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POINCARÉ–MELNIKOV THEORY FOR n-DIMENSIONAL DIFFEOMORPHISMS

Abstract. We consider perturbations of n-dimensional maps having homo-heteroclinic connections of compact normally hyperbolic invariant manifolds. We justify the applicability of the Poincaré–Melnikov method by following a geometric approach. Several examples are included.

1. Introduction. The Poincaré–Melnikov method is a well known tool for evaluating the distance between splitted invariant manifolds of fixed objects (such as fixed points, periodic orbits, invariant tori, ...) when one perturbs a system of differential equations having homo-heteroclinic connections between such objects ([15], [14], [2], [4], [11], [17]). Furthermore, it is also an important tool for determining the transversality at intersection points of invariant manifolds. The method has been developed for two-dimensional maps ([7], [9]) and applied to several examples ([10], [13]). Recently Delshams and Ramírez-Ros [5] have given a systematic approach for evaluating the Melnikov function (an infinite sum, in this context) under some conditions of meromorphy of the functions involved.

A generalization to invariant manifolds associated with fixed points of n-dimensional maps is given in [16] and [3]. The case of exact symplectic maps is considered in [6].

Here we consider the case of perturbations of n-dimensional maps having homo-heteroclinic connections of compact normally hyperbolic invariant manifolds. We justify the applicability of the method by following a geometric approach.

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Since we do not put restrictions on the dimensions of the invariant manifolds we have to consider families of maps with several parameters. We discuss the locus of homo-heteroclinic intersections in the space of parameters.

In Section 2 we describe the setup, in Section 3 we prove the main result (Theorem 3.5). Two particular cases, of unperturbed maps which are interpolated by hamiltonian flows, are considered in Section 4. In Section 5 we present some examples for which we prove or disprove the existence of "clinic" intersections. Some technical details concerning the analytical computation of the Melnikov function are deferred to the Appendix.

2. Description of the setting. We consider families of maps

$$F_{\varepsilon,\mu} : \mathbb{R}^n \supset U \to \mathbb{R}^n$$

of class C^r, $r \geq 3$, depending C^r on two parameters ε and μ with $\varepsilon \in I \subset \mathbb{R}$, $0 \in I$, and $\mu \in V \subset \mathbb{R}^m$. Also we shall use the notation

$$F_{\varepsilon,\mu}(x) = F(x, \varepsilon, \mu).$$

We assume that F has the form

$$F(x, \varepsilon, \mu) = F_0(x) + \varepsilon G(x, \varepsilon, \mu)$$

with F_0 satisfying the following hypotheses:

- H1. F_0 has two C^r normally hyperbolic invariant manifolds P^1, P^2 not necessarily different, which are compact and connected. In particular, P^1, P^2 may be hyperbolic fixed points.
- H2. The stable invariant manifold of P^1, say W_0^{s}, and the unstable invariant manifold of P^2, say W_0^{u}, are d-dimensional.
- H3. There exists a d-dimensional heteroclinic manifold joining P^1 to P^2. (homoclinic if $P^1 = P^2$; in this case n must be even and $d = n/2$).

We are going to define the Melnikov function in this setting. First we recall a result on existence and persistence of normally hyperbolic invariant manifolds ([8], [12]).

Theorem 2.1. *Let $F : \mathbb{R}^n \supset U \to U$ be a C^r diffeomorphism onto its image, $r \geq 1$. Let M be a C^r compact, connected, invariant manifold of F. Let M be r-normally hyperbolic, that is,*

1. *There exists a continuous decomposition $T\mathbb{R}^n_{|M} = TM \oplus N^{\mathrm{s}} \oplus N^{\mathrm{u}}$.*
2. *$TM \oplus N^{\mathrm{s,u}}$ are F-invariant.*
3. *Let $\Pi^{\mathrm{s,u}}$ be the projections on $N^{\mathrm{s,u}}$ respectively. There exists a constant λ, $0 < \lambda < 1$, such that for all $m \in M$, and $0 \leq k \leq r$,*

$$\|DF^{-1}(m)_{|TM}\|^k \|\Pi^{\mathrm{s}} DF(F^{-1}(m))\| < \lambda$$

and

$$\|DF(m)_{|TM}\|^k \|\Pi^{\mathrm{u}} DF^{-1}(F(m))\| < \lambda.$$

Then M has C^r stable and unstable manifolds. Furthermore, there exists a C^1 neighbourhood of F, say $\mathcal{U}$, such that all $F' \in \mathcal{U}$ have an invariant manifold M', C^r-diffeomorphic to M, and M' has stable and unstable invariant manifolds. Furthermore, these objects depend in a differentiable way on parameters.

3. Construction of the Melnikov vector function. Let $I_0 \subset I$ and $V_0 \subset V$ be open sets such that $0 \in I_0$ and, if $(\varepsilon, \mu) \in I_0 \times V_0$, then $F_{\varepsilon,\mu}$ has normally hyperbolic invariant manifolds $P^1_{\varepsilon,\mu}$, $P^2_{\varepsilon,\mu}$ depending C^r on ε, μ and such that $P^1_{0,\mu} = P^1$, $P^2_{0,\mu} = P^2$. Let $W^{\mathrm{s}}_{\varepsilon,\mu}$ and $W^{\mathrm{u}}_{\varepsilon,\mu}$ be the stable and unstable manifolds of $P^1_{\varepsilon,\mu}$ and $P^2_{\varepsilon,\mu}$ respectively.

We consider a point $z \in (W^{\mathrm{s}}_0 - P^1) \cap (W^{\mathrm{u}}_0 - P^2)$ and a neighbourhood D of it in $(W^{\mathrm{s}}_0 - P^1) \cap (W^{\mathrm{u}}_0 - P^2)$ such that $\overline{D} \cap P^{1,2} = \emptyset$.

We decompose

$$T\mathbb{R}^n_{|D} = TD \oplus Q, \tag{3.1}$$

with Q_x orthogonal to $T_x D$ for all $x \in D$. Because of the results on the dependence of the invariant manifolds on parameters we can assume that $W^{\mathrm{s,u}}_{\varepsilon,\mu}$ and $x + Q_x$ are transversal at their intersection point (taking smaller I_0 and V_0 if necessary). Then there exist

$$x^{\mathrm{s,u}} : D \times I_0 \times V_0 \to U$$

defined by

$$x^{\mathrm{s,u}}(x, \varepsilon, \mu) = W^{\mathrm{s,u}}_{\varepsilon,\mu} \cap (x + Q_x) \cap U.$$

We write $x^{\mathrm{s,u}}_{\varepsilon,\mu}(x) = x^{\mathrm{s,u}}(x, \varepsilon, \mu)$. Let $v_1(x), \ldots, v_{n-d}(x)$ be a basis of Q_x depending C^{r-1} on $x \in D$.

Taking $(x, \varepsilon, \mu) \in D \times I_0 \times V_0$, we want to measure the distance between $x^{\mathrm{u}}_{\varepsilon,\mu}(x)$ and $x^{\mathrm{s}}_{\varepsilon,\mu}(x)$. We define

$$\begin{aligned}
\Delta_i(x, \varepsilon, \mu) &= \langle x^{\mathrm{u}}_{\varepsilon,\mu}(x) - x^{\mathrm{s}}_{\varepsilon,\mu}(x), v_i(x)\rangle, \qquad i = 1, \ldots, n-d,\\
\Delta(x, \varepsilon, \mu) &= (\Delta_1(x, \varepsilon, \mu), \ldots, \Delta_{n-d}(x, \varepsilon, \mu)),\\
M(x, \mu) &= D_\varepsilon \Delta(x, \varepsilon, \mu)|_{\varepsilon=0}.
\end{aligned}$$

The vector M is called the *Melnikov function* associated with the basis $v_1, \ldots, v_{n-d}$. It is of class C^{r-2}. For $x \in D$ we define

$$\begin{aligned}
&x^k = F^k_0(x),\\
&x^{\mathrm{s,u}\ k}_{\varepsilon,\mu}(x) = x^{\mathrm{s,u}\ k}(x, \varepsilon, \mu) = F^k_{\varepsilon,\mu}(x^{\mathrm{s,u}}(x, \varepsilon, \mu)),\\
&\xi^{\mathrm{s,u}\ k}_\mu(x) = \frac{\partial}{\partial \varepsilon} x^{\mathrm{s,u}\ k}(x, \varepsilon, \mu)\Big|_{\varepsilon=0},
\end{aligned}$$

for $k \in \mathbb{Z}$, and

$$\xi^{\mathrm{s,u}}_{\mu}(x) = \xi^{\mathrm{s,u}\ 0}_{\mu}(x).$$

Notice that

$$M_i(x,\mu) = \langle \xi^{\mathrm{u}}_{\mu}(x) - \xi^{\mathrm{s}}_{\mu}(x), v_i(x)\rangle. \tag{3.2}$$

LEMMA 3.1. *For any $i \in \{1,\dots,n-d\}$ and $l_1 > 0$, $l_2 > 0$ we have*

$$\begin{aligned} M_i(x,\mu) = &\sum_{k=-l_2}^{l_1-1} \langle DF_0^k(x^{-k})G(x^{-k-1},0,\mu), v_i(x)\rangle \\ &+ \langle DF_0^{l_1}(x^{-l_1})\xi^{\mathrm{u}\ -l_1}_{\mu}(x), v_i(x)\rangle \\ &- \langle DF_0^{-l_2}(x^{l_2})\xi^{\mathrm{s}\ l_2}_{\mu}(x), v_i(x)\rangle. \end{aligned} \tag{3.3}$$

P r o o f. We have

$$x^{\mathrm{s,u}\ k+1}_{\varepsilon,\mu} = F(x^{\mathrm{s,u}\ k}_{\varepsilon,\mu},\varepsilon,\mu) = F_0(x^{\mathrm{s,u}\ k}_{\varepsilon,\mu}) + \varepsilon G(x^{\mathrm{s,u}\ k}_{\varepsilon,\mu},\varepsilon,\mu).$$

Taking the derivative with respect to ε we get

$$\begin{aligned} \frac{d}{d\varepsilon} x^{\mathrm{s,u}\ k+1}_{\varepsilon,\mu} = &DF_0(x^{\mathrm{s,u}\ k}_{\varepsilon,\mu})\frac{d}{d\varepsilon}x^{\mathrm{s,u}\ k}_{\varepsilon,\mu} + G(x^{\mathrm{s,u}\ k}_{\varepsilon,\mu},\varepsilon,\mu) \\ &+ \varepsilon D_x G(x^{\mathrm{s,u}\ k}_{\varepsilon,\mu},\varepsilon,\mu)\frac{d}{d\varepsilon}x^{\mathrm{s,u}\ k}_{\varepsilon,\mu} + \varepsilon\frac{\partial G}{\partial\varepsilon}(x^{\mathrm{s,u}\ k}_{\varepsilon,\mu},\varepsilon,\mu), \end{aligned}$$

and evaluating it at $\varepsilon = 0$ gives

$$\xi^{\mathrm{s,u}\ k+1}_{\mu}(x) = DF_0(x^k)\xi^{\mathrm{s,u}\ k}_{\mu}(x) + G(x^k,0,\mu). \tag{3.4}$$

Now we shall prove that for all $l > 0$,

$$\xi^{\mathrm{u}\ 0}_{\mu}(x) = DF_0^l(x^{-l})\xi^{\mathrm{u}\ -l}_{\mu}(x) + \sum_{k=0}^{l-1} DF_0^k(x^{-k})G(x^{-k-1},0,\mu). \tag{3.5}$$

Indeed, for $l = 1$ it is (3.4) evaluated at $k = -1$. If it is true for l, using (3.4) evaluated at $k = -l-1$,

$$\begin{aligned} \xi^{\mathrm{u}\ 0}(x,\mu) &= DF_0^l(x^{-l})(DF_0(x^{-l-1})\xi^{\mathrm{u}\ -l-1}_{\mu}(x) + G(x^{-l-1},0,\mu)) \\ &\qquad + \sum_{k=0}^{l-1} DF_0^k(x^{-k})G(x^{-k-1},0,\mu) \\ &= DF_0^{l+1}(x^{-l-1})\xi^{\mathrm{u}\ -l-1}_{\mu}(x) + \sum_{k=0}^{l} DF_0^k(x^{-k})G(x^{-k-1},0,\mu) \end{aligned}$$

which proves (3.5).

From (3.4) we have

$$\xi^{\mathrm{s,u}\ k}_{\mu}(x) = (DF_0(x^k))^{-1}(\xi^{\mathrm{s,u}\ k+1}_{\mu}(x) - G(x^k,0,\mu))$$

and equivalently

$$(3.6)\qquad \xi_\mu^{\mathrm{s,u}\ k}(x) = DF_0^{-1}(x^{k+1})\big(\xi_\mu^{\mathrm{s,u}\ k+1}(x) - G(x^k,0,\mu)\big).$$

In the same way as before we check that for all $l > 0$,

$$(3.7)\qquad \xi_\mu^{\mathrm{s}\ 0}(x) = DF_0^{-l}(x^l)\xi_\mu^{\mathrm{s}\ l}(x) - \sum_{k=-1}^{-l} DF_0^k(x^{-k})G(x^{-k-1},0,\mu).$$

Subtracting (3.7) with $l = l_2$ from (3.5) with $l = l_1$ and taking the scalar product with $v_i(x)$, we obtain (3.3). ∎

The next two lemmas will give us sufficient control on the last two terms of formula (3.3).

LEMMA 3.2. *Let μ be fixed, and $\gamma : I \to \mathbb{R}^n$ be a C^1 curve such that $\gamma(\varepsilon) \in W^{\mathrm{s}}_{\varepsilon,\mu}$ for all $\varepsilon \in I$. Let $\gamma_m(\varepsilon) = F^m_{\varepsilon,\mu}(\gamma(\varepsilon))$. Suppose that there exists an open subset U of W^{s}_0, containing P^1, such that $\gamma_m(0) \in U$ for all $m \geq 0$, and there exists a continuous decomposition $T\mathbb{R}^n_{|U} = TU \oplus N$. Let Π be the projection on N. Then $\Pi\gamma'_m(0)$ is bounded by a constant independent of $m \geq 0$.*

Proof. We enlarge all objects by adding the parameter ε. Precisely, we introduce

$$\widetilde{P}^{\mathrm{s}}(\varepsilon) = (P^{\mathrm{s}}(\varepsilon),\varepsilon),\qquad \widetilde{F}(x,\varepsilon,\mu) = (F(x,\varepsilon,\mu),\varepsilon),\qquad \widetilde{\gamma}_m(\varepsilon) = (\gamma_m(\varepsilon),\varepsilon),$$
$$\widetilde{W}^{\mathrm{s}}_{\varepsilon,\mu} = W^{\mathrm{s}}_{\varepsilon,\mu} \times \{\varepsilon\},\qquad \widetilde{W}^{\mathrm{s}}_0 = \widetilde{W}^{\mathrm{s}}_{0,\mu} = W^{\mathrm{s}}_0 \times \{0\},$$
$$\widetilde{U} = U \times \{0\},\qquad \widetilde{W}^{\mathrm{s}}_\mu = \bigcup_\varepsilon \widetilde{W}^{\mathrm{s}}_{\varepsilon,\mu}.$$

From the definitions we have the decomposition

$$T\mathbb{R}^{n+1}_{|\widetilde{U}} = T\widetilde{U} \oplus \widetilde{N}$$

with $\widetilde{N}_x = N_x \oplus \langle(0,\ldots,0,1)\rangle$. Let $\widetilde{q}_0 = (q_0,0) \in \widetilde{U}$ with q_0 being an arbitrary point in U. We define $\widetilde{L}_{\widetilde{q}_0} = \widetilde{q}_0 + \widetilde{N}_{\widetilde{q}_0}$. Since $\widetilde{L}_{\widetilde{q}_0}$ and $\widetilde{W}^{\mathrm{s}}_0$ intersect transversally at $\widetilde{q}_0$, if ε is small enough there exists a C^{r-1} curve $\widetilde{q}(\varepsilon) = (q(\varepsilon),\varepsilon)$ such that $\widetilde{q}(0) = \widetilde{q}_0$, and $\widetilde{W}^{\mathrm{s}}_{\varepsilon,\mu}$ and $\widetilde{L}_{\widetilde{q}_0}$ intersect transversally at $\widetilde{q}(\varepsilon)$.

The tangent vector $(q'(0),1)$ to the curve $\widetilde{q}(\varepsilon)$ at $\varepsilon = 0$ depends continuously on $q_0 \in U$, and hence it has bounded norm in any compact subset of $U \subset W^{\mathrm{s}}_0$.

On the other hand, the vectors of $T_{\widetilde{q}_0}\widetilde{W}^{\mathrm{s}}_0$ have the form $(w,0) \in \mathbb{R}^n \times \{0\}$.

Let $m \geq 0$. We take $\gamma_m(0)$ as q_0. The tangent vector of the curve $\widetilde{\gamma}_m$ at $\widetilde{q}_0$ is $(\gamma'_m(0),1)$.

Since

$$T_{\widetilde{q}_0}\widetilde{W}^{\mathrm{s}}_\mu = T_{\widetilde{q}_0}\widetilde{W}^{\mathrm{s}}_0 \oplus \langle(q'(0),1)\rangle$$

there exist a unique $(w_m, 0) \in T_{\widetilde{q}_0}\widetilde{W}_0^{\mathrm{s}}$ and a unique $a \in \mathbb{R}$ such that $(\gamma_m'(0), 1) = a(q'(0), 1) + (w_m, 0)$. Then we have $a = 1$ and hence $\gamma_m'(0) = q'(0) + w_m$. From $w_m \in T_q W_0^{\mathrm{s}}$ we have $\Pi\gamma_m'(0) = \Pi q'(0)$.

Since $q_0 = \gamma_m(0)$ tends to P^1 as $m \to \infty$ and P^1 is compact it follows that q_0 belongs to a compact subset of W_0^{s} and $\Pi\gamma_m'(0)$ is bounded independently of $m \geq 0$. ∎

LEMMA 3.3. *Let $(g_k)_{k\geq 0}$ be a bounded sequence of vectors of $\mathbb{R}^n$. Then, given ν such that $0 < \lambda < \nu < 1$, there exists $c \geq 0$ such that for all $x \in D$ and $k \geq 0$,*

$$|\langle DF_0^{-k}(x^k)g_k, v_i(x)\rangle| \leq c\nu^k.$$

Proof. Let

$$N^\eta P^1 = \{(x, v) : x \in P^1,\ v \in N_x P^1,\ \|v\| < \eta\}$$

be a tubular neighbourhood of P^1 with $\eta > 0$ small enough so that the map

$$\psi : N^\eta P^1 \to \mathbb{R}^n, \quad (x, v) \mapsto x + v,$$

is a diffeomorphism onto its image. This is possible because P^1 is compact. Let $\pi_1 : N^\eta P^1 \to P^1$ be its first projection. Let $\Omega_0 = N^\eta P^1 \cap W_0^{\mathrm{s}}$.

Furthermore, we can assume that D and Ω_0 are small enough so that there exists k_0 such that $F_0^j(\overline{D}) \cap \Omega_0 = \emptyset$ for $0 \leq j \leq k_0$ and $F_0^k(\overline{D}) \subset \Omega_0$ for $k > k_0$.

We can assume that D is small enough so that $\overline{F_0(D)} \cap \overline{D} = \emptyset$. Let

$$\widehat{D} = \Omega_0 \cup \Big(\bigcup_{0\leq k\leq k_0} F_0^k(D)\Big).$$

We consider the decomposition

$$T\mathbb{R}^n_{|\widehat{D}} = T\widehat{D} \oplus N$$

defined by

$$N_{F_0^k(x)} = DF_0^k(x)Q_x, \quad 0 \leq k \leq k_0,\ x \in D,$$
$$N_x = T_{x'}W_0^{\mathrm{u}}(P^1), \quad x \in \Omega_0,$$

where Q is defined in (3.1), $W_0^{\mathrm{u}}(P^1)$ is the unstable manifold of P^1 and $x' = \pi_1\psi^{-1}x$. The decomposition is continuous because x' depends continuously on x. Let Π be the projection on N.

Let ν be such that $0 < \lambda < \nu < 1$. By continuity, taking a smaller Ω_0 if necessary, we have $\|\Pi DF_0^{-1}(x)\| < \nu$ for all $x \in \Omega_0$.

Here we have $\Pi DF_0^{-1}\Pi = \Pi DF_0^{-1}$. Indeed, let $u = u_{\mathrm{t}} + u_{\mathrm{n}}$ with

$u_{\rm t} \in T_x\widehat{D}$ and $u_{\rm n} \in N_x$. Then

$$\begin{aligned}\Pi DF_0^{-1}(x)u &= \Pi DF_0^{-1}(x)(u_{\rm t}+u_{\rm n})\\ &= \Pi\big(DF_0^{-1}(x)u_{\rm t}+DF_0^{-1}(x)u_{\rm n}\big)\\ &= \Pi DF_0^{-1}(x)u_{\rm n} = \Pi DF_0^{-1}(x)\Pi u,\end{aligned}$$

because $DF_0^{-1}(x) : T_xD \to T_{F^{-1}(x)}D$.

Now let $a_l = \sup_{x\in\overline{D}} \|\Pi DF_0^{-l}(x^l)\|$ and $b_l = a_l\nu^{-l}$. For $k > k_0$,

$$\begin{aligned}&\|\Pi DF_0^{-k}(x^k)\|\\ &\quad= \|\Pi DF_0^{-k_0-1}(x^{k_0+1})DF_0^{-1}(x^{k_0+2})\ldots DF_0^{-1}(x^k)\|\\ &\quad= \|\Pi DF_0^{-k_0-1}(x^{k_0+1})\Pi DF_0^{-1}(x^{k_0+2})\ldots \Pi DF_0^{-1}(x^k)\|\\ &\quad\le \|\Pi DF_0^{-k_0-1}(x^{k_0+1})\|\,\|\Pi DF_0^{-1}(x^{k_0+2})\|\ldots\|\Pi DF_0^{-1}(x^k)\|\\ &\quad\le a_{k_0}\nu^{k-k_0} = b_{k_0}\nu^k.\end{aligned}$$

Hence $\|\Pi DF_0^{-k}(x^k)\| \le b\nu^k$, for all $k \ge 0$, where $b = \max\{b_j : 0 \le j \le k_0\}$. ■

THEOREM 3.4. *We have the following expression for the Melnikov vector*:

$$M_i(x,\mu) = \sum_{k=-\infty}^{\infty} \langle DF_0^k(x^{-k})G(x^{-k-1},0,\mu), v_i(x)\rangle, \quad \forall \mu\in V_0,\ \forall x\in D.$$

Furthermore, *the sum is absolutely convergent.* (*It is geometrically convergent with rate* ν, $0<\lambda<\nu<1$.)

P r o o f. In view of (3.3) we only have to prove that

$$\langle DF_0^{-l_2}(x^{l_2})\xi_\mu^{{\rm s}\ l_2}(x), v_i(x)\rangle \to 0 \quad \text{as } l_2\to\infty.$$

and

$$\langle DF_0^{l_1}(x^{-l_1})\xi_\mu^{{\rm u}\ -l_1}(x), v_i(x)\rangle \to 0 \quad \text{as } l_1\to\infty.$$

Consider the decomposition and the projection Π defined in the proof of Lemma 3.3. Since

$$\langle DF_0^{-l_2}(x^{l_2})\xi_\mu^{{\rm s}\ l_2}(x), v_i(x)\rangle = \langle DF_0^{-l_2}(x^{l_2})\Pi\xi_\mu^{{\rm s}\ l_2}(x), v_i(x)\rangle$$

and, by Lemma 3.2, $\Pi\xi^{{\rm s}\ l_2}$ has bounded norm for each μ and each x independently of $l_2\ge 0$, Lemma 3.3 shows that

$$\langle DF_0^{-l_2}(x^{l_2})\xi_\mu^{{\rm s}\ l_2}(x), v_i(x)\rangle \to 0$$

as $l_2\to\infty$. The other limit is considered in the same way, using F_0^{-1} instead of F_0. ■

THEOREM 3.5. *Let* F_0 *be a map satisfying hypotheses* H1–H3. *Let* $m+2d-n\ge 0$. *Assume there exists* $(x_0,\mu_0)\in D\times V_1$ *such that* $M(x_0,\mu_0) =$

0 and $\operatorname{rk} DM(x_0,\mu_0)$ *is maximum. (Here we consider the derivative with respect to* x *and* μ*.)*

1. *Then there exists a neighbourhood* $\Omega \subset D \times I_0 \times V_0$ *of* $(x_0,0,\mu_0)$ *and a manifold* $S \subset \Omega$, $(x_0,0,\mu_0) \in S$, $S \not\subset \{(x,0,\mu) \in \Omega\}$, *of class* C^{r-2} *such that*

(a) $S \cup \{(x,0,\mu) \in \Omega\} = \{(x,\varepsilon,\mu) \in \Omega : \Delta(x,\varepsilon,\mu) = 0\}$.

(b) $\dim S = 1 + m + 2d - n \geq 1$.

2. *If we further assume that* $\operatorname{rk} DM_{\mu_0}(x_0)$ *is maximum, where* $M_\mu(x) = M(\mu,x)$ *(here we consider the derivative with respect to* x*) then there exists* $\Omega_0 \subset \Omega$ *such that for all* $(\overline{x},\overline{\varepsilon},\overline{\mu}) \in S_0 = S \cap \Omega_0$ *with* $\overline{\varepsilon} \neq 0$ *we have*

$$\dim(T_z W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} + T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}) = \min(n, 2d)$$

where $z = x^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) = x^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})$. *Notice that* $z \in W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} \cap W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}$ *and that if* $n \leq 2d$ *then* $W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}}$ *and* $W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}$ *are transversal at* z, *and if* $n > 2d$ *then*

$$\dim(T_z W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} + T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}) = \dim T_z W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} + \dim T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}.$$

Proof. 1. The function

$$\Delta : D \times I_0 \times V_0 \to \mathbb{R}^{n-d}, \qquad (x,\varepsilon,\mu) \mapsto \Delta(x,\varepsilon,\mu),$$

is of class C^{r-1}, $r \geq 3$. We define

$$\overline{\Delta} : D \times I_0 \times V_0 \to \mathbb{R}^{n-d}$$

by $\overline{\Delta}(x,\varepsilon,\mu) = \Delta(x,\varepsilon,\mu)/\varepsilon$ if $\varepsilon \neq 0$, and $\overline{\Delta}(x,0,\mu) = M(x,\mu)$. It is of class C^{r-2}.

We have

$$\overline{\Delta}(x,\varepsilon,\mu) = M(x,\mu) + O(\varepsilon).$$

Clearly $\Delta(x,\varepsilon,\mu) = 0$ if and only if either $\varepsilon = 0$ or $\overline{\Delta}(x,\varepsilon,\mu) = 0$. Since $\operatorname{rk} DM(x_0,\mu_0)$ is maximum and equal to $n-d$,

$$\operatorname{rk} D\overline{\Delta}(x_0,0,\mu_0) = \operatorname{rk} DM(x_0,\mu_0) = n - d$$

is also maximum. Then

$$S = \{(\varepsilon,\mu,x) : \overline{\Delta}(x,\varepsilon,\mu) = 0\}$$

is a manifold of class C^{r-2} and dimension $1+m+d-(n-d) = 1+m+2d-n \geq 1$ which can be parametrized by ε and $m+2d-n$ variables of the set $(x_1,\ldots,x_{\mathrm{n}},\mu_1,\ldots,\mu_m)$.

2. Let $\overline{\Delta}_{\varepsilon,\mu}(x) = \overline{\Delta}(x,\varepsilon,\mu)$. Let Ω_0 be a neighbourhood of $(x_0,0,\mu_0)$ in Ω such that

$$\operatorname{rk} D\overline{\Delta}_{\varepsilon,\mu}(x) = \operatorname{rk} DM_\mu(x) = \operatorname{rk} DM_{\mu_0}(x_0) \tag{3.8}$$

for all $(x,\varepsilon,\mu) \in \Omega_0$.

Let $\overline{\varepsilon} \neq 0$ and $z \in W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} \cap W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}$, so that

$$z = x^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) = x^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}). \tag{3.9}$$

We claim that

$$\dim(T_z W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} \cap T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}) \le \dim \operatorname{Ker} D\Delta_{\overline{\varepsilon},\overline{\mu}}(\overline{x}). \tag{3.10}$$

Indeed, we may assume that there exists $u \in T_z W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} \cap T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}$, $u \neq 0$, because otherwise the claim is obviously true. We first prove that there exists $v \in T_{\overline{x}} D$ such that

$$u = Dx^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})v = Dx^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})v. \tag{3.11}$$

Indeed, let $v = Dx^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})^{-1}u$. By construction we can write

$$x^{\mathrm{s,u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) = \overline{x} + \sum_{i=1}^{n-d} \alpha^{\mathrm{s,u}}_{i,\overline{\varepsilon},\overline{\mu}}(\overline{x}) v_i(\overline{x}).$$

If we define $u^{\mathrm{s,u}} = Dx^{\mathrm{s,u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})v$ we have

$$u^{\mathrm{s,u}} = v + \sum_{i=1}^{n-d} (D\alpha^{\mathrm{s,u}}_{i,\overline{\varepsilon},\overline{\mu}}(\overline{x})v) v_i(\overline{x}) + \sum_{i=1}^{n-d} \alpha^{\mathrm{s,u}}_{i,\overline{\varepsilon},\overline{\mu}}(\overline{x}) D v_i(\overline{x}) v.$$

Since $u^{\mathrm{s}} = u$ and $\alpha^{\mathrm{s}}_{i,\overline{\varepsilon},\overline{\mu}}(\overline{x}) = \alpha^{\mathrm{u}}_{i,\overline{\varepsilon},\overline{\mu}}(\overline{x})$ because $x^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) = x^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) = z$, we have $u^{\mathrm{s}} - u^{u} = u - u^{\mathrm{u}} \in T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}$ and also

$$u^{\mathrm{s}} - u^{\mathrm{u}} = \sum_{i=1}^{n-d} ((D\alpha^{\mathrm{s}}_{i,\overline{\varepsilon},\overline{\mu}} - D\alpha^{u}_{i,\overline{\varepsilon},\overline{\mu}})(\overline{x})v) v_i(\overline{x}).$$

Since $v_i(\overline{x})$ is transversal to $T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}$ we have $u^{\mathrm{s}} - u^{\mathrm{u}} = 0$, and hence (3.11) follows.

If we write $\Delta_{i,\varepsilon,\mu}(x) = \Delta_i(x, \varepsilon, \mu)$ then

$$\begin{aligned} D\Delta_{i,\overline{\varepsilon},\overline{\mu}}(\overline{x})v &= \langle x^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) - x^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}), Dv_i(\overline{x})v\rangle \\ &\quad + \langle Dx^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})v - Dx^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})v, v_i(\overline{x})\rangle, \end{aligned}$$

so that, by (3.9) and (3.11), we have

$$D\Delta_{\overline{\varepsilon},\overline{\mu}}(\overline{x})v = 0,$$

which proves (3.10) because $Dx^{\mathrm{s,u}}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) : T_{\overline{x}} W^{\mathrm{s,u}}_0 \to T_z W^{\mathrm{s,u}}_{\overline{\varepsilon},\overline{\mu}}$ is one-to-one.

Since $\overline{\varepsilon} \neq 0$ and $\Delta_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) = \varepsilon \overline{\Delta}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})$, we have $\dim \operatorname{Ker} D\Delta_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) = \dim \operatorname{Ker} D\overline{\Delta}_{\overline{\varepsilon},\overline{\mu}}(\overline{x})$. Now by (3.8),

$$\dim \operatorname{Ker} D\overline{\Delta}_{\overline{\varepsilon},\overline{\mu}}(\overline{x}) = \dim \operatorname{Ker} DM_{\overline{\mu}}(\overline{x}),$$

and from (3.10) we deduce

$$\dim(T_z W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} \cap T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}) \le \dim \operatorname{Ker} DM_{\overline{\mu}}(\overline{x}). \tag{3.12}$$

If $2d \le n$, then $\operatorname{rk} DM_{\overline{\mu}}(\overline{x}) = \min(d, n-d) = d$. Therefore $\operatorname{Ker} DM_{\overline{\mu}}(\overline{x}) = \{0\}$ and hence by (3.12),

$$\dim(T_z W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} + T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}) = 2d.$$

If $2d > n$, we have $\dim \operatorname{Ker} DM_{\overline{\mu}}(\overline{x}) = d - (n-d) = 2d - n$. Then

$$\dim(T_z W^{\mathrm{s}}_{\overline{\varepsilon},\overline{\mu}} + T_z W^{\mathrm{u}}_{\overline{\varepsilon},\overline{\mu}}) \ge d + d - (2d - n) = n. \ \blacksquare$$

4. The case when the unperturbed map comes from a Hamiltonian. When the unperturbed map is the time τ map of a Hamiltonian the expression of the Melnikov function is simpler and the method is easier to apply.

THEOREM 4.1. *Consider F_0 satisfying the hypotheses* H1–H3, *and S, its homoclinic or heteroclinic d-dimensional manifold. Suppose that there exists a Hamiltonian $H : \mathbb{R}^{2n} \supset U \to \mathbb{R}$ such that F_0 is the time τ map of H. Let $x \in S \setminus (P^1 \cup P^2)$. Assume that there exist first integrals $H_1, \ldots, H_r$, $r = 2n - d$, functionally independent at x, satisfying*

1. $\{H, H_i\} = 0$, $i = 1, \ldots, r$.
2. *There are constants $c_1, \ldots, c_r$ with $S \subset \{H_1 = c_1\} \cap \ldots \cap \{H_r = c_r\}$.*

Then

1. $\{\operatorname{grad} H_1(x), \ldots, \operatorname{grad} H_r(x)\}$ *is a basis of the orthogonal space to $T_x S$.*
2. *Given a perturbed map*

$$F(x, \varepsilon, \mu) = F_0(x) + \varepsilon G(x, \varepsilon, \mu),$$

the Melnikov function associated with this basis is $M = (M_1, \ldots, M_r)$ with

$$M_i(x, \mu) = \sum_{k=-\infty}^{\infty} \langle G(x^{k-1}, 0, \mu), \operatorname{grad} H_i(x^k) \rangle \tag{4.1}$$

where $x^k = F_0^k(x)$.

P r o o f. We shall not write the parameter μ in order to simplify the notation. The first part is an easy consequence of the fact that $\operatorname{grad} H_1(x), \ldots,$ $\operatorname{grad} H_r(x)$ are independent and generate the orthogonal of $T_x S$. To prove the second part we begin by checking that

$$DF_0^k(x) J \operatorname{grad} H_i(x) = J \operatorname{grad} H_i(F_0^k(x)). \tag{4.2}$$

Indeed, let φ_i^s be the time s map of the vector field $X_{H_i} = J \operatorname{grad} H_i$ and φ^t the time t map of $X_H = J \operatorname{grad} H$. The condition $\{H, H_i\} = 0$ implies that $[X_H, X_{H_i}] = 0$, and hence

$$\varphi_i^s \circ \varphi^t(x) = \varphi^t \circ \varphi_i^s(x).$$

Taking the derivative with respect to s and evaluating it at $s = 0$ we get

$$J \operatorname{grad} H_i(\varphi^t(x)) = D\varphi^t(x) J \operatorname{grad} H_i(x),$$

and putting $t = n\tau$ we have (4.2). Also we shall use the fact that $(DF_0^k(x^{-k}))^T = J^T DF_0^{-k}(x) J$, because F_0 is symplectic.

Finally, from the general expression for the Melnikov function,

$$
\begin{aligned}
(4.3)\qquad M_i(x) &= \sum_{k=-\infty}^{\infty} \langle DF_0^k(x^{-k})G(x^{-k-1},0), \operatorname{grad} H_i(x)\rangle \\
&= \sum_{k=-\infty}^{\infty} \langle G(x^{-k-1},0), (DF_0^k(x^{-k}))^T \operatorname{grad} H_i(x)\rangle \\
&= \sum_{k=-\infty}^{\infty} \langle G(x^{-k-1},0), J^T DF_0^{-k}(x) J \operatorname{grad} H_i(x)\rangle \\
&= \sum_{k=-\infty}^{\infty} \langle G(x^{k-1},0), J^T DF_0^{k}(x) J \operatorname{grad} H_i(x)\rangle \\
&= \sum_{k=-\infty}^{\infty} \langle G(x^{k-1},0), J^T J \operatorname{grad} H_i(x^k)\rangle. \ \blacksquare
\end{aligned}
$$

REMARK 4.2. If $r > n$, although there exist r local first integrals, it may be difficult to find explicit expressions for them in concrete examples.

REMARK 4.3. If F_0 coincides with the time τ map of H on S, $\{H, H_i\}|_S = 0$ and S is H_i-invariant with $d > n$ the theorem is also true. Indeed, since S is invariant, $J \operatorname{grad} H_i(x) \in T_xS$ and therefore (4.2) still holds. In this case we need $d > n$, because we want $H_1, \ldots, H_r$, $r = 2n - d$, to be functionally independent at x, but $r \le d$ because S is H_i-invariant, $i = 1, \ldots, r$.

In some examples, it may happen that the unperturbed system is a projection to the set of some variables of the time τ map associated with a Hamiltonian flow. In this case the form of the Melnikov function can be written in terms of the Hamiltonian.

THEOREM 4.4. *Consider F_0 satisfying the hypothesis* H1–H3 *and S, its homoclinic or heteroclinic manifold of dimension d. Suppose there exists a map $F_0' : \mathbb{R}^{n'} \supset U' \to \mathbb{R}^{n'}$, $U' = U \times V$, V open in $\mathbb{R}^{n'-n}$, $0 \in V$, such that if Π is the projection on U, then $\Pi F_0'(x,0) = F_0(x)$, $x \in U$, and such that there exists a Hamiltonian $H : U' \to \mathbb{R}$ with F_0' being its time τ map. Let $x \in S \setminus (P^1 \cup P^2)$ and $x' = (x,0)$. Assume that there exist first integrals $H_1, \ldots, H_{n-d}$ functionally independent at x', satisfying*

1. $\{H, H_i\} = 0$, $i = 1, \ldots, n-d$.
2. *There exist constants $c_1, \ldots, c_{n-d}$ such that $S' \subset \{H_1 = c_1\} \cap \ldots \cap \{H_{n-d} = c_{n-d}\}$ where $S' = S \times \{0\}$.*

Then

1. *$\{\Pi \operatorname{grad} H_1(x,0), \ldots, \Pi \operatorname{grad} H_{n-d}(x,0)\}$ is a basis of the orthogonal space to T_xS.*

2. *Given a perturbed map*

$$F(x,\varepsilon,\mu) = F_0(x) + \varepsilon G(x,\varepsilon,\mu),$$

the Melnikov function associated with this basis is $M = (M_1, \ldots, M_{n-d})$ *with*

$$M_i(x,\mu) = \sum_{k=-\infty}^{\infty} \langle G(x^{k-1},0,\mu), \Pi \operatorname{grad} H_i(x^k,0)\rangle \tag{4.4}$$

where $x^k = F_0^k(x)$.

P r o o f. (a) Since $S' = S \times \{0\} \subset \{H_1 = c_1\} \cap \ldots \cap \{H_{n-d} = c_{n-d}\}$, the vectors $\operatorname{grad} H_1(x,0), \ldots, \operatorname{grad} H_{n-d}(x,0)$ are orthogonal to $T_xS' = T_xS \times \{0\}$, therefore $\Pi \operatorname{grad} H_1(x,0), \ldots, \Pi \operatorname{grad} H_{n-d}(x,0)$ are orthogonal to T_xS.

(b) Formula (4.4) is proved in an analogous way to (4.1). We only have to take into account that from $F_0' \circ i = i \circ F_0$ where $i : \mathbb{R}^n \to \mathbb{R}^{n'}$ is defined by $i(x) = (x,0)$, we have $DF_0'(i(x))i = iDF_0(x)$.

REMARK 4.5. As in Remark 4.3, for Theorem 4.4 to hold it is enough that H interpolates F_0' just on S', $\{H, H_i\}_{|S'} = 0$ and that S' is H_i-invariant.

5. Examples

EXAMPLE 1. As a first example we consider a very simple two-dimensional map which we shall generalize later. Let $(x_1,y_1) = F(x,y)$ with

$$x_1 = (\beta x + \alpha)/(\beta + \alpha x), \qquad y_1 = y(\beta + \alpha x)^2,$$

where $\alpha = \sinh \tau$, $\beta = \cosh \tau$ and $\tau > 0$. It is easily checked that it has two fixed points, $(1,0)$ and $(-1,0)$, which are hyperbolic, and the line $\{y = 0\}$ is a heteroclinic connection.

This map is the time τ map of the system given by the Hamiltonian $H(x,y) = y(1-x^2)$. Consequently, the associated Hamiltonian system has $(-1,0)$ and $(1,0)$ as hyperbolic saddle points and the unperturbed heteroclinic orbit is given by

$$x(t) = \tanh(t + t_0), \qquad y(t) = 0.$$

Then, if $x_0 = \tanh t_0$, $y_0 = 0$, the iterates $(x_n, y_n) = f^n(x_0,y_0)$ are given by

$$x_n = x(\tau n) = \tanh(\tau n + t_0), \qquad y_n = 0. \tag{5.1}$$

Now we consider the perturbed map F_ε defined by the relations

$$x_1 = (\beta x + \alpha)/(\beta + \alpha x) + \varepsilon h_1(x,y),$$
$$y_1 = y(\beta + \alpha x)^2 + \varepsilon h_2(x,y).$$

By Theorem 4.1 the Melnikov function in the basis given by $\operatorname{grad} H(x,y) = (-2xy, 1-x^2)^T$ is

$$M(x) = \sum_{n=-\infty}^{\infty} h_2(x_{n-1},0)(1-x_n^2)$$

with $x = x_0$. By (5.1) we have

$$M(x) = \sum_{n=-\infty}^{\infty} \frac{h_2(\tanh((n-1)\tau + t_0),0)}{\cosh^2(n\tau + t_0)},$$

with $x = \tanh t_0$.

If we take the particular perturbation $h_2(x,y) = x$ the Melnikov function becomes

$$M(x) = \sum_{n=-\infty}^{\infty} \frac{\tanh(n\tau + t_0 - \tau)}{\cosh^2(n\tau + t_0)}$$

and using formula (6.1) of the Appendix,

$$M(x) = \frac{2}{\sinh^2\tau} - \coth\tau\left((1-\lambda E)\frac{2}{\tau} + \lambda^2 \mathrm{dn}^2(\lambda t_0)\right)$$

where $\lambda = 2K(m)/\tau$ and $K'(m)/K(m) = \pi/\tau$.

Since $\mathrm{dn}^2(\lambda t_0)$ is τ-periodic, and takes its maximum at $t_0 = 0$ which is 1 and its minimum at $t_0 = \tau/2$ which is $1-m$, we have

$$\begin{aligned} M(x) &\le \frac{2}{\sinh^2\tau} - \coth\tau\left((1-\lambda E)\frac{2}{\tau} + \lambda^2(1-m)\right) \\ &= \sum_{n=-\infty}^{\infty} \frac{\tanh(n\tau - \tau/2)}{\cosh^2(n\tau + \tau/2)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{\tanh(n\tau - \tau/2)}{\cosh^2(n\tau + \tau/2)} &= \sum_{n=0}^{\infty} \frac{\tanh(n\tau - \tau/2)}{\cosh^2(n\tau + \tau/2)} + \sum_{n=1}^{\infty} \frac{\tanh(-n\tau - \tau/2)}{\cosh^2(-n\tau + \tau/2)} \\ &= \sum_{n=0}^{\infty} \frac{\tanh(n\tau - \tau/2)}{\cosh^2(n\tau + \tau/2)} - \sum_{n=0}^{\infty} \frac{\tanh(n\tau + 3\tau/2)}{\cosh^2(n\tau + \tau/2)} \\ &= \sum_{n=0}^{\infty} \frac{\tanh(n\tau - \tau/2) - \tanh(n\tau + 3\tau/2)}{\cosh^2(n\tau + \tau/2)} < 0, \end{aligned}$$

because all terms in the last sum are negative. Hence if ε is small the perturbed map does not have heteroclinic points. Also we can have an asymptotic expression of $M(x)$ for τ small.

From the relation $K'(m)/K(m) = \pi/\tau$, τ can be expressed in terms of m through $q = \exp(-\pi K'(m)/K(m))$ (see [1]) as $\tau = -\pi^2/\ln q$. If τ is small,

m is small and since $q = m/16 + O(m^2)$ we have

$$m \sim 16e^{-\pi^2/\tau}.$$

Since $K(m) = \frac{\pi}{2}(1 + m/4 + O(m^2))$ and $E(m) = \frac{\pi}{2}(1 - m/4 + O(m^2))$, for τ small we have

$$M(x) = \frac{2}{\tau^2 \sinh \tau}(-2\tau^3/3 + O(\tau^4)) = -4/3 + O(\tau)$$

uniformly with respect to t_0.

If we take the particular perturbation $h_2(x, y) = x_1 = (\beta x + \alpha)/(\beta + \alpha x)$ the Melnikov function is

$$M(x) = \sum_{n=-\infty}^{\infty} \frac{\tanh(n\tau + t_0)}{\cosh^2(n\tau + t_0)},$$

and using the calculations given in the Appendix gives

$$M(x) = m\lambda^3 \operatorname{sn}(\lambda t_0)\operatorname{cn}(\lambda t_0)\operatorname{dn}(\lambda t_0),$$

where as before $\lambda = 2K(m)/\tau$, and m is such that $K'(m)/K(m) = \pi/\tau$. We have $M(x)_{|t_0=0} = 0$ and $\frac{d(M \circ x)}{dt}\Big|_{t_0=0} = m\lambda^4 \neq 0$, where we have to take into account that $x = \tanh t_0$. Then if ε is small enough the perturbed map has a transversal heteroclinic point near the point $(0, 0)$.

Example 2. Here we consider the product of two maps of the previous example. Let

$$(x_1, u_1, y_1, v_1) = F(x, u, y, v)$$

be defined by

$$\begin{aligned} x_1 &= (\beta x + \alpha)/(\beta + \alpha x), \quad & y_1 &= y(\beta + \alpha x)^2, \\ u_1 &= (\beta u + \alpha)/(\beta + \alpha u), \quad & v_1 &= v(\beta + \alpha u)^2, \end{aligned} \tag{5.2}$$

where $\alpha = \sinh \tau$ and $\beta = \cosh \tau$ with $\tau > 0$.

The map F is the time τ map of the Hamiltonian $H(x, u, y, v) = y(1 - x^2) + v(1 - u^2)$. Both the Hamiltonian system and the map (5.2) have four fixed points $(\pm 1, \pm 1, 0, 0)$ which are hyperbolic. The solutions of the Hamiltonian equations for $(x, u) \in (-1, 1) \times (-1, 1)$ are

$$\begin{aligned} x(t) &= \tanh(t + t_1), \quad & y(t) &= k_1 \cosh^2(t + t_1), \\ u(t) &= \tanh(t + t_2), \quad & v(t) &= k_2 \cosh^2(t + t_2), \end{aligned}$$

with $k_1, k_2, t_1, t_2 \in \mathbb{R}$. The set $\{y = 0, v = 0, |x| < 1, |u| < 1\}$ is a two-dimensional heteroclinic manifold for the points $(-1, -1, 0, 0)$ and $(1, 1, 0, 0)$. If $x = x_0 = \tanh t_1$, $u = u_0 = \tanh t_2$, $y_0 = 0$ and $v_0 = 0$ then the iterates $(x_n, u_n, y_n, v_n) = F^n(x_0, u_0, y_0, v_0)$ are

$$\begin{aligned} x_n &= x(\tau n) = \tanh(\tau n + t_1), \quad & y_n &= 0, \\ u_n &= u(\tau n) = \tanh(\tau n + t_2), \quad & v_n &= 0. \end{aligned} \tag{5.3}$$

We first consider a general perturbation F_ε of F given by

$$\begin{aligned} x_1 &= (\beta x+\alpha)/(\beta+\alpha x)+\varepsilon h_1(x,u,y,v),\\ u_1 &= (\beta u+\alpha)/(\beta+\alpha u)+\varepsilon h_2(x,u,y,v),\\ y_1 &= y(\beta+\alpha x)^2+\varepsilon h_3(x,u,y,v),\\ v_1 &= v(\beta+\alpha u)^2+\varepsilon h_4(x,u,y,v). \end{aligned}$$

The functions $H_1(x,u,y,v)=y(1-x^2)$ and $H_2(x,u,y,v)=v(1-u^2)$ are linearly independent first integrals in involution so that by Theorem 4.1 the Melnikov vector in the basis given by $\operatorname{grad} H_1(x,u,y,v)=(-2xy,0,1-x^2,0)^T$, $\operatorname{grad} H_2(x,u,y,v)=(0,-2uv,0,1-u^2)^T$ is $M(z)=(M_1(z),M_2(z))$ with

$$\begin{aligned} M_1(z) &= \sum_{n=-\infty}^{\infty} h_3(x_{n-1},u_{n-1},0,0)(1-x_n^2),\\ M_2(z) &= \sum_{n=-\infty}^{\infty} h_4(x_{n-1},u_{n-1},0,0)(1-u_n^2), \end{aligned}$$

where $z=(x_0,u_0,y_0,v_0)=(\tanh t_1,\tanh t_2,0,0)$, and substituting (5.3) gives

$$\begin{aligned} M_1(z) &= \sum_{n=-\infty}^{\infty} \frac{h_3(\tanh((n-1)\tau+t_1),\tanh((n-1)\tau+t_2),0,0)}{\cosh^2(n\tau+t_1)},\\ M_2(z) &= \sum_{n=-\infty}^{\infty} \frac{h_4(\tanh((n-1)\tau+t_1),\tanh((n-1)\tau+t_2),0,0)}{\cosh^2(n\tau+t_2)}. \end{aligned}$$

In the particular case where $h_3(x,u,y,v)=u$ and $h_4(x,u,y,v)=x$ the Melnikov vector becomes

$$M_1(z)=\sum_{n=-\infty}^{\infty}\frac{\tanh(n\tau+t_2-\tau)}{\cosh^2(n\tau+t_1)},\qquad M_2(z)=\sum_{n=-\infty}^{\infty}\frac{\tanh(n\tau+t_1-\tau)}{\cosh^2(n\tau+t_2)}.$$

We claim that $M_1(z)$ and $M_2(z)$ cannot vanish simultaneously. Denote them by $M_1(t_1,t_2)$ and $M_2(t_1,t_2)$. Notice that $M_1(t_1,t_2)=M_2(t_2,t_1)$. If we fix t_1, then $\varphi(t_2)=M_1(t_1,t_2)$ has only one zero $t_2=\widetilde{t}_2(t_1)$, and $\widetilde{t}_2$ is continuous. Indeed, from (6.1) we have, writing $\lambda=2K/\tau$,

$$\begin{aligned} \varphi(t_2) = \frac{-1}{\sinh^2(t_2-t_1-\tau)} &\\ \times\Bigg((1-\lambda E)\frac{2}{\tau}(t_2-t_1-\tau)&+\lambda(E(\lambda(t_2-\tau))-E(\lambda t_1))\Bigg)\\ +\coth(t_2-t_1-\tau)\Bigg((1-\lambda E)\frac{2}{\tau}&+\lambda^2\mathrm{dn}^2(\lambda t_1)\Bigg). \end{aligned}$$

The coefficient of $\coth(t_2 - t_1 - \tau)$ can be bounded from below:

$$\begin{aligned}
&\left(1 - \frac{2K}{\tau}E\right)\frac{2}{\tau} + \left(\frac{2K}{\tau}\right)^2 \mathrm{dn}^2\left(\frac{2K}{\tau}t_1\right) \\
&\quad = \frac{2}{\tau}\left(1 - \frac{2K}{\tau}E + \frac{2K^2}{\tau}\mathrm{dn}^2\left(\frac{2K}{\tau}t_1\right)\right) \geq \frac{2}{\tau}\left(1 + \frac{2K'}{\pi}((1-m)K - E)\right) \\
&\quad \geq \frac{2}{\tau}\left(1 + \frac{2(1-m)}{\pi}(K'K - K'E)\right) = \frac{2}{\tau}\left(1 + \frac{2(1-m)}{\pi}(E'K - \pi/2)\right) \\
&\quad = \frac{2}{\tau}\left(\frac{2(1-m)}{\pi}E'K + m\right) > 0.
\end{aligned}$$

Then $\lim_{t_2 \to \infty} \varphi(t_2) = (1 - \lambda E)\frac{2}{\tau} + \lambda^2 \mathrm{dn}^2(\lambda t_1) > 0$ and $\lim_{t_2 \to -\infty} \varphi(t_2) = -(1 - \lambda E)\frac{2}{\tau} - \lambda^2 \mathrm{dn}^2(\lambda t_1) < 0$. On the other hand,

$$\varphi'(t_2) = \sum_{n=-\infty}^{\infty} \frac{1}{\cosh^2(n\tau + t_1)\cosh^2(n\tau + t_2 - \tau)} > 0.$$

Furthermore, since M_1 is of class C^1, and $(\partial M_1/\partial t_2)(t_1, t_2) = \varphi'(t_2) > 0$, by the implicit function theorem $\widetilde{t}_2$ is of class C^1.

Then if $M_1(t_1^0, t_2^0) = 0$ and $M_2(t_1^0, t_2^0) = 0$ we shall also have $M_1(t_2^0, t_1^0) = 0$. Then either $t_1^0 = \widetilde{t}_2(t_1^0)$, in which case $M(t_1^0, t_1^0) = 0$, or $t_1^0 \neq \widetilde{t}_2(t_1^0)$. In the latter case we can suppose that $t_1^0 > \widetilde{t}_2(t_1^0)$ (the other case being analogous). Also $t_1^0 = \widetilde{t}_2(t_2^0) > t_2^0$ and hence by Bolzano's theorem applied to $\widetilde{t}_2(t) - t$ there exists t^* such that $t^* = \widetilde{t}_2(t^*)$ and therefore $M(t^*, t^*) = 0$.

But, as we have seen in the computations for Example 1, $M(t_1, t_1)$ never vanishes. This shows that the Melnikov vector cannot be zero. Hence F_ε, if ε is small, does not have heteroclinic intersections.

Now we consider another perturbation

$$\begin{aligned}
h_3(x, u, y, v) &= u_1 = (\beta u + \alpha)/(\beta + \alpha u), \\
h_4(x, u, y, v) &= x_1 = (\beta x + \alpha)/(\beta + \alpha x).
\end{aligned}$$

For it we have

$$\begin{aligned}
M_1(x_0, u_0) &= \sum_{n=-\infty}^{\infty} \frac{\tanh(n\tau + t_2)}{\cosh^2(n\tau + t_1)}, \\
M_2(x_0, u_0) &= \sum_{n=-\infty}^{\infty} \frac{\tanh(n\tau + t_1)}{\cosh^2(n\tau + t_2)}.
\end{aligned}$$

A closed form for M_1 and M_2 can be obtained from the results in the Appendix, but here it is easier to work directly with the series. If $x_0 = u_0 = 0$, which corresponds to $t_1 = t_2 = 0$, it is easily seen that $M(0, 0) = 0$.

Since $\frac{dx_0}{dt_1}(0,0) = \frac{du_0}{dt_2}(0,0) = 1$ and $\frac{dx_0}{dt_2}(0,0) = \frac{du_0}{dt_1}(0,0) = 0$ we have

$$\det DM(0,0) = \begin{vmatrix} -2\sum_{n=-\infty}^{\infty} \frac{\tanh^2(n\tau)}{\cosh^2(n\tau)} & \sum_{n=-\infty}^{\infty} \frac{1}{\cosh^4(n\tau)} \\ \sum_{n=-\infty}^{\infty} \frac{1}{\cosh^4(n\tau)} & -2\sum_{n=-\infty}^{\infty} \frac{\tanh^2(n\tau)}{\cosh^2(n\tau)} \end{vmatrix}$$
$$= \begin{vmatrix} -2(A-B) & B \\ B & -2(A-B) \end{vmatrix} = (2A-3B)(2A-B),$$

where $A = \sum_{n=-\infty}^{\infty} 1/\cosh^2(n\tau)$ and $B = \sum_{n=-\infty}^{\infty} 1/\cosh^4(n\tau)$. Clearly $2A - B > 0$. In the Appendix it is shown that $2A - 3B < 0$. Then, if ε is small enough, the perturbed invariant manifolds intersect transversally near $(0,0,0,0)$.

EXAMPLE 3. Now we consider the map $(x_1, y_1, \phi_1) = F(x, y, \phi)$ defined by

$$x_1 = (\beta x + \alpha)/(\beta + \alpha x), \qquad y_1 = y(\beta + \alpha x)^2,$$
$$\phi_1 = \phi + \nu, \tag{5.4}$$

where $\alpha = \sinh\tau$, $\beta = \cosh\tau$, $\tau > 0$, $x, y \in \mathbb{R}$ and $\nu, \phi \in \mathbb{T}^k$.

This map has two normally hyperbolic invariant manifolds

$$P_\pm = \{(\pm 1, 0, \phi) : \phi \in \mathbb{T}^k\},$$

joined by a heteroclinic manifold

$$S = \{(x, 0, \phi) : -1 < x < 1,\ \phi \in \mathbb{T}^k\}.$$

First we take $k = 1$. Let $x = x_0 = \tanh t_0$, $y_0 = 0$, $\phi = \phi_0$ and $(x_n, y_n, \phi_n) = F^n(x_0, y_0, \phi_0)$. Then

$$x_n = \tanh(\tau n + t_0), \qquad \phi_n = n\nu + \phi_0. \tag{5.5}$$

We consider the perturbed map F_ε defined by

$$x_1 = (\beta x + \alpha)/(\beta + \alpha x), \qquad y_1 = y(\beta + \alpha x)^2 + \varepsilon \sin\phi,$$
$$\phi_1 = \phi + \nu.$$

It is the projection onto the variables (x, y, ϕ) of the time τ map of the Hamiltonian $H(x, \phi, y, I) = y(1 - x^2) + \frac{\nu}{\tau} I$, so that we can apply Theorem 4.4. The vector $(0, 1 - x^2, 0)$ generates a basis of the orthogonal space to $T_x S$. In this case the Melnikov function in the basis given by this vector is

$$M(x, \phi) = \sum_{n=-\infty}^{\infty} \sin\phi_{n-1}(1 - x_n^2)$$

so that substituting (5.5) we get

$$M(x, \phi) = \sum_{n=-\infty}^{\infty} \frac{\sin(n\nu + \phi_0 - \nu)}{\cosh^2(n\tau + t_0)}.$$

We have $M(0,\nu)=0$. Using the fact that $dx_0/dt_0(0)=1$ we obtain

$$DM(0,\nu)=\Big(-2\sum_{n=-\infty}^{\infty}\frac{\sin(n\nu)\sinh(n\tau)}{\cosh^3(n\tau)},\sum_{n=-\infty}^{\infty}\frac{\cos(n\nu)}{\cosh^2(n\tau)}\Big),$$

whose rank is 1 if τ is large enough. Indeed,

$$\begin{aligned}\varphi(\nu,\tau)&=\sum_{n=-\infty}^{\infty}\frac{\cos(n\nu)}{\cosh^2(n\tau)}=1+2\sum_{n=1}^{\infty}\frac{\cos(n\nu)}{\cosh^2(n\tau)}\\&>1-2\sum_{n=1}^{\infty}\frac{1}{\cosh^2(n\tau)}>1-8\sum_{n=1}^{\infty}e^{-2n\tau}\\&=1-8e^{-2\tau}/(1-e^{-2\tau}).\end{aligned}$$

So, if $\tau\geq(\ln 9)/2$ then $\varphi(\nu,\tau)>0$.

The function φ is 2π-periodic with respect to ν. From numerical computations we believe that for fixed $\tau\neq 0$ it has a global minimum at $\nu=\pi$, where indeed $(\partial\varphi/\partial\nu)(\pi,\tau)=0$. We can compute $\varphi(\pi,\nu)$ explicitly and check that it is positive, which would guarantee the transversality in all cases, if ε is small enough:

$$\varphi(\pi,\tau)=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\cosh^2(n\tau)}=\sum_{n=-\infty}^{\infty}\frac{1}{\cosh^2(n2\tau)}-\sum_{n=-\infty}^{\infty}\frac{1}{\cosh^2(n2\tau+\tau)}.$$

Using formulas (6.8) and (6.9) of the Appendix adapted to this case and choosing m such as $K'(m)/K(m)=\pi/(2\tau)$ we get

$$\begin{aligned}\varphi(\pi,\tau)&=\Big(\frac{K}{\tau}\Big)^2\Big(\mathrm{dn}^2(0)-\mathrm{dn}^2\Big(\frac{K}{\tau}\tau\Big)\Big)\\&=\Big(\frac{K}{\tau}\Big)^2(1-(1-m))=\Big(\frac{K}{\tau}\Big)^2 m>0.\end{aligned}$$

More generally, we consider the map defined by

$$\begin{aligned}x_1&=(\beta x+\alpha)/(\beta+\alpha x),\\y_1&=y(\beta+\alpha x)^2+\varepsilon(a_1\sin\phi^1+\ldots+a_k\sin\phi^k),\\\phi_1&=\phi+\nu,\end{aligned}$$

with $\nu,\phi=(\phi^1,\ldots,\phi^k)\in\mathbb{T}^k$, $k\geq 1$. Now

$$M(x_0,\phi_0)=\sum_{n=-\infty}^{\infty}\frac{a_1\sin(n\nu_1+\phi_0^1-\nu_1)+\ldots+a_k\sin(n\nu_k+\phi_0^k-\nu_k)}{\cosh^2(n\tau+t_0)},$$

with $\phi_0=(\phi_0^1,\ldots,\phi_0^k)$ and $t_0=\operatorname{arctanh}x_0$.

As before, if $t_0=0$ and $\phi_0^j=\nu_j$, then $M(x_0,\phi_0)=0$. If at least one of the derivatives of M is different from zero then, if ε is small, we have transversal intersection of the invariant manifolds associated with the tori.

EXAMPLE 4. Finally, we consider the map $(x_1, \phi_1, y_1) = F(x, \phi, y)$, $(x, y) \in \mathbb{R}^2$, $\phi \in \mathbb{T}^k$, $|\mu| > 1$, defined by

$$x_1 = y, \qquad \phi_1 = \phi + \nu + \varepsilon h(x, \phi, y),$$
$$y_1 = -x + 2y \frac{\mu}{1 + y^2} + \varepsilon g(x, \phi, y).$$

For $k = 0$ it is called the *McMillan map* and has been studied in [10] and [5]. For $\varepsilon = 0$ we consider the normally hyperbolic invariant manifold $\{x = y = 0\}$. Its stable and unstable manifolds form two homoclinic manifolds

$$\Gamma^{\pm} = \{(x^{\pm}(t - \tau), \phi, x^{\pm}(t)) : \phi \in \mathbb{T}^k,\ t \in \mathbb{R}\}$$

where

$$x^{\pm}(t) = \pm \frac{\sqrt{\mu^2 - 1}}{\cosh t} = \pm \frac{\sinh \tau}{\cosh t}.$$

and $\tau = \ln(\mu + \sqrt{\mu^2 - 1})$ or equivalently $\sqrt{\mu^2 - 1} = \sinh \tau$. We consider $S = \Gamma^+$. F_0 restricted to S coincides with the projection onto the variables x, y, ϕ of the time τ map corresponding to the Hamiltonian

$$H(x, \phi, y, I) = \frac{1}{2\sqrt{\mu^2 - 1}}(x^2 - 2\mu xy + y^2 + x^2 y^2) + \frac{\nu}{\tau} I.$$

According to Remark 4.5 we can write the Melnikov function associated with the basis $\Pi \operatorname{grad} H$, with $\Pi(x, \phi, y, I) = (x, \phi, y)$. The flow associated with H on the homoclinic manifold is

$$w(t) = \left(x^+(t - \tau + t_0), \phi_0 + \frac{\nu}{\tau} t, x^+(t + t_0), I_0\right).$$

If $z(t) = \Pi w(t)$ then $\Pi \operatorname{grad} H(z(t)) = (-\dot{x}^+(t + t_0), 0, \dot{x}^+(t - \tau + t_0))$. We define $z_n = z(n\tau)$. Then

$$\begin{aligned} M(z_0) &= \sum_{n=-\infty}^{\infty} \langle (0, h(z_{n-1}), g(z_{n-1})), \Pi \operatorname{grad} H(z_n) \rangle \\ &= -\sum_{n=-\infty}^{\infty} g(z_{n-1}) \dot{x}^+(n\tau - \tau + t_0) = -\sum_{n=-\infty}^{\infty} g(z_n) \dot{x}^+(n\tau + t_0). \end{aligned}$$

In the case $k = 1$ and $g(x, \phi, y) = \cos \phi$ we have

$$M(z_0) = \sinh \tau \sum_{n=-\infty}^{\infty} \cos(n\nu + \phi_0) \frac{\sinh(n\tau + t_0)}{\cosh^2(n\tau + t_0)}.$$

When $t_0 = 0$ and $\phi_0 = 0$ we have $M(z_0) = 0$, that is to say, the stable and unstable manifolds intersect. To study the transversality we have to look at

$$\begin{aligned}&\operatorname{rk}\left(\frac{d}{dt_0}(M(z_0)),\frac{d}{d\phi_0}(M(z_0))\right)\\&\quad=\operatorname{rk}\left(\sum_{n=-\infty}^{\infty}\cos(n\nu)\frac{2-\cosh^2(n\tau)}{\cosh^3(n\tau)},-\sum_{n=-\infty}^{\infty}\sin(n\nu)\frac{\sinh(n\tau)}{\cosh^2(n\tau)}\right)\end{aligned}$$

($\sinh\tau\neq 0$) at $t_0=0$ and $\phi_0=0$. Proceeding as in the previous example we find that if $\tau\geq\ln 5$ the first component of the vector is different from zero for all ν. This implies the transversal intersection of the invariant manifolds.

6. Appendix. We devote this Appendix to some technical computations which provide closed formulas for some series which we have obtained as Melnikov functions or their derivatives. For that we shall use a method developed in [5].

LEMMA 6.1. *The sum*

$$\sum_{n=-\infty}^{\infty}\frac{\tanh(n\tau+t_1)}{\cosh^2(n\tau+t_2)}$$

takes the value

$$\begin{aligned}\text{(6.1)}\quad &\frac{-1}{\sinh^2(t_1-t_2)}\left(\left(1-\frac{2K}{\tau}E\right)\frac{2}{\tau}(t_1-t_2)+\frac{2K}{\tau}\left(E\left(\frac{2K}{\tau}t_1\right)-E\left(\frac{2K}{\tau}t_2\right)\right)\right)\\&\qquad+\coth(t_1-t_2)\left(\left(1-\frac{2K}{\tau}E\right)\frac{2}{\tau}+\left(\frac{2K}{\tau}\right)^2\operatorname{dn}^2\left(\frac{2K}{\tau}t_2\right)\right)\end{aligned}$$

if $t_1\neq t_2$, *and*

$$m\left(\frac{2K}{\tau}\right)^3\operatorname{sn}\left(\frac{2K}{\tau}t\right)\operatorname{cn}\left(\frac{2K}{\tau}t\right)\operatorname{dn}\left(\frac{2K}{\tau}t\right)\tag{6.2}$$

if $t_1=t_2$, *with* m *satisfying* $K'(m)/K(m)=\pi/\tau$.

Proof. First we recall some definitions concerning elliptic functions. See [1]. Let $m\in(0,1)$. The *complete elliptic integrals of first and second kind* are defined by

$$K(m)=\int_0^1((1-y^2)(1-my^2))^{-1/2}\,dy$$

and

$$E(m)=\int_0^1\left(\frac{1-my^2}{1-y^2}\right)^{1/2}dy.$$

Also one introduces the following quantities: $m_1=1-m$, $K=K(m)$, $K'=K(m_1)$, $E=E(m)$ and $E'=E(m_1)$. The *incomplete elliptic integral* *of second kind* is defined by

$$E(u|m) = \int_0^u \mathrm{dn}^2(v|m)\,dv,$$

where dn is the Jacobian elliptic function.

Now we collect some properties of the above functions which will be used in the computations. The function E satisfies $E(-u) = -E(u)$, $E(z+2K) = E(z) + 2E$ and $E(z + 2iK') = E(z) + 2i(K' - E')$. The Legendre equality is

$$EK' + E'K - KK' = \pi/2. \tag{6.3}$$

The functions sn, cn and dn have two periods. The periods of $\mathrm{dn}(v)$ are $2K$, $4K'i$. In a fundamental domain $\mathrm{dn}(v)$ has two poles at $K'i$ and $3K'i$ with residues $-i$ and i respectively, and they are of order 1.

We shall also use the following properties of the elliptic functions: $\mathrm{sn}(-u) = -\mathrm{sn}(u)$, $\mathrm{cn}(-u) = \mathrm{cn}(u)$, $\mathrm{dn}(-u) = \mathrm{dn}(u)$, $\mathrm{sn}(u + 2K'i) = \mathrm{sn}(u)$, $\mathrm{cn}(u + 2K'i) = -\mathrm{cn}(u)$, $\mathrm{dn}(u + 2K'i) = -\mathrm{dn}(u)$, $\mathrm{dn}' = -m\,\mathrm{sn}\,\mathrm{cn}$ (see [1]).

Following [5] we introduce

$$\chi(z) = 2(\tau - 2KE)z + 2KE(2Kz + K'i|m).$$

where the parameter m satisfies

$$\frac{K'(m)}{K(m)} = \frac{\pi}{\tau}. \tag{6.4}$$

The function χ has the following properties:

- χ is $\frac{\pi}{\tau}i$-periodic,
- χ' is 1-periodic,
- the singularities of χ on $\{|\mathrm{Im}\, z| < \pi/\tau\}$ are poles located at $z = n$, $n \in \mathbb{Z}$, they are simple and their residues are 1.

Let

$$g(z) = \frac{\tanh(z\tau + t_1)}{\cosh^2(z\tau + t_2)}.$$

Clearly g is $\frac{\pi}{\tau}i$-periodic.

Now we consider the rectangle R_n with vertices $\pm(n+1/2) + (\pm\pi/(2\tau) + \varepsilon)i$, $0 < \varepsilon < \pi/(2\tau)$. If $0 < \varepsilon < \pi/(2\tau)$ then for any $t_1, t_2 \in \mathbb{R}$ there exists n_0 such that if $n \geq n_0$ then χg does not have singularities on the border of R_n. Let $R = \lim_{n\to\infty} R_n$ and $P = \{\text{poles of } \chi g \text{ on } R\}$. By the residue theorem we have

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial R_n} \chi(z)\, g(z)\, dz = \sum_{z\in P} \mathrm{res}(\chi g, z). \tag{6.5}$$

On the other hand,

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial R_n} \chi(z)\, g(z)\, dz = 0$$

because χg is $\frac{\pi}{\tau}i$-periodic, and the integrals on the vertical paths go to zero as $n \to \infty$ since g decreases exponentially and χ increases at most linearly because χ' is 1-periodic.

We have the following table:

Function	Poles in the interior of R	Laurent series
$\chi(z)$	$\mathbb{Z}$	$\frac{1}{z-n} + \dots,\ n \in \mathbb{Z}$
$\tanh(z\tau + t_1)$	$z_1 = \frac{\pi}{2\tau}i - \frac{t_1}{\tau}$	$\frac{1}{\tau}\frac{1}{z-z_1} + \frac{1}{3}\tau(z-z_1) + \dots$
$\cosh^{-2}(z\tau + t_2)$	$z_2 = \frac{\pi}{2\tau}i - \frac{t_2}{\tau}$	$-\frac{1}{\tau^2}\frac{1}{(z-z_2)^2} + \frac{1}{3} + \dots$
$\tanh(z\tau + t)\cosh^{-2}(z\tau + t)$	$z_0 = \frac{\pi}{2\tau}i - \frac{t}{\tau}$	$-\frac{1}{\tau^3}\frac{1}{(z-z_0)^3} + \frac{1}{15}\tau(z-z_0) + \dots$

From (6.5) we deduce that if $t_1 \neq t_2$ then

$$\sum_{n\in\mathbb{Z}} \operatorname{res}(\chi g, n) + \operatorname{res}(\chi g, z_1) + \operatorname{res}(\chi g, z_2) = 0,$$

which gives

$$\text{(6.6)} \qquad \sum_{n=-\infty}^{\infty} g(n) + \frac{1}{\tau}\chi(z_1)/\cosh^2(z_1\tau + t_2) \\ - \frac{1}{\tau^2}\chi'(z_2)\tanh(z_2\tau + t_1) - \frac{1}{\tau}\chi(z_2)/\cosh^2(z_2\tau + t_1) = 0,$$

and if $t_1 = t_2 = t$ then $\sum_{n\in\mathbb{Z}} \operatorname{res}(\chi g, n) + \operatorname{res}(\chi g, z_0) = 0$, which gives

$$\text{(6.7)} \qquad \sum_{n=-\infty}^{\infty} g(n) - \frac{1}{\tau^3} \cdot \frac{\chi''(z_2)}{2!} = 0.$$

We need the following computations:

$$\chi(z_1) = (2EK' - \pi)i - 2\left(1 - \frac{2K}{\tau}E\right)t_1 - 2KE\left(\frac{2K}{\tau}t_1\right),$$

$$\chi'(z_2) = 2(\tau - 2KE) + (2K)^2 \mathrm{dn}^2\left(\frac{2K}{\tau}t_2\right),$$

$$\text{(6.8)} \qquad \chi''(z_0) = 2m(2K)^3 \mathrm{sn}\left(\frac{2K}{\tau}t\right)\mathrm{cn}\left(\frac{2K}{\tau}t\right)\mathrm{dn}\left(\frac{2K}{\tau}t\right),$$

$$\cosh^2(z_1\tau + t_2) = -\sinh^2(t_1 - t_2),$$

$$\tanh(z_2\tau + t_1) = \coth(t_1 - t_2),$$

$$\cosh^2(z_2\tau + t_1) = -\sinh^2(t_1 - t_2).$$

Finally, substituting the previous calculations into formulas (6.6) and (6.7) we get (6.1) and (6.2). ■

LEMMA 6.2. *If* $A = \sum_{n=-\infty}^{\infty} 1/\cosh^2(n\tau)$ *and* $B = \sum_{n=-\infty}^{\infty} 1/\cosh^4(n\tau)$ *we have*

$$2A - 3B = -m(2K/\tau)^4 < 0.$$

Proof. To compute A we consider the function $g(z) = 1/\cosh^2(z\tau + t)$. It has a pole at

$$z_0 = \frac{\pi}{2\tau} i - \frac{t}{\tau}$$

with Laurent series

$$g(z) = -\frac{1}{\tau^2} \cdot \frac{1}{(z - z_0)^2} + \frac{1}{3} + \ldots$$

As in the previous example we have $\sum_{n\in\mathbb{Z}} \operatorname{res}(\chi g, n) + \operatorname{res}(\chi g, z_0) = 0$ and therefore

$$A = \sum_{n=-\infty}^{\infty} g(n)|_{t=0} = \frac{1}{\tau^2} \chi'(z_0)|_{t=0}. \tag{6.9}$$

To compute B we consider $g(z) = 1/\cosh^4(z\tau + t)$. The function g has a pole at

$$z_0 = \frac{\pi}{2\tau} i - \frac{t}{\tau}$$

with Laurent series

$$g(z) = \frac{1}{\tau^4} \cdot \frac{1}{(z - z_0)^4} - \frac{2}{3\tau^2} \cdot \frac{1}{(z - z_0)^2} + O(1).$$

Similarly, we obtain

$$\sum_{n\in\mathbb{Z}} g(n) + \frac{1}{\tau^4} \cdot \frac{\chi'''(z_0)}{3!} - \frac{2}{3\tau^2} \chi'(z_0) = 0.$$

Using the fact that

$$\chi''(z_0) = 2m(2K)^3 \operatorname{sn}\left(\frac{2K}{\tau} t\right) \operatorname{cn}\left(\frac{2K}{\tau} t\right) \operatorname{dn}\left(\frac{2K}{\tau} t\right)$$

and that

$$\chi'''(z_0)|_{t=0} = \frac{d}{dt}(\chi''(z_0)) \frac{dt}{dz_0}\bigg|_{t=0} = 2m\frac{(2K)^4}{\tau}(-\tau) = -2m(2K)^4,$$

we obtain

$$B = \frac{m}{3}\left(\frac{2K}{\tau}\right)^4 + \frac{2}{3} A,$$

and hence $2A - 3B = -m(2K/\tau)^4 < 0$. ■

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