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CONVERGENCE ACCELERATION BY THE E_{+p}-ALGORITHM

Abstract. A new algorithm which generalizes the E-algorithm is presented. It is called the E_{+p}-algorithm. Some results on convergence acceleration for the E_{+p}-algorithm are proved. Some applications are given.

1. Introduction. Many convergent sequences (s_n) of complex numbers are of the form

$$s_n = s + a_1 g_1(n) + \ldots + a_i g_i(n) + r_n, \tag{1}$$

where $(g_i(n))$, $i = 1, \ldots, k$, are known sequences satisfying for each i, $g_{i+1}(n) = o(g_i(n))$ (i.e. $g_{i+1}(n)/g_i(n) \to 0$ as $n \to \infty$), the limit s of (s_n) and the coefficients a_i, $i = 1, \ldots, k$, are unknown and $r_n = o(g_k(n))$.

When the sequences $(g_i(n))$, $i = 1, \ldots, k$, satisfy $g_i(n+1)/g_i(n) \to b_i$ as $n \to \infty$, with some additional assumptions, the E-algorithm with $(g_i(n))$, $i = 1, \ldots, k$, as auxiliary sequences is effective for accelerating (s_n) (see [2, 3, 5]). However, in general, the E-algorithm cannot accelerate (s_n) when the sequences $(g_i(n+1)/g_i(n)), i = 1, \ldots, k$, are not convergent. This is, for example, the case of the sequence

$$s_n = g_1(n) + r_n,$$

where

$$g_1(2n) = \frac{1}{3^n}, \quad g_1(2n+1) = \frac{1}{3^n} + \frac{1}{5^n}, \quad r_{2n} = \frac{1}{4^n}, \quad r_{2n+1} = 0 \quad \text{for } n = 0, 1, \ldots$$

We have

$$\frac{g_1(2n+1)}{g_1(2n)} \underset{n}{\longrightarrow} 1, \quad \frac{g_1(2n+2)}{g_1(2n+1)} \underset{n}{\longrightarrow} \frac{1}{3}, \quad \frac{g_1(n+2)}{g_1(n)} \underset{n}{\longrightarrow} \frac{1}{3}.$$

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The convergence of (s_n) is linear periodic of period 2. One can easily check that (s_n) is not accelerated by the sequence transformation

$$E_1 : (s_n) \to \left(\frac{g_1(n+1)s_n - g_1(n)s_{n+1}}{g_1(n+1) - g_1(n)} \right)$$

which is the first step of the E-algorithm. However, the sequence transformation

$$E_{+2,1} : (s_n) \to \left(\frac{g_1(n+2)s_n - g_1(n)s_{n+2}}{g_1(n+2) - g_1(n)} \right)$$

does accelerate (s_n).

The sequence transformation $E_{+2,1}$ is a particular case of the sequence transformation

$$E_{+p,1} : (s_n) \to \left(\frac{g_1(n+p)s_n - g_1(n)s_{n+p}}{g_1(n+p) - g_1(n)} \right),$$

where p is a positive integer, $p \geq 1$ and $(g_1(n))$ is an auxiliary sequence. It includes the sequence transformation T_{+p} of Gray and Clark ($g_1(n) = \Delta s_n$) [7] and the process (Δ_p^2) of Delahaye ($g_1(n) = s_{n+p} - s_n$) [4].

In order to accelerate convergence of sequences (s_n) of complex numbers of the form (1), where the g_i are such that

$$\frac{g_i((n+1)p+j)}{g_i(np+j)} \xrightarrow[n]{} b_{j,i} \quad \text{for } j = 0, \ldots, p-1$$

(p is a fixed positive integer), we present in Section 2 a new algorithm called the E_{+p}-algorithm. Its first step is the preceding sequence transformation $E_{+p,1}$. It is a generalization of the E-algorithm.

In Section 3 we establish some results on convergence acceleration for the E_{+p}-algorithm. Section 4 is devoted to some applications of the E_{+p}-algorithm. Numerical examples are given for illustrating the theoretical results.

2. The E_{+p}-algorithm. Let us begin with the following notations:

- $\mathbb{N}$: the set of positive integers.
- $\mathbb{N}^* = \mathbb{N} - \{0\}$.
- $\mathbb{C}$: the set of complex numbers.
- $\operatorname{Re} z$: real part of the complex number z.
- $\operatorname{Conv}(\mathbb{C})$: the set of convergent sequences of complex numbers.
- If $(s_n) \in \operatorname{Conv}(\mathbb{C})$, then s denotes its limit.
- $u_n = o(v_n)$ means that $u_n/v_n \to 0$ as $n \to \infty$.

Let $p \in \mathbb{N}^*$. Let $(s_n) \in \operatorname{Conv}(\mathbb{C})$ be such that for all $n \in \mathbb{N}$,

$$s_n = s + a_1 g_1(n) + \ldots + a_i g_i(n) + \ldots, \tag{2}$$

where the g_i are some known sequences. We have

$$s_n = s + a_1 g_1(n) + \ldots + a_i g_i(n) + \ldots,$$
$$s_{n+p} = s + a_1 g_1(n+p) + \ldots + a_i g_i(n+p) + \ldots$$

Thus

$$\frac{g_1(n+p)s_n - g_1(n)s_{n+p}}{g_1(n+p) - g_1(n)} = s + \sum_{i=2}^{\infty} a_i \frac{g_1(n+p)g_i(n) - g_1(n)g_i(n+p)}{g_1(n+p) - g_1(n)}.$$

Set

$$E_{+p,1}^{(n)} = \frac{g_1(n+p)s_n - g_1(n)s_{n+p}}{g_1(n+p) - g_1(n)},$$
$$g_{1,i}^{(n)} = \frac{g_1(n+p)g_i(n) - g_1(n)g_i(n+p)}{g_1(n+p) - g_1(n)}, \qquad n \geq 0,\ i \geq 2.$$

Thus

$$E_{+p,1}^{(n)} = s + a_2 g_{1,2}^{(n)} + \ldots + a_i g_{1,i}^{(n)} + \ldots$$

The sequence $(E_{+p,1}^{(n)})$ is of the form (2). Consequently, the process can be repeated. Thus, we obtain the following *E_{+p}-algorithm*:

$$E_{+p,0}^{(n)} = s_n, \quad g_{0,i}^{(n)} = g_i(n), \qquad n \geq 0,\ i \geq 1,$$
$$E_{+p,k}^{(n)} = \frac{g_{k-1,k}^{(n+p)} E_{+p,k-1}^{(n)} - g_{k-1,k}^{(n)} E_{+p,k-1}^{(n+p)}}{g_{k-1,k}^{(n+p)} - g_{k-1,k}^{(n)}}, \qquad n \geq 0,\ k \geq 1,$$
$$g_{k,j}^{(n)} = \frac{g_{k-1,k}^{(n+p)} g_{k-1,j}^{(n)} - g_{k-1,k}^{(n)} g_{k-1,j}^{(n+p)}}{g_{k-1,k}^{(n+p)} - g_{k-1,k}^{(n)}}, \qquad n \geq 0,\ k \geq 1,\ j > k.$$

The sequences $(g_i(n))$, $i \geq 1$, are called the *auxiliary sequences* of the E_{+p}-algorithm.

REMARKS. 1. When $p = 1$, we obtain the E-algorithm.

2. For each $j > k$, the sequence $(g_{k,j}^{(n)})_n$ is obtained by applying the sequence transformation $E_{+p,k} : (s_n) \to (E_{+p,k}^{(n)})$ to the sequence $(g_j(n))$.

3. If the sequences $(g_1(n)), \ldots, (g_k(n))$ do not depend (respectively depend) on (s_n), then the sequence transformation $E_{+p,k}$ is linear (respectively nonlinear).

4. The E_{+p}-algorithm can be generalized by replacing p by an integer $p(n,k)$ (depending on n and k) in the rules of the E_{+p}-algorithm.

THEOREM 1. *Let $j \in \{0, \ldots, p-1\}$ and $k \geq 0$. Let $E_{j,k}^{(n)}$, $k \geq 0$, $n \geq 0$, be the quantities obtained by applying the E-algorithm (i.e. the E_{+1}-algorithm) to $(s_{np+j})_n$ with $(h_{j,i}(n)) = (g_i(np+j))$, $i = 1, 2, \ldots,$ as auxiliary sequences. Then $E_{j,k}^{(n)} = E_{+p,k}^{(np+j)}$ for all $n \geq 0$.*

P r o o f. By induction on k with the help of Remarks 1 and 2.

DEFINITION. Let $m \in \mathbb{N}^*$. Let (s_n) be a sequence of complex numbers. We say that (s_n) is *m-periodic* if $s_{n+m} = s_n$ for $n = 0, 1, \ldots$

DEFINITION. Let T be a sequence transformation. The set of sequences (s_n) such that the sequence $(T^{(n)})$ obtained by applying T to (s_n) is 1-periodic is called the *kernel* of T.

THEOREM 2 (see [2]). *The kernel of the sequence transformation $E_{+1,k}$ is the set of sequences (s_n) such that*

$$s_n = s + a_1 g_1(n) + \ldots + a_k g_k(n), \qquad n = 0, 1, \ldots$$

THEOREM 3. *The kernel of the sequence transformation $E_{+p,k}$ is the set of sequences (s_n) of the form*

$$s_n = s + a_1(n) g_1(n) + \ldots + a_k(n) g_k(n), \qquad n \geq 0,$$

where the sequences $(a_i(n))$, $i = 1, \ldots, k$, are p-periodic.

P r o o f. This follows immediately from Theorems 1 and 2.

REMARK. The kernel of $E_{+p,k}$ contains the kernel of $E_{+1,k}$.

THEOREM 4 (see [2]). *If for all n,*

$$s_n = s + a_1 g_1(n) + \ldots + a_i g_i(n) + \ldots,$$

then for all k and n,

$$E_{+1,k}^{(n)} = s + a_{k+1} g_{k,k+1}^{(n)} + \ldots + a_i g_{k,i}^{(n)} + \ldots$$

An immediate consequence of Theorems 1 and 4 is

THEOREM 5. *If for all n,*

$$s_n = s + a_1(n) g_1(n) + \ldots + a_i(n) g_i(n) + \ldots,$$

where the sequences $(a_i(n)), i \geq 1$, are p-periodic, then for all k and n,

$$E_{+p,k}^{(n)} = s + a_{k+1}(n) g_{k,k+1}^{(n)} + \ldots + a_i(n) g_{k,i}^{(n)} + \ldots$$

Let us now establish some results on convergence acceleration for the E_{+p}-algorithm.

3. Convergence acceleration. Let $(s_n) \in \mathrm{Conv}(\mathbb{C})$. Let $k \in \mathbb{N}^*$.

THEOREM 6. *Assume that*:

1. $E_{+p,k-1}^{(n)} \to s$ *as* $n \to \infty$.
2. *There are $\varepsilon > 0$ and n_0 such that for all $n \geq n_0$,*

$$|g_{k-1,k}^{(n+p)} / g_{k-1,k}^{(n)} - 1| \geq \varepsilon.$$

Then $E_{+p,k}^{(n)} \to s$ as $n \to \infty$.

P r o o f. We have

$$E^{(n)}_{+p,k} - s = (E^{(n)}_{+p,k-1} - s) + \frac{(E^{(n)}_{+p,k-1} - s) - (E^{(n+p)}_{+p,k-1} - s)}{g^{(n+p)}_{k-1,k}/g^{(n)}_{k-1,k} - 1}.$$

Thus

$$|E^{(n)}_{+p,k} - s| \le \left(1 + \frac{2}{|g^{(n+p)}_{k-1,k}/g^{(n)}_{k-1,k} - 1|}\right) \max(|E^{(n)}_{+p,k-1} - s|, |E^{(n+p)}_{+p,k-1} - s|),$$

and from assumptions 1 and 2 we get the assertion. ■

THEOREM 7. *Assume that*:

1. $E^{(n)}_{+p,k-1} \to s$ *as* $n \to \infty$.
2. *For each* $j \in \{0, \ldots, p-1\}$, $g^{((n+1)p+j)}_{k-1,k}/g^{(np+j)}_{k-1,k} \to l_j \ne 1$ *as* $n \to \infty$.

Then:

(i) $E^{(n)}_{+p,k} \to s$ *as* $n \to \infty$.

(ii) $E^{(n)}_{+p,k} - s = o(E^{(n)}_{+p,k-1} - s)$ *iff*

$$\forall j \in \{0, \ldots, p-1\}, \qquad \frac{E^{((n+1)p+j)}_{+p,k-1} - s}{E^{(np+j)}_{+p,k-1} - s} \xrightarrow[n]{} l_j.$$

P r o o f. (i) follows from Theorem 6.

(ii) We have

$$\frac{E^{(n)}_{+p,k} - s}{E^{(n)}_{+p,k-1} - s} = \frac{\dfrac{g^{(n+p)}_{k-1,k}}{g^{(n)}_{k-1,k}} - \dfrac{E^{(n+p)}_{+p,k-1} - s}{E^{(n)}_{+p,k-1} - s}}{\dfrac{g^{(n+p)}_{k-1,k}}{g^{(n)}_{k-1,k}} - 1}, \tag{3}$$

$$E^{(n)}_{+p,k} - s = o(E^{(n)}_{+p,k-1} - s) \quad \text{iff} \tag{4}$$
$$\forall j \in \{0, \ldots, p-1\}, \qquad E^{(np+j)}_{+p,k} - s = o(E^{(np+j)}_{+p,k-1} - s).$$

From (3)–(4) and assumption 2 we get the assertion. ■

REMARK. If $\prod_{j=0}^{p-1} l_j \ne 0$, then

$$E^{(n)}_{+p,k} - s = o(E^{(n+p)}_{+p,k-1} - s) \quad \text{iff} \quad E^{(n)}_{+p,k} - s = o(E^{(n)}_{+p,k-1} - s).$$

Let L_p be the set of sequences $(s_n) \in \mathrm{Conv}(\mathbb{C})$ such that for every $j \in \{0, \ldots, p-1\}$,

$$\frac{s_{(n+1)p+j} - s}{s_{np+j} - s} \xrightarrow[n]{} a_j \in [-1, 1[.$$

Then L_p contains the set P_p of sequences $(s_n) \in \mathrm{Conv}(\mathbb{C})$ such that for all $j \in \{0, \ldots, p-1\}$,

$$\frac{s_{np+j+1} - s}{s_{np+j} - s} \xrightarrow[n]{} a_j \neq 1 \quad \text{with } 0 < \Big|\prod_{j=0}^{p-1} a_j\Big| < 1$$

(i.e. the convergence of (s_n) is linear periodic of period p). The sequence transformation (Δ_p^2) accelerates P_p (i.e. (Δ_p^2) accelerates the convergence of each sequence $(s_n) \in P_p$; see [4]).

THEOREM 8. *The sequence transformation* (Δ_p^2) *accelerates* L_p. *The sequence transformation* T_{+p} *accelerates the set of sequences* $(s_n) \in L_p$ *such that for all* $j \in \{0, \ldots, p-1\}$,

$$\lim_{n\to\infty} \frac{s_{(n+1)p+j+1} - s_{(n+1)p+j}}{s_{np+j+1} - s_{np+j}} = \lim_{n\to\infty} \frac{s_{(n+1)p+j} - s}{s_{np+j} - s}.$$

In particular, T_{+p} *accelerates* P_p.

Proof. This follows from Theorem 7.

DEFINITION. We say that the auxiliary sequences $(g_i(n))$, $i \geq 1$, of the E_{+p}-algorithm satisfy the *condition* (b_{+p}) if for all $i \geq 1$ and $j \in \{0, \ldots$ $\ldots, p-1\}$,

$$\frac{g_i((n+1)p+j)}{g_i(np+j)} \xrightarrow[n]{} b_{j,i} \neq 1 \quad \text{with } b_{j,i} \neq b_{j,k} \text{ for } k \neq i.$$

REMARKS. 1. The condition (b_{+1}) is a condition due to Brezinski, under which some results on convergence acceleration for the E-algorithm are proved in [2].

2. If the g_i satisfy the condition (b_{+1}), then the condition (b_{+p}) is satisfied in the following cases:

(i) $|b_{0,i}| \neq |b_{0,j}|$ for all $i \neq j$;
(ii) the numbers b_i are real and $b_{0,i} b_{0,j} > 0$ for all $i \neq j$;
(iii) the numbers $b_{0,i}$ are real and p is odd.

We assume in the sequel that the condition (b_{+p}) is satisfied.

LEMMA. *Let* $j \in \{0, \ldots, p-1\}$. *For each* $k \geq 0$ *and* $i > k$,

$$\frac{g_{k,i}^{((n+1)p+j)}}{g_{k,i}^{(np+j)}} \xrightarrow[n]{} b_{j,i}.$$

Proof. By induction on k.

With the help of Theorem 7 and the Lemma, we can easily prove

THEOREM 9. *Let* $(s_n) \in \mathrm{Conv}(\mathbb{C})$, $k \geq 1$. *Then*:

1. $E_{+p,k}^{(n)} \to s$ *as* $n \to \infty$.

2. $E^{(n)}_{+p,k} - s = o(E^{(n)}_{+p,k-1} - s)$ *iff for all* $j \in \{0, \ldots, p-1\}$,

$$\frac{E^{((n+1)p+j)}_{+p,k-1} - s}{E^{(np+j)}_{+p,k-1} - s} \underset{n}{\longrightarrow} b_{j,k}.$$

Definition. Let $(f_i(n))$, $i = 1, 2, \ldots$, be some sequences of complex numbers. Then $(f_1, \ldots, f_i, \ldots)$ is called an *asymptotic sequence* if for each i, $f_{i+1}(n) = o(f_i(n))$.

Let $(f_1, \ldots, f_i, \ldots)$ be an asymptotic sequence. Let (t_n) be a sequence of complex numbers. The notation

$$t_n \approx a_1(n) f_1(n) + \ldots + a_k(n) f_k(n) + \ldots,$$

where for each i, $(a_i(n))$ is a p-periodic sequence, means that for all $k \geq 1$,

$$t_n = a_1(n) f_1(n) + \ldots + a_k(n) f_k(n) + o(f_k(n)) \quad \text{as } n \to \infty$$

(i.e. for each $j \in \{0, \ldots, p-1\}$, (t_{np+j}) has an asymptotic expansion with respect to $(h_1, \ldots, h_i, \ldots)$ where $(h_i(n)) = (f_i(np+j))$, $i = 1, 2, \ldots$).

By using the previous Lemma, we can easily prove

Theorem 10. *If* $(g_1, \ldots, g_i, \ldots)$ *is an asymptotic sequence, then so is* $(g_{k,k+1}, \ldots, g_{k,i}, \ldots)$ *for each* $k \geq 1$.

Theorem 11. *Let* $(s_n) \in \mathrm{Conv}(\mathbb{C})$. *Assume that*:

1. $(g_1, \ldots, g_i, \ldots)$ *is an asymptotic sequence.*
2. $s_n - s \approx a_1(n) g_1(n) + \ldots + a_i(n) g_i(n) + \ldots$ *where* $(a_i(n))$ *is* p*-periodic for each* i.

For each $k \geq 1$, *we have*:

(i) $E^{(n)}_{+p,k} - s \approx a_{k+1}(n) g^{(n)}_{k,k+1} + \ldots + a_i(n) g^{(n)}_{k,i} + \ldots$

(ii) *If* $a_i(n) = 0$ *for all* $i > k$ *and* n, *then* $E^{(n)}_{+p,k} = s$ *for all* n.

(iii) *Let* $j \in \{0, \ldots, p-1\}$. *If* $a_i(j) = 0$ *for all* $i > k$, *then* $E^{(np+j)}_{+p,k} = s$ *for all* n. *If the coefficients* $a_i(j)$, $i \geq k$, *are not all zero, then*

$$\frac{E^{(np+j)}_{+p,k} - s}{E^{(np+j)}_{+p,k-1} - s} \underset{n}{\longrightarrow} \frac{b_{j,k} - b_{j,i_j}}{b_{j,k} - 1},$$

where i_j *is the smallest index such that* $i_j \geq k$ *and* $a_{i_j}(j) \neq 0$.

(iv) *If* $\prod_{j=0}^{p-1} a_k(j) \neq 0$ *then* $E^{(n)}_{+p,k} - s = o(E^{(n+p)}_{+p,k-1} - s)$.

Proof. (i) By induction on k.

(ii), (iii) and (iv) follow from (i). ∎

Remark. If $\prod_{j=0}^{p-1} a_k(j) \neq 0$ for each $k \geq 1$, then $E^{(n)}_{+p,k} - s = o(E^{(n+p)}_{+p,k-1} - s)$ for all $k \geq 1$.

An immediate consequence of Theorem 11 is

COROLLARY. *Let* $(s_n) \in \mathrm{Conv}(\mathbb{C})$. *Assume that* $(g_1, \ldots, g_i, \ldots)$ *is an asymptotic sequence. If*

$$s_n - s \approx a_1 g_1(n) + \ldots + a_i g_i(n) + \ldots,$$

where $a_i \neq 0$ *for all* $i \geq 1$, *then* $E_{+p,k}^{(n)} - s = o(E_{+p,k-1}^{(n+p)} - s)$ *for all* $k \geq 1$.

THEOREM 12. *Let* $(s_n) \in \mathrm{Conv}(\mathbb{C})$. *Assume that*:

1. *For each* $j \in \{0, \ldots, p-1\}$,

$$s_{np+j} - s \approx \lambda_j^n n^{\alpha_j} \left(a_{j,0} + \frac{a_{j,1}}{n^{\alpha_{j,1}}} + \ldots + \frac{a_{j,i}}{n^{\alpha_{j,i}}} + \ldots \right), \tag{5}$$

where $0 < |\lambda_j| < 1$, $a_{j,0} \neq 0$, $0 < \mathrm{Re}\, \alpha_{j,1} < \mathrm{Re}\, \alpha_{j,2} < \ldots < \mathrm{Re}\, \alpha_{j,i} < \ldots$

2. *The auxiliary sequences* $(g_i(n))$ *of the* E_{+p}*-algorithm are such that for all* $i \geq 1$ *and* $j \in \{0, \ldots, p-1\}$,

$$g_i(np+j) \approx \lambda_j^n n^{\theta_{j,i}} \left(a_{j,i,0} + \frac{a_{j,i,1}}{n^{\alpha_{j,i,1}}} + \ldots + \frac{a_{j,i,k}}{n^{\alpha_{j,i,k}}} + \ldots \right),$$

with $a_{j,i,0} \neq 0$, $0 < \mathrm{Re}\, \alpha_{j,i,1} < \mathrm{Re}\, \alpha_{j,i,2} < \ldots < \mathrm{Re}\, \alpha_{j,i,k} < \ldots$

Then for each $k \geq 1$ *and each* $j \in \{0, \ldots, p-1\}$, *either there exists* n_0 *such that* $E_{+p,k}^{(np+j)} = s$ *for all* $n \geq n_0$, *or* $E_{+p,k}^{(np+j)} - s = o(E_{+p,k-1}^{((n+1)p+j)} - s)$ *and*

$$E_{+p,k}^{(np+j)} - s \approx \lambda_j^n n^{\beta_{j,k}} \left(b_{j,k,0} + \frac{b_{j,i,1}}{n^{\beta_{j,i,1}}} + \ldots + \frac{b_{j,i,k}}{n^{\beta_{j,i,k}}} + \ldots \right),$$

with $b_{j,k,0} \neq 0$, $\mathrm{Re}\, \beta_{j,k} \leq \mathrm{Re}\, \alpha_j - k$, $0 < \mathrm{Re}\, \beta_{j,i,1} < \mathrm{Re}\, \beta_{j,i,2} < \ldots$ $\ldots < \mathrm{Re}\, \beta_{j,i,k} < \ldots$

Proof. By induction on k.

Theorem 12 generalizes a result for the E-algorithm (i.e. $p = 1$) given in [5].

DEFINITION. The E_{+p}-algorithm is called *effective* on (s_n) if for all $k \geq 1$, either $E_{+p,k}$ is exact on (s_n) (i.e. there exists n_0 such that $E_{+p,k}^{(n)} = s$ for all $n \geq n_0$) or $E_{+p,k}^{(n)} - s = o(E_{+p,k-1}^{(n+p)} - s)$.

THEOREM 13. *Assume that* (s_n) *satisfies* (5). *The* E_{+p}*-algorithm with the following particular auxiliary sequences is effective on* (s_n):

I. $g_i(n) = s_{n+ip} - s_{n+(i-1)p}$, $i \geq 1$;

II. $g_1(np+j) = \lambda_j^{np+j} n^{\beta_j}$, $\beta_j \in \mathbb{C}$, $j = 0, \ldots, p-1$ *and* $g_i(n) = s_{n+(i-1)p} - s_{n+(i-2)p}$ *for* $i \geq 2$;

III. $g_i(n) = (s_n - s_{n-p})/n^{i-1}$, $i \geq 1$;

IV. $g_i(n) = (s_n - s_{n-p})/n^{i-2}$, $i \geq 1$;

V. $g_i(n) = \dfrac{(s_{n+p} - s_n)(s_n - s_{n-p})^2}{(s_{n+p} - 2s_n + s_{n-p})n^{i-1}}$, $i \ge 1$;

VI. $g_i(np+j) = \lambda_j^{np+j}(n+i)^{\beta_j}$, $\beta_j \in \mathbb{C}$, $i \ge 1$, $j = 0, \ldots, p-1$.

Proof. This follows immediately from Theorem 12.

Let us mention that in the cases considered in Theorem 13 the E_{+p}-algorithm is a generalization of the ε-algorithm (case I), the process p (case II), the transformation T of Levin (case III), the transformation U of Levin (case IV), the transformation V of Levin (case V), and the G-transformation (case VI).

4. Applications. Let (s_n) be a convergent sequence such that the error $s_n - s$ has an asymptotic expansion of the form

$$s_n - s \approx a_1 g_1(n) + \ldots + a_i g_i(n) + \ldots,$$

where for all $i \ge 1$ and $j \in \{0, \ldots, p-1\}$,

$$\frac{g_i(np+j)}{g_i(np+j-1)} \xrightarrow[n]{} b_{j,i}$$

with

$$\prod_{m=0}^{p-1} b_{m,i} \ne 0, 1 \quad \text{and} \quad \prod_{m=0}^{p-1} b_{m,i} \ne \prod_{m=0}^{p-1} b_{m,k} \quad \text{for } i \ne k.$$

The auxiliary sequences $(g_i(n)), i \ge 1$, satisfy the condition (b_{+p}). Consequently, we can use the E_{+p}-algorithm for accelerating (s_n).

If there exist $i_0 \ge 1$ and $r, s \in \{0, \ldots, p-1\}$ such that $b_{r,i_0} \ne b_{s,i_0}$ then $(g_{i_0(n+1)}/g_{i_0(n)})$ is not convergent. Hence, we cannot use Brezinski's result [2] and Fdil's result [5] for the E-algorithm.

Assume that the auxiliary sequences $(g_i(n))$, $i \ge 1$, of the E-algorithm are such that for all $k \ge 0$ and $i > k$,

$$\frac{g_{k,i}^{(n+1)}}{g_{k,i}^{(n)}} \xrightarrow[n]{} b_i \quad \text{with } 1 > b_1 \ge \ldots \ge b_i \ge \ldots$$

If some numbers b_i are close to 1, the E-algorithm is numerically unstable. Choose a positive integer p^* (odd if there exists i such that $b_i = -1$) such that $b_1^{p^*}$ is not close to 1 (for example, $b_1^{p^*} \le 0.8$). Then the condition (b_{+p^*}) is satisfied and the E_{+p^*}-algorithm with $(g_i(n))$, $i \ge 1$, as auxiliary sequences is numerically stable. Consequently, we can use the E_{+p^*}-algorithm instead of the E-algorithm for accelerating the convergence. For illustration, consider the sequence

$$s_n = \sum_{k=0}^{n} (k+1)(k+2)(.9)^k, \quad n = 0, 1, \ldots$$

It is convergent to $s = 2000$. From a result due to Wimp [14, p. 19], we get

$$s_n - s \approx a_1 g_1(n) + \ldots + a_i g_i(n) + \ldots$$

with $g_i(n) = (.9)^n (n+1)^{2-i+1}, \ i \geq 1, \ n \geq 0.$

Applying the E-algorithm and the E_{+4}-algorithm to (s_n), with $g_i(n)$, $i \geq 1$, as auxiliary sequences, we obtain

n	$E_n^{(0)}$	$E_{+4,n/4}^{(0)}$
4	2000.00000000115	2.934258724233079
8	2000.000000001432	28.04921739106413
12	2000.000000010298	1999.999999999997
16	2000.000000096219	2000.000000000004
20	2000.000060098672	2000.000000000002
24	1999.999887612695	1999.999999999999

Let us now give another application of the E_{+p}-algorithm.

The use of some quadrature formulas for computing the integral $s = \int_0^1 \ldots \int_0^1 f(x_1, \ldots, x_k)\, dx_1 \ldots dx_k$, where f does not have a logarithmic singularity, often leads to an asymptotic expansion of the form

$$T(h) - s \approx a_1 h^{\gamma_1} + \ldots + a_k h^{\gamma_k} + \ldots, \tag{6}$$

where $T(h)$ is an approximate value of s, associated with the step length h ($[0,1]$ is divided into $1/h$ subintervals of length h), $0 < \gamma_1 < \ldots < \gamma_i < \ldots$ (see, for example, [6, 8–12]).

Let (h_n) be a sequence of step lengths. Set $s_n = T(h_n), \ n \geq 0$, and $g_i(n) = h_n^{\gamma_i}$ for $i \geq 1$, $n \geq 0$. Thus

$$s_n - s \approx a_1 g_1(n) + \ldots + a_i g_i(n) + \ldots$$

For the choice $h_n = 1/2^n, \ n \geq 0$ (geometric sequence), the auxiliary sequences $(g_i(n)), \ i \geq 1$, satisfy the condition (b_{+1}) $(b_i = 1/2^{\gamma_i})$ and the E-algorithm is numerically stable. Consequently, we can compute s with high accuracy. The disadvantage of this choice is that the number of function evaluations is doubled from one step to the next.

The choice $h_n = 1/(n+1), \ n \geq 0$ (harmonic sequence), is the most economic in terms of the number of function evaluations. However, the E-algorithm with $(g_i(n)), \ i \geq 1$, as auxiliary sequences is numerically unstable.

Håvie [9] proposed the following general choice:

$$h_{2n} = \frac{1}{\sigma_0 M^n}, \qquad h_{2n+1} = \frac{1}{\sigma_1 M^n}, \qquad n \geq 0,$$

where $\sigma_0, \sigma_1, M \in \mathbb{N}^*$ with $1 \le \sigma_0 < \sigma_1,\ 2 \le M$.

σ_0	σ_1	M	(h_n)
1	2	4	$(\frac{1}{2})^n$
1	2	3	Bauer
2	3	2	Bulirsch

For this general choice, we have, for all $i \ge 1$ and $n \ge 0$,

$$\frac{g_i(2n+1)}{g_i(2n)} = \left(\frac{\sigma_0}{\sigma_1}\right)^{\gamma_i}, \qquad \frac{g_i(2n+2)}{g_i(2n+1)} = \left(\frac{\sigma_1}{\sigma_0 M}\right)^{\gamma_i}.$$

The sequences $(g_i(n)),\ i \ge 1$, satisfy the condition (b_{+2}). Consequently, the E_{+2}-algorithm can be used for accelerating (s_n).

Let p be a positive integer, $p \ge 2$. Put

$$h_j = \frac{1}{2^j}, \qquad h_{p(n+1)+j} = \frac{1}{2^{n+p}+j}, \qquad j = 0, \dots, p-1,\ n = 0, 1, \dots$$

This choice is more economical than Håvie's choice. We have, for all $i \ge 1$,

$$\frac{g_i(np)}{g_i(np-1)} \underset{n}{\longrightarrow} \left(\frac{1}{2}\right)^{\gamma_i}, \qquad \frac{g_i(np+j)}{g_i(np+j-1)} \underset{n}{\longrightarrow} 1 \quad \text{for } j = 1, \dots, p-1,$$

$$\frac{g_i(n+p)}{g_i(n)} \underset{n}{\longrightarrow} b_i = \left(\frac{1}{2}\right)^{\gamma_i}.$$

The condition (b_{+p}) is satisfied and the E_{+p}-algorithm is numerically stable. Thus, we can use the E_{+p}-algorithm for computing s with high accuracy. Note that the E-algorithm with $g_i(n) = h_n^{\gamma_i},\ i \ge 1$, as auxiliary sequences is numerically unstable because for all $i \ge 1$, the sequence $(g_i(n+1)/g_i(n))$ has 1 as an accumulation point.

Example: $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$

Let $T(h)$ be the approximate value of $s = \int_0^1 (1/\sqrt{x})\,dx$ computed by the rectangular method:

$$T(h) = h \sum_{k=0}^{1/h-1} ((k+1/2)h)^{-1/2}.$$

The function $T(h)$ has an asymptotic expansion of the form (6), with $\gamma_i = (2i-1)/2$ for $i = 1, 2, \dots$

Let (h_n) be the preceding sequence of step lengths with $p = 4$. Applying the E-algorithm and the E_{+4}-algorithm to the sequence $s_n = T(h_n)$, we obtain

n	$E_n^{(0)}$	$E_{+4,n/4}^{(0)}$
8	2.000005028488225	2.000267155507016
16	2.000000047703804	2.000001670788114
24	2.000002619190568	2.000000068392044
32	1.99977686651984	2.000000003881382
40	2.203733445234798	2.00000000023691
48	0.5596855849893074	2.000000000014719

We see that the E_{+4}-algorithm is more effective than the E-algorithm for accelerating (s_n).

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