SINGULARITIES SYMPOSIUM — ŁOJASIEWICZ 70
BANACH CENTER PUBLICATIONS, VOLUME 44
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

EXAMPLES OF FUNCTIONS $\mathcal{C}^k$-EXTENDABLE FOR EACH k FINITE, BUT NOT $\mathcal{C}^\infty$-EXTENDABLE

WIESŁAW PAWŁUCKI
Instytut Matematyki, Uniwersytet Jagielloński
ul. Reymonta 4, 30-059 Kraków, Poland
E-mail: pawlucki@im.uj.edu.pl

Dedicated to Professor Stanisław Łojasiewicz

Abstract. In Example 1, we describe a subset X of the plane and a function on X which has a $\mathcal{C}^k$-extension to the whole $\mathbb{R}^2$ for each k finite, but has no $\mathcal{C}^\infty$-extension to $\mathbb{R}^2$. In Example 2, we construct a similar example of a subanalytic subset of $\mathbb{R}^5$; much more sophisticated than the first one. The dimensions given here are smallest possible.

1. Introduction. Let X be any subset of $\mathbb{R}^n$. Consider the following $\mathbb{R}$-algebras of functions on X

$$\mathcal{C}^k(X) = \{f\colon X \longrightarrow \mathbb{R} \mid f = \widetilde{f} \text{ on } X \text{ for some } \mathcal{C}^k\text{-function } \widetilde{f}\colon \mathbb{R}^n \longrightarrow \mathbb{R}\},$$

where $k \in \mathbb{N} \cup \{\infty\}$, and

$$\mathcal{C}^{(\infty)}(X) = \varprojlim_{k\in\mathbb{N}} \mathcal{C}^k(X) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(X).$$

It is clear that $\mathcal{C}^\infty(X) \subset \mathcal{C}^{(\infty)}(X) \subset \mathcal{C}^k(X)$, with $k \in \mathbb{N}$. An interesting question of differential analysis is the following:

When $\mathcal{C}^{(\infty)}(X) = \mathcal{C}^\infty(X)$?

Of course, one can assume that X is closed in $\mathbb{R}^n$. The answer to the above question is affirmative in the following cases:

1) When $n = 1$ (see [9]); it is not so when $n = 2$ (see Example 1 below).

This research has been partially supported by the KBN grant 2 PO3A 013 14.
1991 *Mathematics Subject Classification*: Primary 32B20. Secondary 14P10, 26B05.
Received by the editors: July 30, 1998.
The paper is in final form and no version of it will be published elsewhere.

2) When $X = \overline{\operatorname{int} X}$, because then $\mathcal{C}^k(X)$ is naturally isomorphic to the space $\mathcal{E}^k(X)$ of $\mathcal{C}^k$-Whitney fields on X ($k \in \mathbb{N} \cup \infty$), and so

$$\mathcal{C}^{(\infty)}(X) = \varprojlim_{k\in\mathbb{N}} \mathcal{C}^k(X) = \varprojlim_{k\in\mathbb{N}} \mathcal{E}^k(X) = \mathcal{E}^\infty(X) = \mathcal{C}^\infty(X),$$

(see [8; Chap. I, §4]). Observe that the isomorphisms $\mathcal{C}^k(X) = \mathcal{E}^k(X)$, and thus $\mathcal{C}^{(\infty)}(X) = \mathcal{C}^\infty(X)$, occurs for more general sets than those satisfying the condition $X = \overline{\operatorname{int} X}$; e.g. for the Cantor set in $\mathbb{R}$.

3) When X is a semianalytic or, more generally, Nash subanalytic subset of $\mathbb{R}^n$ (see [4]). The equality $\mathcal{C}^{(\infty)}(X) = \mathcal{C}^\infty(X)$ also holds if X is a subanalytic subset of $\mathbb{R}^n$ of dimension not more than two or of pure codimension one (see [11, 4]). Bierstone and Milman ([1, 2]) give necessary and sufficient conditions for a subanalytic subset X of $\mathbb{R}^n$ to satisfy the equality $\mathcal{C}^{(\infty)}(X) = \mathcal{C}^\infty(X)$. In particular, it follows from their results and [4] that the construction from [10] provides examples of subanalytic subsets X of $\mathbb{R}^5$ of dimension three such that $\mathcal{C}^\infty(X) \subsetneq \mathcal{C}^{(\infty)}(X)$. In Example 2 below, we verify this explicitly, constructing a function $f \in \mathcal{C}^{(\infty)}(X) \setminus \mathcal{C}^\infty(X)$.

2. Example 1. Let X denote the union of the following arcs

$$\lambda_i = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le \epsilon,\ y = x^{i+\frac{1}{2}}\} \qquad (i = 1, 2, \ldots),$$

and of the arc $\lambda_0 = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le \epsilon,\ y = 0\}$, where ϵ is a small positive real number.

We define a function $f : X \longrightarrow \mathbb{R}$ by the following formulae

$$f(x, y) = iy - 1 = ix^{i+\frac{1}{2}} \quad \text{for } (x, y) \in \lambda_i \quad (i = 1, 2, \ldots)$$

and $f(x, y) = 0$ for $(x, y) \in \lambda_0$.

The function f is $\mathcal{C}^k$-extendable to $\mathbb{R}^2$ for each $k \in \mathbb{N}$. To see this, notice that this function on λ_i is defined by the $\mathcal{C}^\infty$-function $f(x, y) = iy - 1$, and by the $\mathcal{C}^k$-function $f(x, y) = ix^{i+\frac{1}{2}}$ on each λ_i with $i \ge k$. Now, it is enough to glue all these $\mathcal{C}^k$-functions together, by using, for example, Whitney's extension theorem.

On the other hand, f has no $\mathcal{C}^\infty$-extension to $\mathbb{R}^2$. The point is that λ_i is $\mathcal{C}^i$ but not $\mathcal{C}^{i+1}$. This implies that if $h \in \mathcal{C}^{i+1}(\lambda_i)$, then each $\mathcal{C}^{i+1}$-extension $\tilde{h} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of h has a uniquely determined derivative $(\partial\tilde{h}/\partial y)(0, 1/i) = i$. It follows that if $\tilde{h}$ were a $\mathcal{C}^\infty$-extension of f, then $(\partial\tilde{h}/\partial y)(0, 1/i) = i$, which is a contradiction.

3. Example 2. In this section we will give an example of a subanalytic subset X of $\mathbb{R}^5$ and of a function $f \in \mathcal{C}^{(\infty)}(X) \setminus \mathcal{C}^\infty(X)$. As for the definitions and basic properties of subanalytic sets, we refer the reader to [5], [6], [7] or [3].

Before describing the example observe that if $\varphi : G \longrightarrow H$ is an analytic mapping, where $G \subset \mathbb{R}^m$ and $H \subset \mathbb{R}^n$ are open subsets, then, for each point $y \in G$, φ induces a homomorphism of the algebras of germs of analytic functions

$$\varphi_y^* : \mathcal{O}_{H,\varphi(y)} \longrightarrow \mathcal{O}_{G,y}, \qquad \varphi_y^*(g) = g \circ \varphi_y.$$

We will also need its completion

$$\widehat{\varphi}_y^* : \widehat{\mathcal{O}}_{H,\varphi(y)} \longrightarrow \widehat{\mathcal{O}}_{G,y}$$

which can be identified with the homomorphism

$$\widehat{\varphi}_y^* : \mathbb{R}[[x_1, \dots, x_n]] \longrightarrow \mathbb{R}[[y_1, \dots, y_m]],$$

defined by the formula $\widehat{\varphi}_y^*(Q) = Q \circ ((T_y\varphi) - \varphi(y))$.

Then $\ker \varphi_y^*$ is the *ideal of analytic relations* among $\varphi_1, \dots, \varphi_n$ at y, and $\ker \widehat{\varphi}_y^*$ is the *ideal of formal relations* at y.

Theorem (see [10]). *Let $I = (-1/2, 1/2)$ and $J = I \times 0 \times 0 \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$. Let $A = \{(a_\nu, 0, 0) \mid \nu = 1, 2, \dots\}$ be any countable subset of J. Then there exists an analytic mapping $\varphi = (\varphi_1, \dots, \varphi_5) : I^3 \longrightarrow \mathbb{R}^5$ such that*

(1) $\ker \widehat{\varphi}_y^* = 0$, *whenever* $y \in A$;
(2) $\ker \varphi_y^* \neq 0$, *whenever* $y \in J \setminus \overline{A}$;
(3) $\ker \varphi_y^* = 0 \neq \ker \widehat{\varphi}_y^*$, *whenever* $y \in J \cap (\overline{A} \setminus A)$.

We are going to recall the construction of $\varphi = \varphi(u, w, t) = (\varphi_1, \dots, \varphi_5)$.

We put $\varphi_1(u, w, t) = u$, $\varphi_2(u, w, t) = t$, $\varphi_3(u, w, t) = tw$. Take two sequences $\{r(n)\}$ $(n = 1, 2, \dots)$ and $\{\rho(n)\}$ $(n = 1, 2, \dots)$ such that $r(n) \in \mathbb{Z}$, $0 < r(n) \leq r(n+1)$, $\limsup r(n)/n = +\infty$, $\rho(n) \in \mathbb{R}$, $0 < \rho(n) \leq n^{-nr(n)}$, for each n, and $\rho(n+1) < \rho(n)$.

Put

$$p_n(u) = \big[(u - a_1) \dots (u - a_n)\big]^{r(n)}, \qquad n = 1, 2, \dots.$$

We define φ_4 by the formula

$$\varphi_4(u, w, t) = t \cdot \sum_{n=1}^{\infty} p_n(u) w^n.$$

To define φ_5 we need the following sequence of rational functions

$$f_n = p_n^{-1}(u) \Big[t^{n-1} y - \sum_{\nu=1}^{n-1} p_\nu(u) t^{n-\nu} x^\nu\Big] \quad (n = 1, 2, \dots).$$

Then

$$f_n(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = t^n \cdot \sum_{\nu=n}^{\infty} p_n^{-1}(u) p_\nu(u) w^\nu,$$

and we define φ_5 by the formula

$$\varphi_5(u, w, t) = \sum_{n=1}^{\infty} \rho(n) f_n(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \sum_{n=1}^{\infty} \rho(n) t^n \cdot \sum_{\nu=n}^{\infty} p_n^{-1}(u) p_\nu(u) w^\nu.$$

The formula

$$F(u, t, x, y, z) = z - \sum_{n=1}^{\infty} \rho(n) f_n(u, t, x, y)$$

defines an analytic function on $(I \setminus \overline{Z}) \times \mathbb{R}^4$, where $Z = \{a_\nu \mid \nu = 1, 2, \dots\}$, and $F(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) = 0$ on $(I \setminus \overline{Z}) \times I^2$.

Now we will choose A in a special way: assume that $0 < a_{n+1} < a_n < 1/4$ and $\lim a_n = 0$.

Let $X = \varphi([-1/4, 1/4]^3)$. Take a sequence $\{\epsilon_n\}$ $(n = 1, 2, \dots)$ such that $\epsilon_n > 0$, $a_{n+1} + \epsilon_{n+1} < a_n - \epsilon_n$.

There are $\mathcal{C}^\infty$-functions $\lambda_n : \mathbb{R} \longrightarrow [0,1]$ $(n = 1,2,\ldots)$ such that $\lambda_n = 1$ in a neighbourhood of a_n, $\lambda_n(u) = 0$ if $|u - a_n| \geq \epsilon_n$ and $|\lambda_n^{(k)}(u)| \leq C_k \cdot \epsilon_n^{-k}$ for each $u \in \mathbb{R}$, where C_k is a constant depending only on k (see [8; Chap. I, Lemma 4.2]).

Consider the following sequence of $\mathcal{C}^\infty$-functions on $\mathbb{R}^5$

$$G_m(u,t,x,y,z) = \Big[z - \sum_{n=1}^{m-1} \rho(n) f_n(u,t,x,y)\Big] \cdot m \cdot \lambda_m(u), \qquad m = 1,2,\ldots.$$

Now we have

$$\begin{aligned} G_m(\varphi_1,\ldots,\varphi_5) &= \Big[\varphi_5 - \sum_{n=1}^{m-1} \rho(n) f_n(\varphi_1,\ldots,\varphi_4)\Big] \cdot m \cdot \lambda_m(\varphi_1) \\ &= \sum_{n=m}^{\infty} m\lambda_m(u)\rho(n)t^n\omega_n(u,w), \quad \text{where} \quad \omega_n(u,w) = \sum_{\nu=n}^{\infty} p_n^{-1} p_\nu(u) w^\nu. \end{aligned}$$

Consider now the function

$$h = \sum_{m=1}^{\infty} G_m(\varphi_1,\ldots,\varphi_5) = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} m\lambda_m(u)\rho(n)t^n\omega_n(u,w).$$

It is a simple matter to check that h is a $\mathcal{C}^\infty$-function on $[-1/4,1/4]^3$. It is easily seen that there is a function $h_0 : X \longrightarrow \mathbb{R}$ such that $h = h_0(\varphi_1,\ldots,\varphi_5)$.

We will show that $h_0 \in \mathcal{C}^{(\infty)}(X)\setminus\mathcal{C}^\infty(X)$. If there were a $\mathcal{C}^\infty$-extension $\widetilde{h}_0$ of h_0 to $\mathbb{R}^5$, then we would have the equality $h = G_m(\varphi_1,\ldots,\varphi_5)$ near $(a_m,0,0)$ for each m, hence, in view of (1), $(\partial\widetilde{h}_0/\partial z)(a_m,0,0) = m$, which should tend to $(\partial\widetilde{h}_0/\partial z)(0,0,0)$, when m tends to infinity, a contradiction.

Now fix any $k \in \mathbb{N}$. We will show that there is a $\mathcal{C}^k$-function H_k on $\mathbb{R}^5$ such that $H_k = h_0$ on X.

Put

$$\Omega = \{(u,t,x) \in \mathbb{R}^3 \,\big|\, |u| < 1/4,\ |t| < 1/4,\ |x| < (1/4)|t|\}.$$

Observe that if $(u,t,x,y,z) \in X$ and $t \neq 0$, then

$$h_0(u,t,x,y,z) = \sum_{m=1}^{k} G_m(u,t,x,y,z) + \sum_{m=k+1}^{\infty}\sum_{n=m}^{\infty}\sum_{\nu=n}^{\infty} \theta_{mn\nu}(u,t,x),$$

where

$$\theta_{mn\nu}(u,t,x) = m\lambda_m(u)\rho(n)(p_n^{-1}p_\nu(u))x^\nu t^{n-\nu}.$$

Let $\alpha,\beta,\gamma \in \mathbb{N}$ be such that $\alpha+\beta+\gamma \leq k$. Then $\partial^{\alpha+\beta+\gamma}\theta_{mn\nu}/\partial u^\alpha\partial t^\beta\partial x^\gamma$ is equal to

$$\sum_{i=0}^{\alpha} \frac{m\alpha!}{i!(\alpha-i)!}\lambda_m^{(i)}(u)\rho(n)(p_n^{-1}p_\nu)^{(\alpha-i)}(u) \cdot \frac{\nu!(n-\nu)!}{(\nu-\gamma)!(n-\nu-\beta)!}(x/t)^{\nu-\gamma}t^{n-\gamma-\beta}.$$

Since $n-\gamma-\beta \geq 1$, this derivative extends continuously to $\overline{\Omega}$. Estimating the absolute value of this derivative on Ω, the reader can easily check that there is a $\mathcal{C}^k$-function $\widetilde{H}_k$ on $\mathbb{R}^3$ such that

$$\widetilde{H}_k(u,t,x) = \sum_{m=k+1}^{\infty}\sum_{n=m}^{\infty}\sum_{\nu=n}^{\infty} \theta_{mn\nu}(u,t,x)$$

on Ω. Thus, the formula

$$H_k(u,t,x,y,z) = \widetilde{H}_k(u,t,x) + \sum_{m=1}^{k} G_m(u,t,x,y,z)$$

defines the required extension.

References

[1] E. Bierstone, P. D. Milman, *Geometric and differential properties of subanalytic sets*, Bull. Amer. Math. Soc. 25 (1991), 385–383.

[2] E. Bierstone, P. D. Milman, *Geometric and differential properties of subanalytic sets*, preprint.

[3] E. Bierstone, P. D. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. 67 (1988), 5–42.

[4] E. Bierstone, P. D. Milman, W. Pawłucki, *Composite differentiable functions*, Duke Math. J. 83 (1996), 607–620.

[5] Z. Denkowska, S. Łojasiewicz, J. Stasica, *Certaines propriétés élémentaires des ensembles sous-analytiques*, Bull. Polish Acad. Sci. Math. 27 (1979), 529–536.

[6] A. Gabrielov, *Projections of semianalytic sets*, Funkcional Anal. i Priložen. 2 no. 4 (1968), 18–30 (in Russian). English transl.: Functional Anal. Appl. 2 (1968), 282–291.

[7] H. Hironaka, *Subanalytic sets*, in: Number Theory, Algebraic Geometry and Commutative Algebra in Honor of Yasuo Akizuki, Kinokuniya, Tokyo, 1973, 453–493.

[8] B. Malgrange, *Ideals of Differentiable Functions*, Oxford University Press, Bombay, 1966.

[9] J. Merrien, *Prolongateurs de fonctions différentiables d'une variable réelle*, J. Math. Pures Appl. (9) 45 (1966), 291–309.

[10] W. Pawłucki, *On relations among analytic functions and geometry of subanalytic sets*, Bull. Polish Acad. Sci. Math. 37 (1989), 117–125.

[11] W. Pawłucki, *On Gabrielov's regularity condition for analytic mappings*, Duke Math. J. 65 (1992), 299–311.