} \int_0^{2\pi} \left[T_2(\Psi)(S_4(\Psi) + \overline{S}_4(\Psi)) + S_2(\Psi)S_4(\Psi) + \tfrac{1}{2}S_3^2(\Psi)\right] d\Psi$$

$$+ \tfrac{1}{4} \operatorname{Re} \int_0^{2\pi} S_3(\Psi)[(S_2(\Psi) + \overline{S}_2(\Psi))^2 - (T_2(\Psi) - \overline{T}_2(\Psi) + 2S_2(\Psi))^2]\, d\Psi$$

$$+ \tfrac{1}{8} \operatorname{Im} \int_0^{2\pi} S_2^2(\Psi)[T_2(\Psi) - \overline{T}_2(\Psi) + S_2(\Psi) - \overline{S}_2(\Psi)]^2 d\Psi,$$

where $S_k(\psi) = e^{-i\psi}F_k(e^{i\psi}, e^{-i\psi})$ *and* $T_2(\Psi) = -i[\widetilde{S}_2(\Psi) - \frac{1}{2\pi}\int_0^{2\pi} \widetilde{S}_2(\theta)d\theta]$

P r o o f. By using Corollary 2 and formulas (5) we get the expression for V_3.

To obtain V_5 we recall that by Corollary 2,

$$V_5 = (R_3(\widetilde{R}_2)^2 + 2R_4\widetilde{R}_2 + R_5)^{\sim}(2\pi).$$

In order to simplify the calculations of V_5 we define, for any real number v,

$$V_5(v) = (R_3(\widetilde{R}_2 + v)^2 + 2R_4(\widetilde{R}_2 + v) + R_5)^{\sim}(2\pi).$$

By using the fact that $V_3 = 0$ ($\widetilde{R}_3(2\pi) = 0$) and also that $u_4(2\pi) = 0$ ($(\widetilde{R}_4 + \widetilde{\widetilde{R}_2 R_3})(2\pi) = 0$), it turns out that $V_5(v) \equiv V_5$. Therefore we can choose any v for computing V_5. We choose it such that

$$\widetilde{R}_2 + v = \operatorname{Re}(\widetilde{S}_2 + v) = \operatorname{Re}(iT_2) = -\operatorname{Im}(T_2).$$

Hence

$$V_5 = \int_0^{2\pi} (R_3(\theta)(\operatorname{Im}(T_2(\theta)))^2 - 2R_4(\theta)\operatorname{Im}(T_2(\theta)) + R_5(\theta))\, d\theta.$$

To get a more suitable expression for the integrated function we again use Corollary 2, obtaining

$$\begin{aligned}
&(\operatorname{Re} S_3 - \operatorname{Re} S_2 \operatorname{Im} S_2)(\operatorname{Im} T_2)^2 \\
&\qquad - 2(\operatorname{Re} S_4 - \operatorname{Re} S_3 \operatorname{Im} S_2 + \operatorname{Re} S_2(\operatorname{Im} S_2)^2 - \operatorname{Re} S_2 \operatorname{Im} S_3)\operatorname{Im} T_2 \\
&\qquad + \operatorname{Re} S_5 - \operatorname{Re} S_4 \operatorname{Im} S_2 - \operatorname{Re} S_2 \operatorname{Im} S_4 + 2\operatorname{Re} S_2 \operatorname{Im} S_2 \operatorname{Im} S_3 \\
&\qquad - \operatorname{Re} S_3 \operatorname{Im} S_3 + \operatorname{Re} S_3(\operatorname{Im} S_2)^2 - \operatorname{Re} S_2(\operatorname{Im} S_2)^3.
\end{aligned}$$

Collecting terms taking into account the number of factors they have, we get

$$\begin{aligned}
&\operatorname{Re} S_5 - 2\operatorname{Re} S_4 \operatorname{Im} T_2 - \operatorname{Re} S_4 \operatorname{Im} S_2 - \operatorname{Re} S_2 \operatorname{Im} S_4 - \operatorname{Re} S_3 \operatorname{Im} S_3 \\
&\qquad + \operatorname{Re} S_3(\operatorname{Im} T_2)^2 + 2\operatorname{Re} S_3 \operatorname{Im} S_2 \operatorname{Im} T_2 + 2\operatorname{Re} S_2 \operatorname{Im} S_3 \operatorname{Im} T_2 \\
&\qquad + 2\operatorname{Re} S_2 \operatorname{Im} S_2 \operatorname{Im} S_3 + \operatorname{Re} S_3(\operatorname{Im} S_2)^2 \\
&\qquad - \operatorname{Re} S_2 \operatorname{Im} S_2(\operatorname{Im} T_2)^2 - 2\operatorname{Re} S_2(\operatorname{Im} S_2)^2 \operatorname{Im} T_2 - \operatorname{Re} S_2(\operatorname{Im} S_2)^3.
\end{aligned}$$

Afterwards we will apply iteratively the formulas (5) to arrive at the final expression of V_5.

Firstly we consider the terms with one, two and three factors. The unique term with exactly one factor is $\operatorname{Re} S_5$, and its integral appears in the expression of V_5. With exactly two factors we have

$$-2\operatorname{Re} S_4 \operatorname{Im} T_2 - \operatorname{Re} S_4 \operatorname{Im} S_2 - \operatorname{Re} S_2 \operatorname{Im} S_4 - \operatorname{Re} S_3 \operatorname{Im} S_3.$$

The use of formulas (5) gives

$$-\operatorname{Im}\left[T_2(S_4+\overline{S}_4)+S_2S_4+\tfrac{1}{2}S_3^2\right],$$

which is the result that appears in the expression of V_5.

We have the following terms with exactly three factors:

$$\operatorname{Re} S_3(\operatorname{Im} T_2)^2 + 2\operatorname{Re} S_3 \operatorname{Im} S_2 \operatorname{Im} T_2 + 2\operatorname{Re} S_2 \operatorname{Im} S_3 \operatorname{Im} T_2 \\ + 2\operatorname{Re} S_2 \operatorname{Im} S_2 \operatorname{Im} S_3 + \operatorname{Re} S_3(\operatorname{Im} S_2)^2.$$

Transforming this expression term after term by applying formulas (5), we have

$$\begin{aligned}
&\operatorname{Re} S_3(\operatorname{Im} T_2)^2 = -\tfrac{1}{4}\operatorname{Re}(S_3(T_2-\overline{T}_2)^2),\\
&2\operatorname{Re} S_3 \operatorname{Im} S_2 \operatorname{Im} T_2 + 2\operatorname{Re} S_2 \operatorname{Im} S_3 \operatorname{Im} T_2 = -\operatorname{Re}(S_2S_3(T_2-\overline{T}_2)),\\
&2\operatorname{Re} S_2 \operatorname{Im} S_2 \operatorname{Im} S_3 + \operatorname{Re} S_3(\operatorname{Im} S_2)^2 = \tfrac{1}{4}\operatorname{Re}(S_3[(\overline{S}_2+S_2)^2-4S_2^2]).
\end{aligned}$$

Integrating the sum of the last three expressions we obtain the corresponding term that appears in V_5.

The computations involving the terms with four factors are tedious but straightforward and we omit them. ■

As a consequence of the previous proposition we can prove our main result.

Proof of Theorem A. If we express $S_2(\theta)$, $S_3(\theta)$, $S_4(\theta)$, $S_5(\theta)$ and $T_2(\theta)$ in terms of the coefficients of the differential equation we get

$$\begin{aligned}
S_2(\theta) &= Ae^{i\theta}+Be^{-i\theta}+Ce^{-3i\theta},\\
S_3(\theta) &= De^{2i\theta}+E+Fe^{-2i\theta}+Ge^{-4i\theta},\\
S_4(\theta) &= He^{3i\theta}+Ie^{i\theta}+Je^{-i\theta}+Ke^{-3i\theta}+Le^{-5i\theta},\\
S_5(\theta) &= Me^{4i\theta}+Ne^{2i\theta}+O+Pe^{-2i\theta}+Qe^{-4i\theta}+Re^{-6i\theta},\\
T_2(\theta) &= -Ae^{i\theta}+Be^{-i\theta}+\frac{C}{3}e^{-3i\theta}.
\end{aligned}$$

To compute V_3, from Proposition 3, we need to calculate

$$\operatorname{Re}\int_0^{2\pi}(De^{2i\theta}+E+Fe^{-2i\theta}+Ge^{-4i\theta})\,d\theta$$
$$-\tfrac{1}{2}\operatorname{Im}\int_0^{2\pi}(A^2e^{2i\theta}+2AB+(B^2+2AC)e^{-2i\theta}+2BCe^{-4i\theta}+C^2e^{-6i\theta})\,d\theta.$$

Hence, it suffices to obtain the terms with no exponential factors. This is because the other terms have 2π-periodic primitives and consequently, when we integrate between 0 and 2π, they vanish. Therefore, we have V_3.

To obtain V_5, first we obtain the trigonometric polynomial expressions of the integrands in V_5 of Proposition 3, and then we utilize the argument used in the calculus of V_3. That is, we are only interested in the terms of the resulting trigonometric polynomials without exponential factors. This argument allows computing V_5 by hand. Anyway, observe that by changing $e^{i\theta}$ and $e^{-i\theta}$ to x and $1/x$ respectively, the problem is reduced to the study of a product of polynomials in x and $1/x$, which is done extremely fast by computer. In any case, we get the following expression for V_5:

$$\begin{aligned}(6)\quad V_5=2\pi\big[&\operatorname{Re}(O)-\operatorname{Im}\big(\tfrac{1}{2}E^2+DF-A\overline{I}+2BI+B\overline{J}+\tfrac{4}{3}CH+\tfrac{1}{3}C\overline{K}\big)\\&+\tfrac{1}{4}\operatorname{Re}\big(\tfrac{32}{9}C\overline{C}E+\tfrac{8}{3}AC\overline{F}+4A\overline{B}F+4B\overline{C}F\\&-8B^2D-\tfrac{8}{3}ACD-4A\overline{B}\,\overline{D}+\tfrac{20}{3}B\overline{C}\,\overline{D}\\&+\tfrac{8}{3}A\overline{C}G+\tfrac{4}{3}BC\overline{G}+4E\overline{A}\,\overline{B}+8B\overline{B}E-4ABE\big)\\&+\tfrac{1}{8}\operatorname{Im}\big(8A\overline{B}^2C+4A^2B^2-\tfrac{16}{3}A^2\overline{B}C\\&+\tfrac{16}{3}\overline{B}^3C-16AB^2\overline{B}-\tfrac{160}{9}ABC\overline{C}\big)\big],\end{aligned}$$

but this expression can be reduced by using the fact that $V_3=0$. We note that this fact has already been partially used.

To simplify the expression for V_5 we proceed as follows. Take the terms

$$\begin{aligned}(7)\quad 2\pi\big\{&-\operatorname{Im}\big(\tfrac{1}{2}E^2\big)+\tfrac{1}{4}\operatorname{Re}\big(\tfrac{32}{9}C\overline{C}E+4E\overline{A}\,\overline{B}+8B\overline{B}E-4ABE\big)\\&+\tfrac{1}{8}\operatorname{Im}\big(-16AB^2\overline{B}-\tfrac{160}{9}ABC\overline{C}\big)\big\}.\end{aligned}$$

Using (5) and the fact that $\operatorname{Re}(E)=\operatorname{Im}(AB)$ (i.e. that $V_3=0$), we have

$$\tfrac{1}{4}\operatorname{Re}(8B\overline{B}E)+\tfrac{1}{8}\operatorname{Im}(-16AB^2\overline{B})=0,$$
$$2\pi\big\{\tfrac{1}{4}\operatorname{Re}\big(\tfrac{32}{9}C\overline{C}E\big)-\tfrac{1}{8}\operatorname{Im}\big(\tfrac{160}{9}ABC\overline{C}\big)\big\}=\tfrac{-8}{3}\pi C\overline{C}\operatorname{Re}(E),$$

and

$$2\pi\big\{-\operatorname{Im}\big(\tfrac{1}{2}E^2\big)+\tfrac{1}{4}\operatorname{Re}(4E\overline{A}\,\overline{B}-4ABE)\big\}=\pi\operatorname{Im}(E^2).$$

Hence, (7) is equal to

$$2\pi\big\{\tfrac{1}{2}\operatorname{Im}(E^2)-\tfrac{4}{3}\operatorname{Re}(C\overline{C}E)\big\}.$$

Therefore, by substituting this last expression in (6) we get the final formula for V_5. ■

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Received on 24.5.1995;
revised version on 15.7.1996