APPLICATIONES MATHEMATICAE
25,1 (1998), pp. 55–72

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ON OPTION PRICING IN THE MULTIDIMENSIONAL COX–ROSS–RUBINSTEIN MODEL

Abstract. Option pricing in the multidimensional case, i.e. when the contingent claim paid at maturity depends on a number of risky assets, is considered. It is assumed that the prices of the risky assets are in discrete time subject to binomial disturbances. Two approaches to option pricing are studied: geometric and analytic. A numerical example is also given.

1. Introduction. Assume we are given d risky assets (called *stocks*) with prices S_k^i, $i = 1, \ldots, d$, and a riskless *bond* with price B_k, at time $k = 1, \ldots, T$ respectively. The price of the risky assets is subject to random changes according to the rule

$$S_{k+1}^i = (1 + \varrho_k^i) S_k^i \quad \text{for } i = 1, \ldots, d$$

where $\varrho_i := \varrho_k^i$ stand for i.i.d. random variables defined on a given probability space $(\Omega, \mathcal{F}, P)$ and concentrated at two points $-1 < a_i < b_i$, i.e.

$$P(\varrho_k^i = a_i) > 0, \quad P(\varrho_k^i = b_i) = 1 - P(\varrho_k^i = a_i) > 0 \quad \text{for } k = 1, \ldots, T.$$

Furthermore, we assume that the bond price is deterministic and

$$B_{k+1} = (1 + r) B_k$$

where the interest rate r is positive and $a_i < r < b_i$.

We study the problem of pricing a contingent claim called *option* that guarantees the buyer a return equal to $\varphi(S_T^1, \ldots, S_T^d)$ at time T, where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a given Borel measurable function. It is commonly considered that the price of such a contingent claim should be equal to the minimal value of the capital which invested in an optimal way in bonds and stocks

1991 *Mathematics Subject Classification*: Primary 90A60; Secondary 90A12, 93E20.
Key words and phrases: option pricing, contingent claim, self-financing strategies, super-hedging.

will replicate the potential loss of the seller, i.e. at time T we obtain

$$X_T \geq \varphi(S_T^1, \ldots, S_T^d) \quad P\text{-a.s.}$$

where X_T is the value of the seller's portfolio at time T.

In what follows we assume that all assets are infinitely divisible, that is, we allow possession of a part of an asset.

Let γ_k^i denote the number of shares of the ith stock and β_k the number of bonds that the seller of the option owns at time k. Let X_k be the corresponding value of the seller's portfolio. We clearly have

$$X_k = \beta_k B_k + \sum_{i=1}^{d} \gamma_k^i S_k^i. \tag{1}$$

We assume that the investor can change his portfolio at time k from $(\beta_k, \gamma_k^1, \ldots, \gamma_k^d)$ to $(\beta_{k+1}, \gamma_{k+1}^1, \ldots, \gamma_{k+1}^d)$, but we only admit self-financing portfolio strategies, i.e. neither consumption nor exogeneous income is allowed, so that we have

$$X_k = \beta_{k+1} B_k + \sum_{i=1}^{d} \gamma_{k+1}^i S_k^i. \tag{2}$$

Since

$$X_{k+1} = \beta_{k+1} B_{k+1} + \sum_{i=1}^{d} \gamma_{k+1}^i S_{k+1}^i$$

using (2) we obtain

$$X_{k+1} = (1+r) X_k + \gamma_{k+1}^1 (\varrho_1 - r) S_k^1 + \ldots + \gamma_{k+1}^d (\varrho_d - r) S_k^d.$$

Denote by S_k the column vector $(S_k^1, \ldots, S_k^d)'$. Then

$$S_{k+1} = \Gamma_\varrho S_k$$

where Γ_ϱ is a random $d \times d$ diagonal matrix with the (i,i) entry equal to $1 + \varrho_i$. Consequently, the process $Z_k = (X_k, S_k)$ can be considered as a controlled Markov chain on the state space $E = \{(x,s) : x \in \mathbb{R}_+,\ s \in \mathbb{R}_+^d\}$. Denote by P_γ the transition probability of Z_k under a portfolio strategy $\gamma = (\gamma^1, \ldots, \gamma^d)$. Let

$$\mathcal{K} := \{(x,s) : x \geq \varphi(s)\}, \quad \mathcal{K}_{-1} := \{(x,s) : \exists_\gamma\ P_\gamma((x,s); \mathcal{K}) = 1\}.$$

Denote by $\mathcal{B}$ the space of Borel measurable functions from $\mathbb{R}^d$ into $\mathbb{R}$. Notice that if there exists a transformation Q of $\mathcal{B}$ into itself such that

$$\mathcal{K}_{-1} = \{(x,s) : x \geq Q(\varphi(s))\}$$

then the price x_0 of the contingent claim $\varphi(S_T^1, \ldots, S_T^d)$ is equal to

$$x_0 = Q^T(\varphi(s))$$

where Q^T stands for the Tth iteration of the transformation Q. The problem is to find the form of Q.

Given a strategy $\gamma = (\gamma^1, \ldots, \gamma^d)$ the condition

$$P_\gamma((x,s);\mathcal{K}) = 1$$

is equivalent to the system of 2^d inequalities

$$(3)\qquad \begin{cases} (1+r)x + \gamma^1(a_1 - r)s^1 + \ldots + \gamma^d(a_d - r)s^d \geq \overline{\varphi}(a_1, \ldots, a_d, s), \\ \qquad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ (1+r)x + \gamma^1(b_1 - r)s^1 + \ldots + \gamma^d(b_d - r)s^d \geq \overline{\varphi}(b_1, \ldots, b_d, s), \end{cases}$$

where $\overline{\varphi}$ is defined as follows:

$$\overline{\varphi}(i_1, \ldots, i_d, s) = \varphi((1+i_1)s^1, \ldots, (1+i_d)s^d)$$

with $i_j \in \{a_j, b_j\}$ for $j = 1, \ldots, d$, and $s = (s^1, \ldots, s^d)'$.

We determine the value of x_0 and the form of Q using two independent approaches: a geometric one that is based on the study of the location of d-dimensional hyperplanes in $\mathbb{R}^{d+1}$, and an analytic one in which inequalities (3) are solved using direct methods.

Our results can be extended to the case of bounded disturbances assuming additionally that the contingent claim function φ is convex. Using the analytic approach in the particular case $d = 2$ we obtain an explicit formula for the transformation Q.

The paper generalizes the results of the famous paper of Cox–Ross–Rubinstein [1], as well as more recent approaches of Shiryaev, Kabanov, Kramkov and Mel'nikov [3], and Tessitore and Zabczyk [4] where the case of $d = 1$ was studied.

It is worth pointing out that in the one-dimensional case ($d = 1$) we have perfect replication, i.e. we can find an initial capital x_0 and a portfolio strategy $(\gamma_1^1, \gamma_2^1, \ldots, \gamma_T^1)$ for which with probability one we have

$$X_T = \varphi(S_T^1).$$

If $d > 1$ we do not have perfect replication, and therefore we can only guarantee a so-called *super-hedging*, i.e. that

$$X_T \geq \varphi(S_T^1, \ldots, S_T^d) \quad P\text{-a.s.}$$

Finally, notice that, in the continuous time multidimensional lognormal model, contrary to the discrete time case, we do have perfect replication (see Karatzas–Schreve [2], Section 5.8.B), which extends the famous Black–Scholes formula.

2. Geometric approach. In this section we consider a geometric approach which is useful for two reasons: we obtain a formula for the transformation Q in a general d-dimensional case (Section 2.1) and we can extend our model to the case of bounded disturbances with convex contingent claim function φ (Section 2.2).

2.1. *The case of binomial disturbances.* Let

$$V = \{v = (i_1, \ldots, i_d) : \forall_{1\le j\le d}\ i_j \in \{a_j, b_j\}\}$$

be the set of vertices of a d-dimensional cube and

$$C_V = \{c = \{v_1, \ldots, v_{d+1}\} : \forall_{1\le i\le d+1}\ v_i \in V\}$$

the set of $(d+1)$-combinations of elements of V.

The system (3) of inequalities may then be written equivalently as

$$\forall_{v\in V} \quad h(v,s) \ge \overline{\varphi}(v,s) \tag{4}$$

where $h : \mathbb{R}^d \times \mathbb{R}^d_+ \to \mathbb{R}$ is an affine function, defined for $s \in \mathbb{R}^d_+$ as follows:

$$h(x_1, \ldots, x_d, s) = (1+r)x + \sum_{i=1}^{d} \gamma^i (x_i - r)s^i.$$

Let

$$D(s) := \operatorname{conv}\{(v, \overline{\varphi}(v,s)) : v \in V\}$$

and $H(s) \subset \mathbb{R}^{d+1}$ be the hyperplane given for fixed s by the equation

$$x_{d+1} = h(x_1, \ldots, x_d, s).$$

Geometrically, condition (4) means that the convex polyhedron $D(s)$ lies below the hyperplane $H(s)$.

Remark 1. If h is given as above then

$$x = \frac{1}{1+r} h(r, \ldots, r, s),$$
$$\gamma^i = \frac{\partial h}{\partial x_i} \cdot \frac{1}{s^i} \quad \text{for } i = 1, \ldots, d.$$

Therefore $\gamma \in \mathbb{R}^d$, $x \in \mathbb{R}$ are parameters of the hyperplane $H(s)$; x is a position parameter and $\gamma^1, \ldots, \gamma^d$ are direction parameters. Moreover, for fixed s, there is a one-to-one correspondence between the set of parameters and the set of hyperplanes determined by affine functions. This means that we can now reformulate the problem in geometrical terms looking for a suitable hyperplane, instead of x and γ. Therefore the problem of finding the smallest level x of capital for which there exist strategies $\gamma^1, \ldots, \gamma^d$ satisfying (3) is equivalent to determining the hyperplane that is tangent to $D(s)$ at the point $(r, \ldots, r, u)$ for some u and lies above $D(s)$.

DEFINITION 1. We say that a combination $c = \{v_1, \ldots, v_{d+1}\}$ belongs to the set of *admissible combinations* $C_V^{\mathrm{a}} \subset C_V$ if there exists a unique sequence $\lambda_1(c), \ldots, \lambda_{d+1}(c)$ of nonnegative coordinates such that

$$\sum_{i=1}^{d+1} \lambda_i(c) = 1 \quad \text{and} \quad \sum_{i=1}^{d+1} \lambda_i(c) v_i = \mathbf{r}$$

with $\mathbf{r} = [r, \ldots, r]'$.

The above conditions have the following geometrical interpretation: the polyhedron spanned by the set $c = \{v_1, \ldots, v_{d+1}\}$ of vertices is precisely d-dimensional and it contains the point $(r, \ldots, r)$.

Let $f : C_V^{\mathrm{a}} \times \mathbb{R}_+^d \to \mathbb{R}$ be defined as follows:

$$f(c, s) = \sum_{i=1}^{d+1} \overline{\varphi}(v_i, s) \lambda_i(c)$$

where $c = \{v_1, \ldots, v_{d+1}\}$. We have

THEOREM 1. *The smallest x for which there exists a strategy $(\gamma^1, \ldots, \gamma^d)$ such that system* (3) *is satisfied is equal to $Q(\varphi(s))$ and is of the form*

$$\frac{1}{1+r} \max_{c \in C_V^{\mathrm{a}}} f(c, s).$$

P r o o f. Let $\mathcal{H}_u(s)$ be the family of d-dimensional hyperplanes H in $\mathbb{R}^{d+1}$ such that

$$(r, \ldots, r, u) \in H \tag{5}$$

and

$$H = \{(x_1, \ldots, x_{d+1}) : x_{d+1} = h(x_1, \ldots, x_d)\} \Rightarrow \forall_{v \in V}\ h(v) \geq \overline{\varphi}(v, s). \tag{6}$$

According to Remark 1,

$$Q(\varphi(s)) = \frac{1}{1+r} \min\{u : \mathcal{H}_u(s) \neq \emptyset\}.$$

Therefore it is sufficient to show the following conditions:

$$\mathcal{H}_{u_0(s)}(s) \neq \emptyset \qquad \text{for } u_0(s) = \max_{c \in C_V^{\mathrm{a}}} f(c, s), \tag{7}$$

$$\mathcal{H}_u(s) = \emptyset \qquad \text{for } u < u_0(s). \tag{8}$$

We first show (7). Let $c_0(s) \in C_V^{\mathrm{a}}$ be a combination for which the maximum is achieved, $c_0(s) = \{v_1^0(s), \ldots, v_{d+1}^0(s)\}$. Consider the hyperplane $\widehat{H}(s)$ determined by the points $p_1^0(s) = (v_1^0(s), \overline{\varphi}(v_1^0(s), s)), \ldots, p_{d+1}^0(s) =$

$(v^0_{d+1}(s), \overline{\varphi}(v^0_{d+1}(s), s))$. Since $c_0(s) \in C^{\mathrm{a}}_V$ the points $p^0_1(s), \ldots, p^0_{d+1}(s)$ define the d-dimensional hyperplane $\widehat{H}(s)$ in a unique way. Then

$$\widehat{H}(s) = \Big\{(x_1, \ldots, x_{d+1}) : \exists (k_1, \ldots, k_{d+1}) \in \mathbb{R}^{d+1} \quad \sum_{i=1}^{d+1} k_i = 1,\ (x_1, \ldots, x_{d+1}) = \sum_{i=1}^{d+1} k_i p^0_i(s)\Big\}$$

or equivalently

$$(x_1, \ldots, x_d) = \sum_{i=1}^{d+1} k_i v^0_i(s), \qquad x_{d+1} = \sum_{i=1}^{d+1} k_i \overline{\varphi}(v^0_i(s), s).$$

We claim that $\widehat{H}(s) \in \mathcal{H}_{u_0(s)}(s)$.

Since $c_0(s) \in C^{\mathrm{a}}_V$ there exist coordinates $(\lambda_i(c_0(s)))_{i=1,\ldots,d+1}$ that sum to 1 and

$$\mathbf{r} = \sum_{i=1}^{d+1} \lambda_i(c_0(s)) v^0_i(s).$$

Therefore due to the definition of $c_0(s)$ we have

$$u_0(s) = f(c_0(s)) = \sum_{i=1}^{d+1} \lambda_i(c_0(s)) \overline{\varphi}(v^0_i(s), s).$$

Consequently, $(r, \ldots, r, u_0(s)) \in \widehat{H}(s)$ and (5) is satisfied.

We now show that $\widehat{H}(s)$ satisfies (6). Let $J(s)$ be the interval in $\mathbb{R}^{d+1}$ defined as follows:

$$J(s) = \{(x_1, \ldots, x_{d+1}) \in D(s) : x_i = r \text{ for } i = 1, \ldots, d\}.$$

Moreover, let

$$\overline{u}(s) = \max_{p \in J(s)} \pi_{d+1}(p)$$

where π_{d+1} is the natural projection onto the $(d+1)$st coordinate. Since the point $(r, \ldots, r, u_0(s))$ belongs to $J(s)$ we clearly have $u_0(s) \le \overline{u}(s)$. On the other hand, the point $\overline{p}(s) = (r, \ldots, r, \overline{u}(s))$ is on $\mathrm{Fr}(D(s))$ and therefore it belongs to a d-dimensional polyhedron $\overline{P}(s) \subset \mathrm{Fr}(D(s))$. If $\overline{P}(s)$ is determined by the vertices $\overline{p}_1(s) = (\overline{v}_1(s), \overline{\varphi}(\overline{v}_1(s), s)), \ldots, \overline{p}_{d+1}(s) = (\overline{v}_{d+1}(s), \overline{\varphi}(\overline{v}_{d+1}(s), s))$ then it is easy to see that the combination $\overline{c}(s) = \{\overline{v}_1(s), \ldots, \overline{v}_{d+1}(s)\}$ is admissible. Consequently,

$$\overline{u}(s) = \sum_{i=1}^{d+1} \lambda_i(\overline{c}(s)) \overline{\varphi}(\overline{v}_i(s)) = f(\overline{c}(s), s)$$

and by the definition of $u_0(s)$ we obtain $u_0(s) \ge \overline{u}(s)$. This means that $u_0(s) = \overline{u}(s)$ and the d-dimensional polyhedron with vertices $p^0_1(s), \ldots$

$\dots, p^0_{d+1}(s)$ is contained in $\mathrm{Fr}(D(s))$. Therefore the hyperplane $\widehat{H}(s)$, spanned by the same points, is tangent to $D(s)$. Consequently, $\widehat{H}(s)$ satisfies condition (6) and $\widehat{H}(s) \in \mathcal{H}_{u_0(s)}(s)$.

To complete the proof of the theorem it remains to show (8). When $u < u_0(s)$ we have $(r, \dots, r, u) \in \mathrm{int}(D(s))$, and every hyperplane containing $(r, \dots, r, u)$ crosses $D(s)$ and hence (6) is not satisfied. Thus we have (8) and the proof is complete. ∎

Remark 2. The vector γ of optimal controls may not be unique, which means that it is possible that the seller has several alternative ways to invest his money to super-hedge the option.

2.2. *The case of bounded noises*. In this section we relax the assumption on the distribution of disturbances of the stock prices. Namely, we only assume that the support of the distribution is bounded, i.e.

$$P(a_i \le S^i_k \le b_i) = 1$$

where

$$a_i = \inf \operatorname{supp} \varrho^i_k > -1, \qquad b_i = \sup \operatorname{supp} \varrho^i_k < \infty,$$

for all $k = 1, \dots, T$ and $i = 1, \dots, d$.

Moreover, we additionally assume that the contingent claim function φ is convex. Many popular options (e.g. European call and put) have convex contingent claim functions. One can see that in this new model, due to the convexity of φ, the construction of the set $\mathcal{K}_{-1}$ is also equivalent to solving the system (3), and we obtain the same formula for the transformation Q. However, to iterate this algorithm we have to verify that $Q(\varphi)$ is a convex function of s.

Corollary 1. *The function $Q(\varphi)$ is of the form*

$$Q(\varphi(s)) = \frac{1}{1+r} \max_{c \in C^{\mathrm{a}}_V} \sum_{i=1}^{d+1} \overline{\varphi}(v_i, s)\lambda_i(c)$$

where $c = \{v_1, \dots, v_{d+1}\}$, and is a convex function of s.

P r o o f. The above form of $Q(\varphi)$ is clearly a straightforward conclusion from Theorem 1.

Since φ is a convex function of s, for any $s, s' \in \mathbb{R}^d$ and any combination $c = \{v_1, \dots, v_{d+1}\} \in C^{\mathrm{a}}_V$ we have

$$\sum_{i=1}^{d+1} \overline{\varphi}(v_i, \alpha s + (1-\alpha)s')\lambda_i(c) \le \sum_{i=1}^{d+1} (\alpha\overline{\varphi}(v_i, s) + (1-\alpha)\overline{\varphi}(v_i, s'))\lambda_i(c).$$

The set C^{a}_V does not depend on s so taking the maximum over the set of admissible combinations on both sides of the above inequality we obtain the convexity of the function $Q(\varphi)$. ∎

Remark 3. Notice that the bounds a_i, b_i, $i = 1, \ldots, d$, of the disturbances may also depend on time $k = 1, \ldots, T$. More precisely, the construction of Q and all arguments above are still valid when we assume that $(a_{i,k})_{k=1,\ldots,T}$ and $(b_{i,k})_{k=1,\ldots,T}$ are predictable processes on $(\Omega, \mathcal{F}, P)$ for $i = 1, \ldots, d$.

3. Analytic approach. It turns out that to get an explicit formula for the option price in the multidimensional case is difficult. We managed to find it only in the two-dimensional case. In the case when $d \geq 3$ the pricing of an option can be reduced to a certain optimization problem which can be solved numerically.

3.1. *Two-dimensional case.* The two-dimensional case has an interesting interpretation in the market of derivatives of foreign securities. Namely, options on foreign securities depend both on the value of the asset as well as on the current exchange rate, which are random. This corresponds exactly to the two-dimensional case which we solve below.

For $d = 2$ the system (3) is of the form

$$\begin{cases} (1+r)x + \gamma^1(a_1 - r)s^1 + \gamma^2(a_2 - r)s^2 \geq \overline{\varphi}(a_1, a_2, s), \\ (1+r)x + \gamma^1(b_1 - r)s^1 + \gamma^2(a_2 - r)s^2 \geq \overline{\varphi}(b_1, a_2, s), \\ (1+r)x + \gamma^1(a_1 - r)s^1 + \gamma^2(b_2 - r)s^2 \geq \overline{\varphi}(a_1, b_2, s), \\ (1+r)x + \gamma^1(b_1 - r)s^1 + \gamma^2(b_2 - r)s^2 \geq \overline{\varphi}(b_1, b_2, s). \end{cases} \tag{9}$$

Multiplying the first and third inequalities by $b_1 - r$ and the second and fourth by $r - a_1$ and adding the first inequality to the second and the third to the fourth we get two inequalities without the component γ^1 from which we obtain a lower and an upper bound for γ^2. Analogously we can eliminate the component γ^2 and obtain bounds for γ^1. We have

$$\gamma^1 \geq \frac{\big(\frac{b_2-r}{b_2-a_2}\overline{\varphi}(b_1, a_2, s) + \frac{r-a_2}{b_2-a_2}\overline{\varphi}(b_1, b_2, s)\big) - (1+r)x}{(b_1 - r)s^1},$$

$$\gamma^1 \leq \frac{(1+r)x - \big(\frac{b_2-r}{b_2-a_2}\overline{\varphi}(a_1, a_2, s) + \frac{r-a_2}{b_2-a_2}\overline{\varphi}(a_1, b_2, s)\big)}{(r - a_1)s^1},$$

and

$$\gamma^2 \geq \frac{\big(\frac{b_1-r}{b_1-a_1}\overline{\varphi}(a_1, b_2, s) + \frac{r-a_1}{b_1-a_1}\overline{\varphi}(b_1, b_2, s)\big) - (1+r)x}{(b_2 - r)s^2},$$

$$\gamma^2 \leq \frac{(1+r)x - \big(\frac{b_1-r}{b_1-a_1}\overline{\varphi}(a_1, a_2, s) + \frac{r-a_1}{b_1-a_1}\overline{\varphi}(b_1, a_2, s)\big)}{(r - a_2)s^2}.$$

From the bounds on γ^1 or γ^2 we can obtain the same formula (in both cases) for the lower bound of x, namely

$$x \geq \frac{1}{1+r}\left[\frac{(b_1-r)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\overline{\varphi}(a_1,a_2,s) + \frac{(b_1-r)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\overline{\varphi}(a_1,b_2,s) + \frac{(r-a_1)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\overline{\varphi}(b_1,a_2,s) + \frac{(r-a_1)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\overline{\varphi}(b_1,b_2,s)\right].$$

Let $G_\varphi(s^1,s^2)$ be the expression in brackets on the right side of the above inequality and let $L_\varphi(s)$ be a nonnegative number such that: *if*

$$x = \frac{1}{1+r}(G_\varphi(s^1,s^2) + L_\varphi(s^1,s^2))$$

then there exist strategies γ^1, γ^2 *satisfying system* (9).

Denote by $L^*_\varphi(s)$ the infimum of the set $\{L_\varphi(s)\}$. Clearly,

$$Q(\varphi(s^1,s^2)) = \frac{1}{1+r}(G_\varphi(s^1,s^2) + L^*_\varphi(s^1,s^2)).$$

Furthermore, we have

THEOREM 2. *If*

$$\Delta(s) := \overline{\varphi}(b_1,b_2,s) - \overline{\varphi}(a_1,b_2,s) - \overline{\varphi}(b_1,a_2,s) + \overline{\varphi}(a_1,a_2,s) = 0$$

then $L^*_\varphi(s) = 0$ *and we have perfect replication.*

Proof. Let γ^1 be a linear combination of its lower and upper bound with coefficients α and $1-\alpha$ respectively, for $\alpha \in [0,1]$, and similarly for γ^2 with coefficients β and $1-\beta$ for $\beta \in [0,1]$. Substituting γ^1 and γ^2 in the left hand sides of (9) and using the formula for x with $L_\varphi(s) = 0$, we obtain

$$\begin{cases} (1+r)x + \gamma^1(a_1-r)s^1 + \gamma^2(a_2-r)s^2 = \overline{\varphi}(a_1,a_2,s) \\ \qquad\qquad - \dfrac{(r-a_1)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\Delta(s), \\ (1+r)x + \gamma^1(b_1-r)s^1 + \gamma^2(a_2-r)s^2 = \overline{\varphi}(b_1,a_2,s) \\ \qquad\qquad + \dfrac{(b_1-r)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\Delta(s), \\ (1+r)x + \gamma^1(a_1-r)s^1 + \gamma^2(b_2-r)s^2 = \overline{\varphi}(a_1,b_2,s) \\ \qquad\qquad + \dfrac{(r-a_1)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\Delta(s), \\ (1+r)x + \gamma^1(b_1-r)s^1 + \gamma^2(b_2-r)s^2 = \overline{\varphi}(b_1,b_2,s) \\ \qquad\qquad - \dfrac{(b_1-r)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\Delta(s). \end{cases}$$

Since $\Delta(s) = 0$ by assumption, we have equalities in (9), which means perfect replication. Consequently, $L^*_\varphi(s) = 0$. ∎

Theorem 3. *The smallest* L_φ *for which there exist strategies* (γ^1,γ^2) *such that* (9) *holds is*

$$L^*_\varphi(s)=\begin{cases}\dfrac{(r-a_1)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\Delta(s) & \textit{if } \dfrac{r-a_1}{b_1-a_1}\le\dfrac{r-a_2}{b_2-a_2},\\[2ex] \dfrac{(b_1-r)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\Delta(s) & \textit{if } \dfrac{r-a_1}{b_1-a_1}\ge\dfrac{r-a_2}{b_2-a_2},\end{cases}$$

when $\Delta(s)>0$, *and*

$$L^*_\varphi(s)=\begin{cases}\dfrac{(b_1-r)(b_2-r)}{(b_1-a_1)(b_2-a_2)}|\Delta(s)| & \textit{if } \dfrac{r-a_1}{b_1-a_1}\ge\dfrac{b_2-r}{b_2-a_2},\\[2ex] \dfrac{(r-a_1)(r-a_2)}{(b_1-a_1)(b_2-a_2)}|\Delta(s)| & \textit{if } \dfrac{r-a_1}{b_1-a_1}\le\dfrac{b_2-r}{b_2-a_2},\end{cases}$$

when $\Delta(s)<0$. *The replicating strategies* γ^1,γ^2 *are*

$$\gamma^1=$$
$$\begin{cases}\dfrac{(1+r)x-\big(\frac{b_2-r}{b_2-a_2}\overline{\varphi}(a_1,a_2,s)+\frac{r-a_2}{b_2-a_2}\overline{\varphi}(a_1,b_2,s)\big)}{(r-a_1)s^1} & \textit{if } \dfrac{r-a_1}{b_1-a_1}\le\dfrac{r-a_2}{b_2-a_2},\\[2ex] \dfrac{\big(\frac{b_2-r}{b_2-a_2}\overline{\varphi}(b_1,a_2,s)+\frac{r-a_2}{b_2-a_2}\overline{\varphi}(b_1,b_2,s)\big)-(1+r)x}{(b_1-r)s^1} & \textit{if } \dfrac{r-a_1}{b_1-a_1}\ge\dfrac{r-a_2}{b_2-a_2},\end{cases}$$

$$\gamma^2=$$
$$\begin{cases}\dfrac{\big(\frac{b_1-r}{b_1-a_1}\overline{\varphi}(a_1,b_2,s)+\frac{r-a_1}{b_1-a_1}\overline{\varphi}(b_1,b_2,s)\big)-(1+r)x}{(b_2-r)s^2} & \textit{if } \dfrac{r-a_1}{b_1-a_1}\le\dfrac{r-a_2}{b_2-a_2},\\[2ex] \dfrac{(1+r)x-\big(\frac{b_1-r}{b_1-a_1}\overline{\varphi}(a_1,a_2,s)+\frac{r-a_1}{b_1-a_1}\overline{\varphi}(b_1,a_2,s)\big)}{(r-a_2)s^2} & \textit{if } \dfrac{r-a_1}{b_1-a_1}\ge\dfrac{r-a_2}{b_2-a_2},\end{cases}$$

when $\Delta(s)>0$, *and*

$$\gamma^1=$$
$$\begin{cases}\dfrac{\big(\frac{b_2-r}{b_2-a_2}\overline{\varphi}(b_1,a_2,s)+\frac{r-a_2}{b_2-a_2}\overline{\varphi}(b_1,b_2,s)\big)-(1+r)x}{(b_1-r)s^1} & \textit{if } \dfrac{r-a_1}{b_1-a_1}\ge\dfrac{b_2-r}{b_2-a_2},\\[2ex] \dfrac{(1+r)x-\big(\frac{b_2-r}{b_2-a_2}\overline{\varphi}(a_1,a_2,s)+\frac{r-a_2}{b_2-a_2}\overline{\varphi}(a_1,b_2,s)\big)}{(r-a_1)s^1} & \textit{if } \dfrac{r-a_1}{b_1-a_1}\le\dfrac{b_2-r}{b_2-a_2},\end{cases}$$

$$\gamma^2=$$
$$\begin{cases}\dfrac{\big(\frac{b_1-r}{b_1-a_1}\overline{\varphi}(a_1,b_2,s)+\frac{r-a_1}{b_1-a_1}\overline{\varphi}(b_1,b_2,s)\big)-(1+r)x}{(b_2-r)s^2} & \textit{if } \dfrac{r-a_1}{b_1-a_1}\ge\dfrac{b_2-r}{b_2-a_2},\\[2ex] \dfrac{(1+r)x-\big(\frac{b_1-r}{b_1-a_1}\overline{\varphi}(a_1,a_2,s)+\frac{r-a_1}{b_1-a_1}\overline{\varphi}(b_1,a_2,s)\big)}{(r-a_2)s^2} & \textit{if } \dfrac{r-a_1}{b_1-a_1}\le\dfrac{b_2-r}{b_2-a_2}.\end{cases}$$

when $\Delta(s) < 0$. Furthermore, for $\Delta(s) \neq 0$ we do not have perfect replication.

Proof. We prove the case $\Delta(s) > 0$ only, since the proof in the case $\Delta(s) < 0$ is based on similar considerations. Substituting, in the left hand sides of (9), γ^1, γ^2 as convex combinations of their bounds with coefficients α, $1-\alpha$ and β, $1-\beta$ respectively, and letting $(1+r)x = G_\varphi(s^1, s^2) + L_\varphi(s^1, s^2)$ we obtain the expressions

$$\begin{cases} \overline{\varphi}(a_1, a_2, s) - \dfrac{(r-a_1)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\Delta(s) \\ \qquad\qquad + L_\varphi(s)\left(\alpha\dfrac{b_1-a_1}{b_1-r} + \beta\dfrac{b_2-a_2}{b_2-r} - 1\right), \\ \overline{\varphi}(b_1, a_2, s) + \dfrac{(b_1-r)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\Delta(s) \\ \qquad\qquad + L_\varphi(s)\left(\dfrac{b_1-r}{r-a_1} - \alpha\dfrac{b_1-a_1}{r-a_1} + \beta\dfrac{b_2-a_2}{b_2-r}\right), \\ \overline{\varphi}(a_1, b_2, s) + \dfrac{(r-a_1)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\Delta(s) \\ \qquad\qquad + L_\varphi(s)\left(\alpha\dfrac{b_1-a_1}{b_1-r} + \dfrac{b_2-r}{r-a_2} - \beta\dfrac{b_2-a_2}{r-a_2}\right), \\ \overline{\varphi}(b_1, b_2, s) - \dfrac{(b_1-r)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\Delta(s) \\ \qquad\qquad + L_\varphi(s)\left((1-\alpha)\dfrac{b_1-a_1}{r-a_1} + \dfrac{b_2-r}{r-a_2} - \beta\dfrac{b_2-a_2}{r-a_2}\right). \end{cases}$$

Comparing the first and fourth lines to the corresponding right hand sides of (9) we see that we should have

$$\tag{10} \begin{cases} L_\varphi(s)\left(\alpha\dfrac{b_1-a_1}{b_1-r} + \beta\dfrac{b_2-a_2}{b_2-r} - 1\right) \ge \dfrac{(r-a_1)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\Delta(s), \\ L_\varphi(s)\left((1-\alpha)\dfrac{b_1-a_1}{r-a_1} + \dfrac{b_2-r}{r-a_2} - \beta\dfrac{b_2-a_2}{r-a_2}\right) \\ \qquad\qquad\qquad \ge \dfrac{(b_1-r)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\Delta(s). \end{cases}$$

Let

$$\begin{aligned} A_1 &= \left\{(x_1, x_2) \in [0,1]^2 : x_1\frac{b_1-a_1}{b_1-r} + x_2\frac{b_2-a_2}{b_2-r} - 1 > 0\right\}, \\ A_2 &= \left\{(x_1, x_2) \in [0,1]^2 : (1-x_1)\frac{b_1-a_1}{r-a_1} + \frac{b_2-r}{r-a_2} - x_2\frac{b_2-a_2}{r-a_2} > 0\right\}, \\ A &= A_1 \cap A_2. \end{aligned}$$

The set A is nonempty since it contains the pairs $(0,1)$ and $(1,0)$. Moreover, A is a convex and open subset of $\mathbb{R}^2$ as the intersection of convex and open sets A_1, A_2. Therefore the diagonal $\{(\alpha,\beta)\in[0,1]^2 : \alpha+\beta=1\}$ is contained in A together with its open neighbourhood.

It is easy to check that for $(\alpha,\beta)\in A$, (10) can be rewritten equivalently as

$$L_\varphi(s) \geq \frac{(r-a_2)(b_2-r)(r-a_1)(b_1-r)}{(b_2-a_2)(b_1-a_1)}\Delta(s)\cdot\max\{k_1(\alpha,\beta),k_2(\alpha,\beta)\}$$

where $k_1(\alpha,\beta)$, $k_2(\alpha,\beta)$ are fractions of the form

$$k_1(\alpha,\beta) = \frac{1}{\alpha(b_1-a_1)(b_2-r)+\beta(b_2-a_2)(b_1-r)-(b_1-r)(b_2-r)},$$

$$k_2(\alpha,\beta) = \frac{1}{(1-\alpha)(b_1-a_1)(r-a_2)+(b_2-r)(r-a_1)-\beta(b_2-a_2)(r-a_1)}.$$

Let

$$M = \inf_{(\alpha,\beta)\in A}\max\{k_1(\alpha,\beta),k_2(\alpha,\beta)\}. \tag{11}$$

Comparing the denominators of k_1 and k_2, after algebraic transformations we obtain

$$\begin{aligned}\alpha(b_1-a_1)(b_2-r)&+\beta(b_2-a_2)(b_1-r)-(b_1-r)(b_2-r)\\ &\geq (1-\alpha)(b_1-a_1)(r-a_2)+(b_2-r)(r-a_1)-\beta(b_2-a_2)(r-a_1)\end{aligned}$$

for

$$\alpha+\beta\geq 1.$$

Therefore, when the infimum in (11) is achieved for $\alpha+\beta\geq 1$ we have $M=M_1$ with

$$M_1 = \inf_{\substack{(\alpha,\beta)\in A\\ \alpha+\beta\geq 1}} k_2(\alpha,\beta)$$

and the form of k_2 implies that the infimum is attained for $\alpha+\beta=1$. Consequently,

$$\begin{aligned} M_1 &= \inf_{\alpha\in[0,1]}\frac{1}{(b_1-r)(r-a_2)-\alpha[(b_1-a_1)(r-a_2)-(b_2-a_2)(r-a_1)]}\\ &= \begin{cases} \dfrac{1}{(b_1-r)(r-a_2)} & \text{if } \dfrac{r-a_1}{b_1-a_1}\leq\dfrac{r-a_2}{b_2-a_2},\\[2ex] \dfrac{1}{(r-a_1)(b_2-r)} & \text{if } \dfrac{r-a_1}{b_1-a_1}\geq\dfrac{r-a_2}{b_2-a_2}.\end{cases}\end{aligned}$$

If the infimum in (11) is achieved for $\alpha+\beta\leq 1$ we have $M=M_2$ with

$$M_2 = \inf_{\substack{(\alpha,\beta)\in A\\ \alpha+\beta\leq 1}}\frac{1}{\alpha(b_1-a_1)(b_2-r)+\beta(b_2-a_2)(b_1-r)-(b_1-r)(b_2-r)}$$

and the infimum is also attained when $\alpha + \beta = 1$, which means that $M = M_1 = M_2$.

Let $\overline{L}_\varphi(s)$ be the lower level for which there exist strategies γ^1, γ^2 such that the first and fourth inequalities of (9) are satisfied. Then

$$\overline{L}_\varphi(s) = \begin{cases} \dfrac{(r-a_1)(b_2-r)}{(b_1-a_1)(b_2-a_2)}\Delta(s) & \text{if } \dfrac{r-a_1}{b_1-a_1} \le \dfrac{r-a_2}{b_2-a_2}, \\ \dfrac{(b_1-r)(r-a_2)}{(b_1-a_1)(b_2-a_2)}\Delta(s) & \text{if } \dfrac{r-a_1}{b_1-a_1} \ge \dfrac{r-a_2}{b_2-a_2}, \end{cases}$$

and the corresponding strategies are linear combinations of their bounds with coefficients:

$$\alpha = \begin{cases} 0 & \text{if } \dfrac{r-a_1}{b_1-a_1} \le \dfrac{r-a_2}{b_2-a_2}, \\ 1 & \text{if } \dfrac{r-a_1}{b_1-a_1} \ge \dfrac{r-a_2}{b_2-a_2}, \end{cases}$$

for γ_1, and

$$\beta = \begin{cases} 1 & \text{if } \dfrac{r-a_1}{b_1-a_1} \le \dfrac{r-a_2}{b_2-a_2}, \\ 0 & \text{if } \dfrac{r-a_1}{b_1-a_1} \ge \dfrac{r-a_2}{b_2-a_2}, \end{cases}$$

for γ_2.

Furthermore, notice that for $\alpha = 0$, $\beta = 1$ we have a strict inequality in the second line and equality in the third line of (9), while for $\alpha = 1$, $\beta = 0$ we have equality in the second line and strict inequality in the third line of (9). Hence, $L^*_\varphi(s) = \overline{L}_\varphi(s)$. The proof for $\Delta(s) > 0$ is thus complete. ∎

3.2. *Multidimensional case*. In this section we present some analytical results obtained for the general case $d \ge 1$.

We now leave the geometric language and denote the vertices $(i_1, \ldots, i_d)$ by ξ. Let

$$\Xi = \{\xi = (\xi_1, \ldots, \xi_d) : \xi_j \in \{a_j, b_j\} \text{ for } j \in I\}$$

where $I = \{1, \ldots, d\}$.

Definition 2. For a given sequence $(i^*_{j_1}, \ldots, i^*_{j_n})$, $i^*_{j_l} \in \{a_{j_l}, b_{j_l}\}$ for $l = 1, \ldots, n$, $n \le d$, the *projection set* $\Xi(i^*_{j_1}, \ldots, i^*_{j_n})$ is the subset of Ξ defined as follows:

$$\Xi(i^*_{j_1}, \ldots, i^*_{j_n}) = \{\xi : \xi_{j_l} = i^*_{j_l} \text{ for } l = 1, \ldots, n\}.$$

Let $\{F^{\overline{\varphi}}_{j_1,\ldots,j_k} : \{j_1, \ldots, j_k\} \subset I\}$ be the family of functions defined by the formula

$$F^{\overline{\varphi}}_{j_1,\ldots,j_k}(\xi, s) = \prod_{l=1}^{d-k} \left(1 - \frac{|\xi_{k_l} - r|}{b_{k_l} - a_{k_l}}\right)\overline{\varphi}(\xi, s)$$

where $k_l \in I \setminus \{j_1, \ldots, j_k\}$ for $l = 1, \ldots, d-k$. Denote $F^{\overline{\varphi}}_{\emptyset}$ by $F^{\overline{\varphi}}$.

REMARK 4. Notice that for any set $\{j_1, \ldots, j_{k-1}\}$ of indices and $j_k \in I \setminus \{j_1, \ldots, j_{k-1}\}$ the following identity holds:

$$\begin{aligned}\sum_{\xi \in \Xi(i_{j_1}, \ldots, i_{j_{k-1}})} & F^{\bar{\varphi}}_{j_1, \ldots, j_{k-1}}(\xi, s) \\ &= \frac{b_{j_k} - r}{b_{j_k} - a_{j_k}} \sum_{\xi \in \Xi(i_{j_1}, \ldots, i_{j_{k-1}}, a_{j_k})} F^{\bar{\varphi}}_{j_1, \ldots, j_k}(\xi, s) \\ &\quad + \frac{r - a_{j_k}}{b_{j_k} - a_{j_k}} \sum_{\xi \in \Xi(i_{j_1}, \ldots, i_{j_{k-1}}, b_{j_k})} F^{\bar{\varphi}}_{j_1, \ldots, j_k}(\xi, s).\end{aligned}$$

LEMMA 1. *If x and $(\gamma^1, \ldots, \gamma^d)$ satisfy the system* (3) *of 2^d inequalities then*

$$\left\{\begin{aligned}(1+r)x + \gamma^j(a_j - r)s^j &\geq \sum_{\xi \in \Xi(a_j)} F^{\bar{\varphi}}_j(\xi, s), \\ (1+r)x + \gamma^j(b_j - r)s^j &\geq \sum_{\xi \in \Xi(b_j)} F^{\bar{\varphi}}_j(\xi, s),\end{aligned}\right. \tag{12}$$

for all $j = 1, \ldots, d$.

Proof. The proof is by induction on d. For $d = 1$, (12) holds because then the systems (3) and (12) are identical. Assume that the induction hypothesis is true for $d-1$. We show that (12) is true for any k, $1 \leq k \leq d$. Let j be an integer, $1 \leq j \leq d$, different from k. Assume x and $(\gamma^1, \ldots, \gamma^d)$ satisfy (3). Multiply each inequality in (3) corresponding to the sequence $(i_1, \ldots, a_j, \ldots, i_d)$ by $b_j - r > 0$, and each inequality corresponding to the sequence $(i_1, \ldots, b_j, \ldots, i_d)$ by $r - a_j > 0$. Adding the inequalities obtained for every $(i_1, \ldots, i_{j-1}, i_{j+1} \ldots, i_d)$ we find that the coefficient of $\gamma^j s^j$ is equal to $(b_j - r)(a_j - r) - (r - a_j)(b_j - r) = 0$. Consequently, we obtain a system of 2^{d-1} inequalities with variables x and $\gamma^1, \ldots, \gamma^{j-1}, \gamma^{j+1}, \ldots, \gamma^d$. Dividing each inequality by $b_j - a_j$ we can write the inequality corresponding to the sequence $(i_1, \ldots, i_{j-1}, i_{j+1} \ldots, i_d)$ in the form

$$\begin{aligned}(1+r)x + \sum_{l=1}^{j-1} \gamma^l(i_l - r)s^l + \sum_{l=j+1}^{d} \gamma^l(i_l - r)s^l \\ \geq \psi(i_1, \ldots, i_{j-1}, i_{j+1} \ldots, i_d, s)\end{aligned} \tag{13}$$

where

$$\begin{aligned}\psi(i_1, \ldots, i_{j-1}, i_{j+1} \ldots, i_d, s) := & \frac{b_j - r}{b_j - a_j} \overline{\varphi}(i_1, \ldots, a_j, \ldots, i_d, s) \\ & + \frac{r - a_j}{b_j - a_j} \overline{\varphi}(i_1, \ldots, b_j, \ldots, i_d, s).\end{aligned}$$

Consider now (13) as a new system of 2^{d-1} inequalities with the right side function ψ. Since $k \neq j$ we can apply the induction hypothesis for k in the $(d-1)$st case. Therefore we have

$$\begin{cases} (1+r)x + \gamma^k (a_k - r)s^k \geq \sum_{\bar{\xi} \in \bar{\Xi}(a_k)} F_k^{\psi}(\bar{\xi}, s), \\ (1+r)x + \gamma^k (b_k - r)s^k \geq \sum_{\bar{\xi} \in \bar{\Xi}(b_k)} F_k^{\psi}(\bar{\xi}, s), \end{cases}$$

where $\overline{\xi}$ and $\overline{\Xi}(\cdot)$ in the system above are from $\mathbb{R}^{d-1}$ because the jth coordinate was omitted. Note that, by the definition of ψ and Remark 4,

$$\begin{aligned} \sum_{\bar{\xi} \in \bar{\Xi}(i_k)} F_k^{\psi}(\bar{\xi}, s) &= \frac{b_j - r}{b_j - a_j} \sum_{\xi \in \Xi(a_j, i_k)} F_{j,k}^{\bar{\varphi}}(\xi, s) + \frac{r - a_j}{b_j - a_j} \sum_{\xi \in \Xi(b_j, i_k)} F_{j,k}^{\bar{\varphi}}(\xi, s) \\ &= \sum_{\xi \in \Xi(i_k)} F_k^{\bar{\varphi}}(\xi, s) \end{aligned}$$

where $i_k \in \{a_k, b_k\}$ and $\xi \in \mathbb{R}^d$, so that the induction hypothesis is also true for d, which completes the proof. ∎

Using Lemma 1 we formulate necessary conditions for solutions of (3) in terms of the bounds on x and $\gamma^1, \ldots, \gamma^d$.

PROPOSITION 1. *If x and $(\gamma^1, \ldots, \gamma^d)$ satisfy the system* (3) *of 2^d inequalities for given $s \in \mathbb{R}^d$ then*

$$x \geq \frac{1}{1+r} \sum_{\xi \in \Xi} F^{\bar{\varphi}}(\xi, s),$$

and, for $j = 1, \ldots, d$,

$$\frac{\sum_{\xi \in \Xi(b_j)} F_j^{\bar{\varphi}}(\xi, s) - (1+r)x}{(b_j - r)s^j} \leq \gamma^j \leq \frac{(1+r)x - \sum_{\xi \in \Xi(a_j)} F_j^{\bar{\varphi}}(\xi, s)}{(r - a_j)s^j}.$$

Proof. The form of the bounds on γ^j is a straightforward consequence of (12). In fact, dividing the first inequality of (12) by $(a_j - r)s^j < 0$ and changing the sign we obtain the upper bound. Similarly from the second inequality we obtain the lower bound.

In order to get the lower bound on x, we multiply the first inequality of (12) by $b_j - r > 0$, and the second by $r - a_j > 0$. Adding them together, dividing the result by $b_j - a_j$ and using Remark 4 we obtain

$$\begin{aligned} (1+r)x &\geq \frac{b_j - r}{b_j - a_j} \sum_{\xi \in \Xi(a_j)} F_j^{\bar{\varphi}}(\xi, s) + \frac{r - a_j}{b_j - a_j} \sum_{\xi \in \Xi(b_j)} F_j^{\bar{\varphi}}(\xi, s) \\ &= \sum_{\xi \in \Xi} F^{\bar{\varphi}}(\xi, s) \end{aligned}$$

and the proof is complete. ∎

The bounds on x and $\gamma^1,\dots,\gamma^d$ may not be sharp. Analogously to the two-dimensional case we shall try to determine a sharp bound for x, i.e. to find the smallest $L_\varphi(s)\ge 0$ such that for

$$(1+r)x=\sum_{\xi\in\Xi}F^{\bar\varphi}(\xi,s)+L_\varphi(s) \tag{14}$$

there exist strategies $\gamma^1,\dots,\gamma^d$ satisfying (3). Let γ^i be linear combinations of their lower and upper bounds for $i=1,\dots,d$ and let x be of the form (14). Consider now the left hand side of the inequality in (3) for fixed $\xi^*=(\xi_1^*,\dots,\xi_d^*)$. We have

$$\begin{aligned}
&\gamma^j(\xi_j^*-r)s^j\\
&\quad=\bigg[\alpha_j\cdot\frac{\sum_{\xi\in\Xi(b_j)}F_j^{\bar\varphi}(\xi,s)-\sum_{\xi\in\Xi}F^{\bar\varphi}(\xi,s)-L_\varphi(s)}{(b_j-r)s^j}\\
&\qquad+(1-\alpha_j)\cdot\frac{\sum_{\xi\in\Xi}F^{\bar\varphi}(\xi,s)+L_\varphi(s)-\sum_{\xi\in\Xi(a_j)}F_j^{\bar\varphi}(\xi,s)}{(r-a_j)s^j}\bigg](\xi_j^*-r)s^j\\
&\quad=\bigg[L_\varphi(s)\Big(-\frac{\alpha_j}{b_j-r}+\frac{1-\alpha_j}{r-a_j}\Big)+\frac{\alpha_j}{b_j-r}\sum_{\xi\in\Xi(b_j)}F_j^{\bar\varphi}(\xi,s)\\
&\qquad-\Big(\frac{\alpha_j}{b_j-r}-\frac{1-\alpha_j}{r-a_j}\Big)\sum_{\xi\in\Xi}F^{\bar\varphi}(\xi,s)-\frac{1-\alpha_j}{r-a_j}\sum_{\xi\in\Xi(a_j)}F_j^{\bar\varphi}(\xi,s)\bigg](\xi_j^*-r).
\end{aligned}$$

Let $k_j(\alpha_j)$ be the affine coefficient of L_φ. According to Remark 4 we can rewrite the expression above in the form

$$\begin{aligned}
&\gamma^j(\xi_j^*-r)s^j\\
&\quad=\bigg[L_\varphi(s)k_j(\alpha_j)+\Big(\frac{\alpha_j}{b_j-r}+\frac{1-\alpha_j}{b_j-a_j}-\frac{\alpha_j}{b_j-r}\cdot\frac{r-a_j}{b_j-a_j}\Big)\sum_{\xi\in\Xi(b_j)}F_j^{\bar\varphi}(\xi,s)\\
&\qquad+\Big(\frac{\alpha_j}{b_j-a_j}-\frac{1-\alpha_j}{r-a_j}\cdot\frac{b_j-r}{b_j-a_j}+\frac{1-\alpha_j}{r-a_j}\Big)\sum_{\xi\in\Xi(a_j)}F_j^{\bar\varphi}(\xi,s)\bigg](\xi_j^*-r).
\end{aligned}$$

Since

$$\frac{\alpha_j}{b_j-r}+\frac{1-\alpha_j}{b_j-a_j}-\frac{\alpha_j}{b_j-r}\cdot\frac{r-a_j}{b_j-a_j}=\frac{1}{b_j-a_j}$$

and

$$\frac{\alpha_j}{b_j-a_j}-\frac{1-\alpha_j}{r-a_j}\cdot\frac{b_j-r}{b_j-a_j}+\frac{1-\alpha_j}{r-a_j}=\frac{1}{b_j-a_j}$$

letting

$$\Delta(j,s):=\sum_{\xi\in\Xi(b_j)}F_j^{\bar\varphi}(\xi,s)-\sum_{\xi\in\Xi(a_j)}F_j^{\bar\varphi}(\xi,s)$$

we obtain

$$\gamma^j(\xi_j^* - r)s^j = \left[L_\varphi(s)k_j(\alpha_j) + \frac{\Delta(j,s)}{b_j - a_j}\right](\xi_j^* - r).$$

Consequently, the inequality in (3) corresponding to the noise $\xi = (\xi_1, \ldots, \xi_d)$ is satisfied when

$$(15)\qquad L_\varphi(s)\Big(1 + \sum_{j=1}^{d} k_j(\alpha_j)(\xi_j - r)\Big) \geq \overline{\varphi}(\xi, s) - \sum_{j=1}^{d} \frac{\xi_j - r}{b_j - a_j}\Delta(j,s) - \sum_{\xi\in\varXi} F^{\bar{\varphi}}(\xi, s).$$

The problem to determine the transformation Q is then reduced to an optimization problem consisting in finding the smallest $L_\varphi(s)$ for which there exist $(\alpha_1, \ldots, \alpha_d)$ satisfying the system of inequalities described by (15) for each $\xi \in \varXi$.

It seems to be difficult to obtain an explicit formula for Q. However, it can be performed using numerical methods.

4. Numerical examples. We present some numerical results for the case $d = 2$ basing on an explicit formula for the transformation Q obtained in Section 3.1.

Example 1. Consider an asset on a foreign stock exchange market with price at time 0 equal to $S_0 = 60$ in foreign currency. Let Y be a current exchange rate with initial value $Y_0 = 3$. The form of the contingent claim is $\varphi(S_T, Y_T) = (S_T \cdot Y_T - K)^+$. In the table below we present the prices of options expressed in domestic currency units for various strike prices K and maturities T.

	$K = 170$	$K = 180$	$K = 190$	$K = 200$
$T = 20$	50.36	44.48	39.8	35.3
$T = 30$	63.1	57.26	52.45	48.71
$T = 40$	74.81	69.04	64.97	61.13
$T = 50$	86.11	81.07	76.93	72.82

The volatility of the asset is equal to 10% (i.e. $a = -0.1$, $b = 0.1$ for S) and that of the currency exchange rate is 1%. The effective overnight interest rate is taken to be 0.03%. One can see that the option prices (i.e. the prices of the super-hedging position) are in the range of 20% to 45% of the initial value of the stock in domestic currency.

Example 2. We now present numerical results of computation of an option for a branch index. Let us assume that on Warsaw Stock Exchange market there exists an option for the brewery index. Let S^1 denote an asset of Okocim Co. and S^2 an asset of Żywiec Co. Initial prices of the assets are $S_0^1 = 16.9$ and $S_0^2 = 149.5$ respectively (quotation of 27.11.96). Let the contingent claim for such an option be $\varphi(S_T^1, S_T^2) = (\alpha_1 S_T^1 + \alpha_2 S_T^2 - K)^+$ where $\alpha_1 = 346$, $\alpha_2 = 50$.

	$K = 12800$	$K = 13322$	$K = 13600$	$K = 14000$
$T = 20$	2728	2443	2327	2161
$T = 30$	3247	2967	2847	2674
$T = 40$	3666	3388	3299	3170
$T = 50$	4032	3812	3718	3582

The option prices are given in index points (10 pts = 0.12 PLN). In this case the volatility of disturbances is equal to 10% so the price of the option is relatively higher than in the previous example. The effective overnight interest rate r is 0.048%.

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Received on 2.12.1996;
revised version on 23.5.1997