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LOCAL EXISTENCE OF SOLUTIONS OF THE FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF COMPRESSIBLE BAROTROPIC VISCOUS SELF-GRAVITATING FLUIDS

Abstract. Local existence of solutions is proved for equations describing the motion of a viscous compressible barotropic and self-gravitating fluid in a domain bounded by a free surface. First by the Galerkin method and regularization techniques the existence of solutions of the linearized momentum equations is proved, next by the method of successive approximations local existence to the nonlinear problem is shown.

1. Introduction. In this paper we prove the existence of local solutions to equations describing the motion of a viscous compressible barotropic fluid under the self-gravitating force in a bounded domain $\Omega_t \subset \mathbb{R}^3$ with a free boundary S_t. Let $v = v(x,t)$ be the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $p = p(\varrho)$ the pressure, μ and ν the constant viscosity coefficients and p_0 the external constant pressure. Then the problem is described by the following system of equations (see [1], Chs. 1,2):

$$
\begin{aligned}
&\varrho(v_t + v\cdot\nabla v) - \operatorname{div}\mathbb{T}(v,p) = \varrho\nabla U && \text{in } \widetilde{\Omega}^T = \textstyle\bigcup_{0\le t\le T}\Omega_t\times\{t\},\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\cdot\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T = \textstyle\bigcup_{0\le t\le T}S_t\times\{t\},\\
&\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0 && \text{in } \Omega = \Omega_0,\\
&v\cdot\overline{n} = -\frac{\phi_t}{|\nabla\phi|} && \text{on } \widetilde{S}^T,
\end{aligned}
\tag{1.1}
$$

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where $\phi(x,t)=0$ describes S_t, $\overline{n}$ is the unit outward vector normal to S_t, $\overline{n}=\nabla\phi/|\nabla\phi|$, Ω_t is the domain at time t, $S_t=\partial\Omega_t$, $\Omega=\Omega_t|_{t=0}=\Omega_0$, $S=\partial\Omega$.

By $\mathbb{T}=\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T}(v,p)=\{T_{ij}\}_{i,j=1,2,3}=\{-p\delta_{ij}+D_{ij}(v)\}_{i,j=1,2,3}, \tag{1.2}$$

where

$$\mathbb{D}(v)=\{D_{ij}\}_{i,j=1,2,3}=\{\mu(\partial_{x_i}v_j+\partial_{x_j}v_i)+(\nu-\mu)\delta_{ij}\operatorname{div}v\}_{i,j=1,2,3} \tag{1.3}$$

is the deformation tensor.

Moreover, $U(x,t)$ is the self-gravitating potential such that

$$U(x,t)=k\int_{\Omega_t}\frac{\varrho(y,t)}{|x-y|}\,dy, \tag{1.4}$$

where k is the gravitation constant.

By the continuity equation $(1.1)_2$ and the kinematic condition $(1.1)_5$ the total mass is conserved, so

$$\int_{\Omega_t}\varrho(x,t)\,dx=\int_{\Omega}\varrho_0(x)\,dx=M, \tag{1.5}$$

where M is a given constant.

Let Ω be given. We introduce the Lagrangian coordinates ξ as the initial data for the Cauchy problem

$$\frac{dx}{dt}=v(x,t),\quad x|_{t=0}=\xi\in\Omega,\quad \xi=(\xi_1,\xi_2,\xi_3). \tag{1.6}$$

Integrating (1.6) we obtain a transformation between the Eulerian x and the Lagrangian ξ coordinates,

$$x=x(\xi,t)\equiv\xi+\int_0^t u(\xi,\tau)\,d\tau\equiv x_u(\xi,t), \tag{1.7}$$

where $u(\xi,t)=v(x_u(\xi,t),t)$ and the index u in $x_u(\xi,t)$ will be omitted in evident cases.

Then, by $(1.1)_5$, $\Omega_t=\{x\in\mathbb{R}^3: x=x(\xi,t),\ \xi\in\Omega\}$ and $S_t=\{x\in\mathbb{R}^3: x=x(\xi,t),\ \xi\in S\}$.

Let $\eta(\xi,t)=\varrho(x(\xi,t),t)$, $q(\xi,t)=p(x(\xi,t),t)$, $\nabla_u=\partial_x\xi_i\nabla_{\xi_i}$, $\partial_{\xi_i}=\nabla_{\xi_i}$, $\mathbb{T}_u(u,q)=-qI+\mathbb{D}_u(u)$, $I=\{\delta_{ij}\}_{i,j=1,2,3}$ is the unit matrix, $\mathbb{D}_u(u)=\{\mu(\partial_{x_i}\xi_k\nabla_{\xi_k}u_j+\partial_{x_j}\xi_k\nabla_{\xi_k}u_i)+(\nu-\mu)\delta_{ij}\nabla_u\cdot u\}$, where $\nabla_u\cdot u=\partial_{x_i}\xi_k\nabla_{\xi_k}u_i$ and summation over repeated indices is assumed.

Since S_t is determined (at least locally) by the equation $\phi(x,t)=0$, S is described by $\phi(x(\xi,t),t)|_{t=0}=\widetilde{\phi}(\xi)=0$. Moreover, we have

$$\overline{n}_u=\overline{n}(x_u(\xi,t),t)=\frac{\nabla_x\phi(x,t)}{|\nabla_x\phi(x,t)|}\bigg|_{x=x_u(\xi,t)},$$
$$\overline{n}_0=\overline{n}_{u0}(\xi,t)=\frac{\nabla_\xi\widetilde{\phi}(\xi)}{|\nabla_\xi\widetilde{\phi}(\xi)|}.$$

In Lagrangian coordinates the problem (1.1) takes the form

$$\begin{aligned}
&\eta u_t-\operatorname{div}_u\mathbb{T}_u(u,q)=U_u(\eta) && \text{in } \Omega^T=\Omega\times(0,T),\\
&\eta_t+\eta\operatorname{div}_u u=0 && \text{in } \Omega^T,\\
&\mathbb{T}_u(u,q)\cdot\overline{n}_u=-p_0\overline{n}_u && \text{on } S^T=S\times(0,T),\\
&u|_{t=0}=v_0 && \text{in } \Omega,\\
&\eta|_{t=0}=\varrho_0 && \text{in } \Omega,
\end{aligned}\tag{1.8}$$

where

$$U_u(\eta)=k\int\limits_\Omega\frac{\eta(\vartheta,t)}{|x_u(\xi,t)-x_u(\vartheta,t)|}A(x_u(\vartheta,t))\,d\vartheta\tag{1.9}$$

and A is the Jacobian determinant of the transformation $x=x(\xi,t)$.

The proof of the existence of solutions of problem (1.8) is divided into a few steps. First we consider the problem

$$\begin{aligned}
&u_t-\operatorname{div}\mathbb{D}(u)=f_1,\\
&\mathbb{D}(u)\cdot\overline{n}=b_1,\\
&u|_{t=0}=v_0.
\end{aligned}\tag{1.10}$$

At the second step we examine the problem with a given positive function $\eta(\xi,t)$:

$$\begin{aligned}
&\eta u_t-\operatorname{div}\mathbb{D}(u)=f && \text{in } \Omega^T,\\
&\mathbb{D}(u)\cdot\overline{n}=g && \text{on } S^T,\\
&u|_{t=0}=v_0 && \text{in } \Omega.
\end{aligned}\tag{1.11}$$

To examine the nonlinear problem (1.8) we need an existence result for the problem

$$\begin{aligned}
&\eta u_t-\operatorname{div}_w\mathbb{D}_w(u)=f_3,\\
&\mathbb{D}_w(u)\cdot\overline{n}_w=g_3,\\
&u|_{t=0}=u_0,
\end{aligned}\tag{1.12}$$

where $\eta>0$ and $w=w(\xi,t)$ are given functions.

Finally, we prove the existence of solutions to (1.8), hence also to (1.1), by the following method of successive approximations:

$$
\begin{aligned}
&\eta_m\partial_t u_{m+1} - \operatorname{div}_{u_m}\mathbb{D}_{u_m}(u_{m+1}) = -\nabla_{u_m} q(\eta_m) + \eta_m U_{u_m}(\eta_m) && \text{in } \Omega^T,\\
&\mathbb{D}_{u_m}(u_{m+1})\cdot\overline{n}_{u_m} = -(q(\eta_m)-p_0)\overline{n}_{u_m} && \text{on } S^T,\\
&u_{m+1}|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}
\tag{1.13}
$$

where η_m and u_m are treated as given, and

$$
\begin{aligned}
&\partial_t\eta_m + \eta_m \operatorname{div}_{u_m} u_m = 0 && \text{in } \Omega^T,\\
&\eta_m|_{t=0} = \varrho_0 && \text{in } \Omega,
\end{aligned}
\tag{1.14}
$$

where u_m is treated as given, and $m = 0, 1, \ldots$

We want to point out that the presented proof of existence uses the Galerkin method and some regularization techniques because the case considered is singular in potential theory. This is related to $H^3(\Omega)$ regularity. Ordinarily the Galerkin and regularization methods are connected with the energy method which is much more natural for (1.1) than the potential technique. Moreover, this technique is applied in the stability proof for (1.1) in [4]. We have also to emphasize that $H^3(\Omega)$ regularity for v is the lowest possible regularity in spaces with integer derivatives for solutions of nonlinear problems such as (1.1) to exist. As follows from [3] the existence of solutions to (1.1) can be shown in the spaces $H^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\alpha\in(1/2,1)$, but the norm of these spaces contains fractional derivatives and is not convenient for our considerations in [4].

In [2] local existence of solutions for the free boundary problem for the equations of a viscous compressible heat-conducting self-gravitating fluid is proved. However, the proof is done in a different way and the regularity obtained is not suitable for our considerations in [4].

2. Notation. To simplify considerations we introduce the following notation:

$$
\begin{aligned}
\|u\|_{s,Q} &= \|u\|_{H^s(Q)}, \quad s\in\mathbb{N}\cup\{0\},\ Q\in\{\Omega,\Omega^t,S,S^t\},\\
\Omega^t &= \Omega\times(0,t), \quad S^t = S\times(0,t),\\
|u|_{p,Q} &= \|u\|_{L_p(Q)}, \quad p\in[1,\infty],\\
\|u\|_{s,p,q,\Omega^T} &= \|u\|_{L_q(0,T;W_p^s(\Omega))}, \quad p,q\in[1,\infty],\ 0\le s\in\mathbb{Z}.
\end{aligned}
$$

We define the space $\Gamma_l^k(\Omega)$ as part of $\bigcap_{i=0}^{k-l} C^i([0,T];H^{k-i}(\Omega))$ with the norm $\|u\|_{\Gamma_l^k(\Omega)} = \sum_{i=0}^{k-l}\|\partial_t^i u\|_{k-i,\Omega}$.

Then we denote by $L_p(0,T;\Gamma_l^k(\Omega))$ the closure of $C^\infty(\Omega^T)$ with the norm

$$\Big(\int_0^T\Big(\sum_{i=0}^{k-l}\|\partial_t^i u\|_{k-i,\Omega}\Big)^p dt\Big)^{1/p}, \qquad p\in[1,\infty].$$

Moreover, we introduce

$$|u|_{k,l,p,\Omega^T}=\|u\|_{L_p(0,T;\,\Gamma_l^k(\Omega))}.$$

3. Existence of solutions. We prove the existence of solutions to problem (1.1) by the method of successive approximations described by problems (1.13) and (1.14). Therefore, we first consider the following auxiliary problem:

$$\begin{aligned} &\eta u_t-\mathrm{div}_w\,\mathbb{D}_w(u)=F &&\text{in } \Omega^T,\\ &\mathbb{D}_w(u)\cdot\overline{n}_w=G &&\text{on } S^T,\\ &u|_{t=0}=v_0 &&\text{in } \Omega, \end{aligned}\tag{3.1}$$

where η and w are given functions. Moreover, η is such that

$$0<\varrho_*\le\eta\le\varrho^*<\infty\tag{3.2}$$

and $w=w(\xi,t)$ is such that

$$x=\xi+\int_0^t w(\xi,\tau)\,d\tau\equiv x_w(\xi,t)\equiv x(\xi,t),\tag{3.3}$$

and

$$\frac{\partial x}{\partial\xi},\ \frac{\partial\xi}{\partial x}\tag{3.4}$$

are matrices with determinants close to 1 for $t\in[0,T]$.

Definition 3.1. By a *weak solution* to problem (3.1) we mean a function u which satisfies the integral identity

$$\int_\Omega[\eta u_t\varphi+\mathbb{D}'_w(u)\cdot\mathbb{D}'_w(\varphi)-F\cdot\varphi]J_w\,d\xi-\int_S G\varphi J_w\,d\xi_s=0\tag{3.5}$$

for any sufficiently smooth function φ, where $\mathbb{D}'_w(u)\cdot\mathbb{D}'_w(\varphi)=\frac{\mu}{2}(\nabla_{w_i}u_j+\nabla_{w_j}u_i)(\nabla_{w_i}\varphi_j+\nabla_{w_j}\varphi_i)+(\nu-\mu)\nabla_w\cdot u\nabla_w\cdot\varphi$ and J_w is the Jacobian determinant of the transformation $x=x_w(\xi,t)$.

To obtain the integral formula for (3.1) we use the following integration by parts:

$$\begin{aligned}
&\int_\Omega \operatorname{div}_w \mathbb{D}_w(v(x_w(\xi,t),t)\varphi(x_w(\xi,t),t)J_w\,d\xi \\
&\qquad\qquad = \int_{\Omega_t} \operatorname{div}\mathbb{D}(v(x,t))\varphi(x,t)\,dx \\
&\qquad\qquad = -\int_{\Omega_t} \mathbb{D}'(v)\cdot\mathbb{D}'(\varphi)dx + \int_{S_t} \overline{n}\cdot\mathbb{D}(v)\varphi\,ds \\
&\qquad\qquad = -\int_\Omega \mathbb{D}'_w(v(x_w(\xi,t),t))\cdot\mathbb{D}'_w(\varphi(x_w(\xi,t),t))J_w\,d\xi \\
&\qquad\qquad\quad + \int_S \overline{n}_w\cdot\mathbb{D}_w(v(x_w(\xi,t),t))\varphi(x_w(\xi,t),t)J_w\,d\xi_S,
\end{aligned}$$

where $\mathbb{D}'(v)\cdot\mathbb{D}'(\varphi) = \frac{\mu}{2}(\partial_{x_i}v_j + \partial_{x_j}v_i)(\partial_{x_i}\varphi_j + \partial_{x_j}\varphi_i) + (\nu-\mu)\operatorname{div} v \operatorname{div}\varphi$.

Take a basis $\{\varphi_k\}$ in $L_2(\Omega)$. Then we are looking for an approximate solution of (3.5) in the form

$$u_n = \sum_{i=1}^n c_{in}(t)\varphi_i(\xi), \tag{3.6}$$

where $c_{in}, i = 1,\dots,n$, are solutions of the following system of ordinary differential equations:

$$\begin{aligned}
&\int_\Omega [\eta u_{nt}\varphi_i + \mathbb{D}'_w(u_n)\cdot\mathbb{D}'_w(\varphi_i) - F\cdot\varphi_i]J_w\,d\xi - \int_S G\varphi_i J_w\,d\xi_S = 0, \\
&u_n|_{t=0} = \sum_{i=1}^n c_{in}(0)\varphi_i(\xi), \qquad c_{in}(0) = \int_\Omega v_{i0}\varphi_n\,d\xi,
\end{aligned}\tag{3.7}$$

where $i = 1,\dots,n$, and existence follows from the theory of ordinary differential equations.

Next we have to obtain estimates for solutions of (3.7).

LEMMA 3.2. *Assume that* $\varrho_* \le \eta$, $\eta_t \in L_2(0,T;H^1(\Omega))$, $F \in L_2(\Omega^T)$, $G \in L_2(S^T)$, $w \in L_2(0,T;H^3(\Omega))$. *Assume that*

$$\sup_{t\in[0,T]}\sup_{\xi\in\Omega} |I - \xi_x| \le \delta, \tag{3.8}$$

where δ *is sufficiently small and* I *is the unit matrix. Then for solutions of* (3.7) *the following inequality holds*:

$$\begin{aligned}
\|u_n\|^2_{0,\Omega} + c_0\|u_n\|^2_{1,2,2,\Omega^t} &\le \psi_1(1/\varrho_*, t, \|\eta_t\|_{1,2,2,\Omega^t}, a(w,t)) \\
&\quad\times \Big[\int_\Omega \varrho_0 v_0^2\,dx + \|F\|^2_{0,\Omega^t} + \|G\|^2_{0,S^t}\Big],
\end{aligned}\tag{3.9}$$

where ψ_1 *is an increasing positive function*, $a(w,t) = t^{1/2}\|w\|_{3,2,2,\Omega^t}$, *and* $t \le T$.

P r o o f. Multiplying (3.7) by c_{in} and summing over i from 1 to n we get

$$(3.10)\quad \frac{1}{2}\int_\Omega \left(\eta\frac{d}{dt}u_n^2 + |\mathbb{D}'_w(u_n)|^2\right)J_w\,d\xi = \int_\Omega F\cdot u_n J_w\,d\xi + \int_S G\cdot u_n J_w\,d\xi_S.$$

Using the Korn inequality

$$(3.11)\quad \|u\|_{1,\Omega}^2 \le c(\|\mathbb{D}''(u)\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2)$$

and $|\mathbb{D}'(v)|^2 \ge c_0|\mathbb{D}''(v)|^2$, $c_0 = \min\{\frac{3}{4}(\nu - \frac{1}{3}\mu), \frac{\mu}{2}\}$, where $\mathbb{D}''(u) = \{\mu(\partial_{x_i}u_j + \partial_{x_j}u_i)\}$, we have

$$\|u\|_{1,\Omega}^2 \le c(\|\mathbb{D}''_w(u)\|_{0,\Omega}^2 + \|\mathbb{D}''(u) - \mathbb{D}''_w(u)\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2),$$

so in view of (3.8) we get

$$(3.12)\quad \|u\|_{1,\Omega}^2 \le c(\delta)(\|\mathbb{D}''_w(u)\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2).$$

Using (3.12) in (3.10) implies

$$(3.13)\quad \frac{d}{dt}\int_\Omega \eta u_n^2 J_w\,d\xi + c_0\|u_n\|_{1,\Omega}^2 \le c\int_\Omega (|\eta_t| + \eta|\mathrm{div}_w\,w|)|u_n|^2 J_w\,d\xi + c(\|u_n\|_{0,\Omega}^2 + \|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2).$$

Estimating the first term on the r.h.s. by

$$\varepsilon\|u_{n\xi}\|_{0,\Omega}^2 + \left(\frac{c(\varepsilon)}{\varrho_*}\|\eta_t\|_{1,\Omega}^2 + |\mathrm{div}_w\,w|_{\infty,\Omega}\right)\int_\Omega \eta u_n^2 J_w d\xi, \qquad \varepsilon\in(0,1),$$

from (3.13) we get

$$(3.14)\quad \frac{d}{dt}\int_\Omega \eta u_n^2 J_w\,d\xi + c_0\|u_n\|_{1,\Omega}^2 \le \left[\frac{c}{\varrho_*}(1 + \|\eta_t\|_{1,\Omega}^2) + |\mathrm{div}_w\,w|_{\infty,\Omega}\right]\int_\Omega \eta u_n^2 J_w\,d\xi + c(\|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2).$$

Integrating (3.14) with respect to time yields

$$(3.15)\quad \int_\Omega \eta u_n^2 J_w\,d\xi + c_0\|u_n\|_{1,2,2,\Omega^t}^2 \le \exp\left[\frac{c}{\varrho_*}(t + \|\eta_t\|_{1,2,2,\Omega^t}^2 + \varphi(a(w,t)))\right]\left[\int_\Omega \varrho_0 v_0^2\,d\xi + \|F\|_{0,\Omega^t}^2 + \|G\|_{0,S^t}^2\right],$$

where φ is an increasing positive function. From (3.15) we obtain (3.9). This concludes the proof.

From (3.9) we can prove the existence of weak solutions such that $u \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;\,H^1(\Omega))$. However, we want to obtain more regular weak solutions simultaneously. Therefore we show

LEMMA 3.3. *Assume that* $\varrho_* \le \eta$, $F_t \in L_2(\Omega^T)$, $G_t \in L_2(S^T)$, $F \in L_\infty(0,T;L_2(\Omega))$, $G \in L_\infty(0,T;L_2(S))$, $w \in L_2(0,T;H^3(\Omega))$, $\eta_t \in L_2(0,T;H^1(\Omega))$ *and* $\int_\Omega \varrho_0 u_t^2(0)\,d\xi < \infty$. *Assume* (3.8). *Then*

$$\|u_{nt}\|_{0,\Omega}^2 + c_0\|u_{nt}\|_{1,2,2,\Omega^t}^2 \le \psi_2(1/\varrho_*, a(w,t), t, \|\eta_t\|_{1,2,2,\Omega^t}) \times \Big[\int_\Omega \varrho_0 u_t^2(0)\,d\xi + \|F_t\|_{0,\Omega^t}^2 + \|G_t\|_{0,S^t}^2 + \sup_t(\|u_n\|_{1,\Omega}^2 + \|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2) \times \int_0^t (\varepsilon_1\|w\|_{3,\Omega}^2 + c(\varepsilon_1)\|w\|_{0,\Omega}^2)\,dt\Big], \tag{3.16}$$

where ψ_2 *is an increasing positive function and* $\varepsilon_1 \in (0,1)$.

Proof. Differentiating (3.7) with respect to t, multiplying by $\dot c_{in}$ and summing up over i from 1 to n we get

$$\frac{d}{dt}\int_\Omega \eta u_{nt}^2 J_w\,d\xi + c_0\|u_{nt}\|_{1,\Omega}^2 \le c\int_\Omega |\eta_t| u_{nt}^2 J_w\,d\xi + \varphi_1(a(w,t))\int_\Omega \eta u_{nt}^2 J_w |w_\xi|\,d\xi + c(\|u_{nt}\|_{0,\Omega}^2 + \|F_t\|_{0,\Omega}^2 + \|G_t\|_{0,S}^2) + \varphi_1(a(w,t))|w_\xi|_{\infty,\Omega}^2(\|u_n\|_{1,\Omega}^2 + \|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2), \tag{3.17}$$

where φ_1 is an increasing positive function, $a(w,t)$ was defined in Lemma 3.2, and the Korn inequality and condition (3.8) were used.

Estimating the first term on the r.h.s. of (3.17) by

$$\frac{c_0}{2}\|u_{nt}\|_{1,\Omega}^2 + \frac{c(\mu,\nu)}{\varrho_*}\|\eta_t\|_{1,\Omega}^2\int_\Omega \eta u_{nt}^2 J_w\,d\xi$$

we can write (3.17) in the form

$$\frac{d}{dt}\int_\Omega \eta u_{nt}^2 J_w\,d\xi + c_0\|u_{nt}\|_{1,\Omega}^2 \le \varphi_2(a(w,t))(1+1/\varrho_*)(1+\|\eta_t\|_{1,\Omega}^2 + |w_\xi|_{\infty,\Omega}) \times \int_\Omega \eta u_{nt}^2 J_w d\xi + c(\|F_t\|_{0,\Omega}^2 + \|G_t\|_{0,S}^2) + \varphi_1(a(w,t))(\varepsilon\|w\|_{3,\Omega}^2 + c(\varepsilon)\|w\|_{0,\Omega}^2) \times (\|u_n\|_{1,\Omega}^2 + \|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2). \tag{3.18}$$

Integrating (3.18) with respect to t we get

$$\int_\Omega \eta u_{nt}^2 J_w d\xi + c_0\|u_{nt}\|_{1,2,2,\Omega^t}^2 \le \exp[\varphi_2(a(w,t))(1+1/\varrho_*) \times (t+\|\eta_t\|_{1,2,2,\Omega^t}^2 + a(w,t))] \times \Big[\int_\Omega \varrho_0 u_t^2(0)\, d\xi + \|F_t\|_{0,\Omega^t}^2 + \|G_t\|_{0,S^t}^2 + \varphi_1(a(w,t))\sup_t(\|u_n\|_{1,\Omega}^2 + \|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2) \times \int_0^t (\varepsilon\|w\|_{3,\Omega}^2 + c(\varepsilon)\|w\|_{0,\Omega}^2)\, dt\Big]. \tag{3.19}$$

From (3.19) we have (3.16). This concludes the proof.

To estimate the expression $\sup_t \|u_n\|_{1,\Omega}^2$ on the r.h.s. of (3.16) we need the following result.

LEMMA 3.4. *Let the assumptions of Lemma* 3.3 *be satisfied. Then*

$$\|u_{nt}\|_{0,\Omega^t}^2 + c_0\|u_n\|_{1,\Omega}^2 \le \psi_3\Big(t, 1/\varrho_*, a(w,t), \int_0^t (\varepsilon_1\|w\|_{3,\Omega}^2 + c(\varepsilon_1)\|w\|_{0,\Omega}^2)\, dt\Big) \times \Big[\|u_0\|_{1,\Omega}^2 + \int_\Omega \varrho_0 v_0^2\, dx + \|F\|_{0,\Omega^t}^2 + c(\varepsilon_2)\|G\|_{0,S^t}^2 + \varepsilon_2\|u_{nt}\|_{1,2,2,\Omega^t}^2\Big], \tag{3.20}$$

where ψ_3 is an increasing positive function and ε_1, $\varepsilon_2 \in (0,1)$.

Proof. Multiplying (3.7) by $\dot c_{in}$ and summing over i from 1 to n we get

$$\int_\Omega \eta u_{nt}^2 J_w\, d\xi + \int_\Omega \mathbb{D}'_w(u_n)\cdot \mathbb{D}'_w(u_{nt}) J_w\, d\xi = \int_\Omega F\cdot u_{nt} J_w\, d\xi + \int_S G\cdot u_{nt} J_w d\xi_S. \tag{3.21}$$

From (3.21) in view of the Hölder and Young inequalities we obtain

$$\int_\Omega \eta u_{nt}^2 J_w\, d\xi + \frac{d}{dt}\int_\Omega |\mathbb{D}'_w(u_n)|^2 J_w\, d\xi \le c(|w_\xi|_{\infty,\Omega}^2 + 1)\int_\Omega |\mathbb{D}'_w(u_n)|^2 J_w\, d\xi + c\int_\Omega |\nabla_\xi u_n|^2 J_w\, d\xi + \varepsilon\|u_{nt}\|_{1,\Omega}^2 + c(\varepsilon)\|G\|_{0,S}^2 + c(\varrho_*)\|F\|_{0,\Omega}^2. \tag{3.22}$$

Integrating (3.22) with respect to time, using the Korn inequality and (3.8) we get

$$
\begin{aligned}
(3.23)\qquad & \int_{\Omega^t} \eta u_{nt}^2 J_w \, d\xi + c_0 \|u_n\|_{1,\Omega}^2 \\
&\le \exp\Big[c\Big(\int_0^t (\varepsilon_1\|w\|_{3,\Omega}^2 + c(\varepsilon_1)\|w\|_{0,\Omega}^2)\, dt + t\Big)\Big] \\
&\quad \times [\|u_0\|_{1,\Omega}^2 + \|u_n\|_{1,2,2,\Omega^t}^2 + \varepsilon_2\|u_{nt}\|_{1,2,2,\Omega^t}^2 \\
&\quad + c(\varepsilon_2)\|G\|_{0,S^t}^2 + c(\varrho_*)\|F\|_{0,\Omega^t}^2] + c\|u_n\|_{0,\Omega}^2.
\end{aligned}
$$

Using (3.9) in (3.23) yields

$$
\begin{aligned}
(3.24)\qquad & \|u_{nt}\|_{0,\Omega^t}^2 + c_0\|u_n\|_{1,\Omega}^2 \\
&\le \exp\Big[c\Big(\int_0^t (\varepsilon_1\|w\|_{3,\Omega}^2 + c(\varepsilon_1)\|w\|_{0,\Omega}^2)\, dt + t\Big)\Big] \\
&\quad \times \Big[\|u_0\|_{1,\Omega}^2 + \psi_1\Big(\int_\Omega \varrho_0 v_0^2\, dx + \|F\|_{0,\Omega^t}^2 + \|G\|_{0,S^t}^2\Big) \\
&\quad + \varepsilon_2\|u_{nt}\|_{1,2,2,\Omega^t}^2 + c(\varepsilon_2)\|G\|_{0,S^t}^2 + c(\varrho_*)\|F\|_{0,\Omega^t}^2\Big].
\end{aligned}
$$

From (3.24) we obtain (3.20). This concludes the proof.

Inserting the estimate for $\|u_n\|_{1,\Omega}^2$ from (3.20) into the r.h.s. of (3.16) and assuming that

$$
\varepsilon_2 \frac{\psi_3}{c_0}\psi_2 b(t,\varepsilon_1,w) = \frac{c_0}{2},
$$

where

$$
(3.25)\qquad b(t,\varepsilon_1,w) = \int_0^t (\varepsilon_1\|w\|_{3,\Omega}^2 + c(\varepsilon_1)\|w\|_{0,\Omega}^2)\, dt,
$$

we obtain

$$
\begin{aligned}
(3.26)\qquad & \|u_{nt}\|_{0,\Omega}^2 + \frac{c_0}{2}\|u_{nt}\|_{1,2,2,\Omega^t}^2 \\
&\le \psi_2\Big[\int_\Omega \varrho_0 u_t^2(0)\, d\xi + \|F_t\|_{0,\Omega^t}^2 + \|G_t\|_{0,S^t}^2 \\
&\quad + \sup_t(\|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2) b(t_1,\varepsilon_1,w)\Big] \\
&\quad + \frac{\psi_2\psi_3}{\mu} b\Big[\|u_0\|_{1,\Omega}^2 + \int_\Omega \varrho_0 v_0^2\, d\xi + \|F\|_{0,\Omega^t}^2 + c\Big(\frac{\psi_2\psi_3 b}{\mu^2}\Big)\|G\|_{0,S^t}^2\Big].
\end{aligned}
$$

Simplifying the expression we get

LEMMA 3.5.

$$
\begin{aligned}
(3.27)\qquad & \|u_{nt}\|_{0,\Omega}^2 + c_0\|u_{nt}\|_{1,2,2,\Omega^t}^2 \\
&\le \psi_4(t, 1/\varrho_*, a(w,t), b(t,\varepsilon_1,w), \|\eta_t\|_{1,2,2,\Omega^t}) \\
&\quad \times \Big[\int_\Omega \varrho_0 v_0^2\,d\xi + \int_\Omega \varrho_0 u_t^2(0)\,d\xi + \|u_0\|_{1,\Omega}^2 + \|F_t\|_{0,\Omega^t}^2 \\
&\quad + \|G_t\|_{0,S^t}^2 + \|F\|_{0,\Omega^t}^2 + \|G\|_{0,S^t}^2 + \sup_t(\|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2)\Big].
\end{aligned}
$$

From (3.27), (3.20) and (3.9) we get

LEMMA 3.6. *Let the assumptions of Lemmas* 3.2–3.4 *be satisfied. Then*

$$
\begin{aligned}
(3.28)\qquad & \|u_n\|_{0,\Omega}^2 + \|u_n\|_{1,\Omega}^2 + \|u_{nt}\|_{0,\Omega}^2 + \|u_n\|_{1,2,2,\Omega^t}^2 + \|u_{nt}\|_{1,2,2,\Omega^t}^2 \\
&\le \psi_5(t, 1/\varrho_*, a(w,t), b(t,\varepsilon_1,w), \|\eta_t\|_{1,2,2,\Omega^t}) \\
&\quad \times \Big[\int_\Omega \varrho_0 v_0^2\,dx + \int_\Omega \varrho_0 u_t^2(0)\,dx + \|u_0\|_{1,\Omega}^2 + \|F_t\|_{0,\Omega^t}^2 \\
&\quad + \|G_t\|_{0,\Omega^t}^2 + \sup_t(\|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2)\Big],
\end{aligned}
$$

where ψ_5 is an increasing positive function of its arguments.

Now choosing a subsequence and passing with n to infinity we get

LEMMA 3.7. *Assume that $\varrho_* \le \eta \le \varrho^*$, $w \in L_2(0,T;H^3(\Omega))$, $\eta_t \in L_2(0,T;H^1(\Omega))$, $v_0 \in H^1(\Omega)$, $u_t(0) \in L_2(\Omega)$, $F_t \in L_2(\Omega^T)$, $G_t \in L_2(S^T)$, $F \in L_\infty(0,T;L_2(\Omega))$ and $G \in L_\infty(0,T;L_2(S))$. Then there exists a weak solution of problem* (3.1) *such that $u \in L_\infty(0,T;H^1(\Omega)) \cap L_2(0,T;H^1(\Omega))$, $u_t \in L_2(0,T;H^1(\Omega)) \cap L_\infty(0,T;L_2(\Omega))$, and*

$$
\begin{aligned}
(3.29)\qquad & \|u\|_{1,\Omega}^2 + \|u_t\|_{0,\Omega}^2 + \|u\|_{1,2,2,\Omega^t}^2 + \|u_t\|_{1,2,2,\Omega^t}^2 \\
&\le \psi_5(t, 1/\varrho_*, a(w,t), b(t,\varepsilon_1,w), \|\eta_t\|_{1,2,2,\Omega^t}) \\
&\quad \times \Big[\int_\Omega \varrho_0 v_0^2\,dx + \int_\Omega \varrho_0 u_t^2(0)\,dx + \|v_0\|_{1,\Omega}^2 \\
&\quad + \|F_t\|_{0,\Omega^t}^2 + \|G_t\|_{0,S^t}^2 + \sup_t(\|F\|_{0,\Omega}^2 + \|G\|_{0,S}^2)\Big].
\end{aligned}
$$

Having proved the existence of weak solutions to problem (3.1) expressed by Lemma 3.7 we obtain by regularization techniques (see Appendix, Theorem 4.1 and Remark 4.2) the following result:

LEMMA 3.8. *Let the assumptions of Lemma* 3.7 *be satisfied. Let* $v_0 \in H^2(\Omega)$, $w \in L_2(0,T;H^3(\Omega))$, $F \in L_2(0,T;H^1(\Omega))$, $G \in L_2(0,T;H^{3/2}(S))$, $\eta \in L_\infty(0,T;H^2(\Omega))$, *and* $S \in H^{5/2}$. *Then there exists a unique solution to problem* (3.1) *such that* $u \in L_\infty(0,T;H^1(\Omega)) \cap L_2(0,T;H^3(\Omega))$, $u_t \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^1(\Omega))$, *and*

$$\begin{aligned}(3.30)\qquad & \|u_t\|^2_{0,\Omega} + \|u\|^2_{1,\Omega} + \|u\|^2_{3,2,2,\Omega^t} + \|u_t\|^2_{1,2,2,\Omega^t} \\ & \le \psi_6(t, 1/\varrho_*, a(w,t), b(t,\varepsilon_1,w), \|\eta_t\|_{1,2,2,\Omega^t}, \|\eta\|_{2,2,\infty,\Omega^t}) \\ & \quad \times \Big[\int\limits_\Omega \varrho_0 v_0^2\,dx + \int\limits_\Omega \varrho_0 u_t^2(0)\,dx + \|v_0\|^2_{2,\Omega} + \|F\|^2_{1,\Omega^t} + \|G_t\|^2_{0,S^t} \\ & \quad + \|G\|^2_{3/2,2,2,S^t} + \|F\|^2_{0,2,\infty,\Omega^t} + \|G\|^2_{0,2,\infty,S^t}\Big],\end{aligned}$$

where ψ_6 *is an increasing positive function.*

Now we prove the existence of solutions of (1.1) by the method of successive approximations determined by problems (1.13) and (1.14). First we show the boundedness of the sequence described by (1.13) and (1.14) in the norm defined in Lemma 3.8.

To simplify considerations let us introduce

$$(3.31)\qquad \alpha_m(t) = \|u_{mt}\|^2_{0,\Omega} + \|u_m\|^2_{1,\Omega} + \|u_m\|^2_{3,2,2,\Omega^t} + \|u_{mt}\|^2_{1,2,2,\Omega^t}.$$

LEMMA 3.9. *Assume that* $v_0 \in H^2(\Omega)$, $\varrho_0 \in H^2(\Omega)$, *and there exist two positive constants* ϱ_* *and* ϱ^*, $\varrho_* < \varrho^*$ *and* $\varrho_* \le \varrho_0 \le \varrho^*$.

$$\mathbb{T}(v_0, p(\varrho_0))\overline{n} = -p_0\overline{n} \quad \text{on } S.$$

Then for A *such that* $G(0,0,F_0) < A$, $\alpha_m(0) \le A$, *where* $F_0 = \|v_0\|^2_{2,\Omega} + \|u_t(0)\|^2_{0,\Omega} + \|\varrho_0\|^2_{2,\Omega}$ *and* G *is defined by the r.h.s. of* (3.41), *there exists* T_* *such that for* $t \le T_*$,

$$(3.32)\qquad \alpha_m(t) \le A, \quad m = 1, 2, \ldots$$

Moreover, in view of $(1.1)_1$ *we have*

$$\|u_t(0)\|_{0,\Omega} \le c\|v_0\|^2_{2,\Omega} + \frac{p'(\varrho^*)}{\varrho_*}\|\varrho_0\|_{1,\Omega} + \frac{\mu+\nu}{\varrho_*}\|v_0\|_{2,\Omega} + c\|\varrho_0\|_{0,\Omega}.$$

Proof. First we obtain estimates for solutions of (1.14). Integrating (1.14) we get

$$(3.33)\qquad \eta_m(\xi,t) = \varrho_0(\xi)\exp\Big[-\int\limits_0^t \operatorname{div}_{u_m} u_m\,d\tau\Big].$$

From (3.33) we have

$$\begin{aligned}\sup_{\Omega^t}\eta_m+\sup_{\Omega^t}\frac{1}{\eta_m}&\le\|\varrho_0\|_{2,\Omega}\varphi_1(a(u_m,t)),\\ \sup_t\|\eta_m\|_{2,\Omega}&\le\|\varrho_0\|_{2,\Omega}\varphi_1(a(u_m,t))\varphi_2(a(u_m,t)),\end{aligned}\tag{3.34}$$

where $a(u_m,t)=t^{1/2}(\int_0^t\|u_m\|_{3,\Omega}^2\,dt)^{1/2}$.

Moreover,

$$\eta_{mt}=\varrho_0(\xi)\exp\Big[-\int_0^t\operatorname{div}_{u_m}u_m\,d\tau\Big](-\operatorname{div}_{u_m}u_m).$$

Therefore

$$\|\eta_{mt}\|_{1,2,2,\Omega^t}\le\|\varrho_0\|_{2,\Omega}\varphi_3(a(u_m,t))b(t,\varepsilon,u_m),\tag{3.35}$$

where $b(t,\varepsilon,u_m)$ is defined by (3.25).

Comparing (3.1) with (1.13) we have

$$F=-\nabla_{u_m}q(\eta_m)+\eta_mU_{u_m}(\eta_m),\qquad G=-(p(\eta_m)-p_0)\overline{n}_{u_m}.\tag{3.36}$$

From (3.36) we have

$$\|F\|_{1,2,2,\Omega^t}^2+\|F\|_{0,2,\infty,\Omega^t}^2\le\varphi_4(t,a(u_m,t),\|\varrho_0\|_{2,\Omega})\tag{3.37}$$

and

$$\|F_t\|_{0,\Omega^t}^2\le\varphi_5(a(u_m,t),\sup_t\|\eta_m\|_{2,\Omega})b(t,\varepsilon,u_m).\tag{3.38}$$

Moreover,

$$\|G\|_{3/2,2,2,S^t}^2+\|G\|_{0,2,\infty,\Omega^t}^2\le\varphi_6(a(u_m,t),t,\sup_t\|\eta_m\|_{2,\Omega})\tag{3.39}$$

and

$$\|G_t\|_{0,S^t}^2\le\varphi_7(a(u_m,t),\sup_t\|\eta_m\|_{2,\Omega})b(t,\varepsilon,u_m).\tag{3.40}$$

Using the fact that

$$a^2(u_m,t)\le t\alpha_m,\qquad b(t,\varepsilon,u_m)\le t^a\alpha_m+cF_0,\qquad a>0,$$

and inserting all the above estimates into (3.30) we get

$$\alpha_{m+1}(t)\le G(t,t^a\alpha_m(t),F_0),\tag{3.41}$$

where $a>0$, $F_0=\|v_0\|_{2,\Omega}^2+\|u_t(0)\|_{0,\Omega}^2+\|\varrho_0\|_{2,\Omega}^2$, and G is an increasing positive function.

Let A be such that $G(0,0,F_0)<A$. Since G is a continuous increasing function of its arguments there exists $T_*>0$ such that for $t\le T_*$ we have

$$G(t,t^a\,A,F_0)\le A.\tag{3.42}$$

From (3.42) we see that if $\alpha_m(t) \le A$ then $\alpha_{m+1}(t) \le A$ for $t \le T_*$. Here A must be so large that $\alpha_m(0) \le A$.

To end the proof we have to construct the zero approximation function u_0. We use the solution of the problem

$$\begin{aligned} &u_{0t} - \operatorname{div} \mathbb{D}(u_0) = 0 && \text{in } \Omega^T,\\ &\overline{n} \cdot \mathbb{D}(u_0) = (p(\varrho_0) - p_0)\overline{n} && \text{on } S^T,\\ &u_0|_{t=0} = v_0 && \text{in } \Omega. \end{aligned}$$

The existence of solutions to the above problem follows from the Galerkin method and can be proved in the classes determined by $\alpha_0(t) < \infty$. Moreover, the compatibility condition is satisfied. Finally, A must be so large that $\alpha_0(t) \le A$, $t \le T_*$. This concludes the proof.

Now we prove the convergence of the sequence $\{u_m, \eta_m\}$.

To show this we obtain from (1.13) and (1.14) the following system of problems for the differences $U_m = u_m - u_{m-1}$ and $H_m = \eta_m - \eta_{m-1}$:

$$\begin{aligned} &\eta_m \partial_t U_{m+1} - \operatorname{div}_{u_m} \mathbb{D}_{u_m} U_{m+1}\\ &\qquad = -H_m \partial_t u_m - (\operatorname{div}_{u_m} \mathbb{D}_{u_m}(u_m) - \operatorname{div}_{u_{m-1}} \mathbb{D}_{u_{m-1}}(u_m))\\ &\qquad\quad - (\nabla_{u_m} - \nabla_{u_{m-1}}) q(\eta_m) - \nabla_{u_{m-1}}(q(\eta_m) - q(\eta_{m-1}))\\ &\qquad\quad + H_m U_{u_m}(\eta_m) + \eta_{m-1} U_{u_m}(H_m)\\ &\qquad\quad + \eta_{m-1}(U_{u_m}(\eta_{m-1}) - U_{u_{m-1}}(\eta_{m-1}))\\ &\qquad \equiv \sum_{i=1}^{7} F_i \equiv \widetilde{F},\\ &\mathbb{D}_{u_m}(U_{m+1}) \cdot \overline{n}_{u_m} = -(\mathbb{D}_{u_m}(u_m) \cdot \overline{n}_{u_m} - \mathbb{D}_{u_{m-1}}(u_m) \cdot \overline{n}_{u_{m-1}})\\ &\qquad\quad - q(\eta_m)(\overline{n}_{u_m} - \overline{n}_{u_{m-1}})\\ &\qquad\quad - (q(\eta_m) - q(\eta_{m-1}))\overline{n}_{u_{m-1}}\\ &\qquad\quad + p_0(\overline{n}_{u_m} - \overline{n}_{u_{m-1}})\\ &\qquad \equiv \sum_{i=1}^{4} G_i \equiv \widetilde{G},\\ &U_{m+1}|_{t=0} = 0, \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} &\partial_t H_m + H_m \operatorname{div}_{u_m} u_m = -\eta_{m-1}(\operatorname{div}_{u_m} u_m - \operatorname{div}_{u_{m-1}} u_{m-1}),\\ &H_m|_{t=0} = 0. \end{aligned} \tag{3.44}$$

Now we write the expressions on the r.h.s. of $(3.43)_1$ in qualitative forms:

$$
\begin{aligned}
F_2 &= f_1 \int_0^t U_{m\xi}\,d\tau\, u_{m\xi\xi} + f_2 \int_0^t U_{m\xi}\,d\tau \int_0^t d\tau\, u_{m\xi\xi} u_{m\xi}\\
&\quad + f_3 \int_0^t U_{m\xi}\,d\tau \int_0^t u_{m-1,\xi\xi}\,d\tau\, u_{m\xi} + f_4 \int_0^t U_{m\xi\xi}\,d\tau\, u_{m\xi},\\
F_3 &= f_5 f_1' \int_0^t U_{m\xi}\,d\tau\, \eta_{m\xi},\\
F_4 &= f_6 f_2' (\eta_{m\xi} + \eta_{m-1\xi}) H_m + f_7 f_3' H_{m\xi},\\
G_1 &= f_8 \int_0^t U_{m\xi}\,d\tau\, u_{m\xi},\\
G_2 &= f_9 f_4' \int_0^t U_{m\xi}\,d\tau,\\
G_3 &= p_0 f_{10} \int_0^t U_{m\xi}\,d\tau,\\
G_4 &= f_{11} f_5' H_m,
\end{aligned}
\tag{3.45}
$$

where $f_i = f_i(I + \int_0^t u_{m\xi}\,d\tau, I + \int_0^t u_{m-1,\xi}\,d\tau)$, $i = 1, \ldots, 11$, $f_j' = f_j'(\eta_m, \eta_{m-1})$, $j = 1, \ldots, 5$, are C^∞ functions of their arguments and I is the unit matrix. Moreover, we have the estimates

$$
|f_i| \le \overline{\varphi}_1(A), \qquad |f_j'| \le \overline{\varphi}_2(A)
\tag{3.46}
$$

where $\overline{\varphi}_1, \overline{\varphi}_2$ are increasing positive functions, for i, j as above.

Therefore we have

LEMMA 3.10. *Let the assumptions of Lemma* 3.9 *be satisfied. Then there exists* $0 < T^{**}$ *sufficiently small such that*

$$
\|U_{m+1}\|_{1,\Omega}^2 + \|U_{m+1,t}\|_{0,\Omega^t}^2 + \|U_{m+1}\|_{2,2,2,\Omega^t}^2 \le \delta \|U_m\|_{2,2,2,\Omega^t}^2,
\tag{3.47}
$$

where $\delta = \delta(t) < 1$ *for* $t \le T^{**}$.

Proof. To show (3.47) we multiply (3.43) by $U_{m+1} J_{u_m}$ and integrate over Ω. Therefore after integration by parts we get

$$
\begin{aligned}
\frac{1}{2} \int_\Omega \eta_m \frac{d}{dt} U_{m+1}^2 J_{u_m}\,d\xi &+ \int_\Omega |\mathbb{D}_{u_m}'(U_{m+1})|^2 J_{u_m}\,d\xi\\
&= \int_\Omega \widetilde{F} U_{m+1} J_{u_m}\,d\xi + \int_S \widetilde{G} U_{m+1} J_{u_m}\,d\xi_S.
\end{aligned}
\tag{3.48}
$$

First we estimate all terms on the r.h.s.:

$$\Big|\int\limits_{\Omega^t} H_m u_{mt} U_{m+1} J_{u_m}\,d\xi\,dt\Big| \le \varepsilon\|U_{m+1}\|^2_{1,2,2,\Omega^t} + c(\varepsilon)\varphi(A)t\sup_t\|u_{mt}\|^2_{0,\Omega}\sup_t\|H_m\|^2_{1,\Omega},$$

$$\Big|\int\limits_{\Omega^t} F_2 U_{m+1} J_{u_m}\,d\xi\,dt\Big| \le \varepsilon\|U_{m+1}\|^2_{1,2,2,\Omega^t} + c(\varepsilon)\varphi(A)t\int\limits_0^t\|U_m\|^2_{2,\Omega}\,dt,$$

$$\Big|\int\limits_{\Omega^t} F_3 U_{m+1} J_{u_m}\,d\xi\,dt\Big| \le \varepsilon\|U_{m+1}\|^2_{1,2,2,\Omega^t} + c(\varepsilon)\varphi(A)t\int\limits_0^t\|U_m\|^2_{2,\Omega}\,dt,$$

$$\Big|\int\limits_{\Omega^t} F_4 U_{m+1} J_{u_m}\,d\xi\,dt\Big| \le \varepsilon\|U_{m+1}\|^2_{1,2,2,\Omega^t} + c(\varepsilon)\varphi(A)t\sup_t\|H_m\|^2_{1,\Omega},$$

$$\Big|\int\limits_{\Omega^t} (F_5+F_6) U_{m+1} J_{u_m}\,d\xi\,dt\Big| \le \varepsilon\|U_{m+1}\|^2_{0,\Omega^t} + c(\varepsilon)\varphi(A)t\sup_t\|H_m\|^2_{0,\Omega},$$

$$\Big|\int\limits_{\Omega^t} F_7 U_{m+1} J_{u_m}\,d\xi\,dt\Big| \le \varepsilon\|U_{m+1}\|^2_{0,\Omega^t} + c(\varepsilon)\varphi(A)t\|U_m\|^2_{1,2,2,\Omega^t}.$$

Next we estimate the boundary term (3.48):

$$\Big|\int\limits_{S^t} (G_1+G_2+G_3) U_{m+1} J_{u_m}\,d\xi_s\,dt\Big| \le \varepsilon\|U_{m+1}\|^2_{1,2,2,\Omega^t} + c(\varepsilon)\varphi(A)t\|U_m\|^2_{2,2,2,\Omega^t},$$

$$\Big|\int\limits_{S^t} G_4 U_{m+1} J_{u_m}\,d\xi_s\,dt\Big| \le \varepsilon\|U_{m+1}\|^2_{1,2,2,\Omega^t} + c(\varepsilon)\varphi(A)t\sup_t\|H_m\|^2_{1,\Omega}.$$

Using the Korn inequality in (3.48), integrating with respect to time, using the above estimates and taking ε sufficiently small we obtain

$$\|U_{m+1}\|^2_{0,\Omega}+\|U_{m+1}\|^2_{1,2,2,\Omega^t} \le \varphi(A)t(\|U_m\|^2_{2,2,2,\Omega^t}+\sup_t\|H_m\|^2_{1,\Omega}). \tag{3.49}$$

Multiplying $(3.43)_1$ by $U_{m+1,t}J_{u_m}$, integrating over Ω and by parts we have

$$\int\limits_\Omega \eta_m|U_{m+1,t}|^2J_{u_m}\,d\xi + \int\limits_\Omega \mathbb{D}'_{u_m}(U_{m+1})\cdot\mathbb{D}'_{u_m}(U_{m+1,t})J_{u_m}\,d\xi = \int\limits_S \widetilde{G}U_{m+1,t}J_{u_m}\,d\xi + \int\limits_\Omega \widetilde{F}\cdot U_{m+1,t}J_{u_m}\,d\xi. \tag{3.50}$$

Continuing, we have

$$\begin{aligned}\int\limits_\Omega \eta_m|U_{m+1,t}|^2J_{u_m}\,d\xi &+ \int\limits_\Omega \mathbb{D}'_{u_m}(U_{m+1})\cdot\frac{d}{dt}\mathbb{D}'_{u_m}(U_{m+1})J_{u_m}\,d\xi\\ &- \int\limits_\Omega \mathbb{D}'_{u_m}(U_{m+1})\cdot\partial_t(\mathbb{D}'_{u_m})(U_{m+1})J_{u_m}\,d\xi\\ &= \frac{d}{dt}\int\limits_S \widetilde{G}\cdot U_{m+1}J_{u_m}\,d\xi_S - \int\limits_S \widetilde{G}_t\cdot U_{m+1}J_{u_m}\,d\xi_S\\ &- \int\limits_S \widetilde{G}\cdot U_{m+1}J_{u_m}\operatorname{div}_{u_m}u_m\,d\xi_S + \int\limits_\Omega \widetilde{F}\cdot U_{m+1,t}J_{u_m}\,d\xi.\end{aligned} \tag{3.51}$$

In view of the Hölder and Young inequalities we get

$$\begin{aligned}\int\limits_\Omega \eta_m|U_{m+1,t}|^2J_{u_m}\,d\xi + \frac{d}{dt}\int\limits_\Omega |\mathbb{D}'_{u_m}(U_{m+1})|^2J_{u_m}\,d\xi &\le \frac{d}{dt}\int\limits_S \widetilde{G}\cdot U_{m+1}J_{u_m}\,d\xi_S\\ &+ c\|u_m\|^2_{3,\Omega}\cdot\int\limits_\Omega |\mathbb{D}'_{u_m}(U_{m+1})|^2J_{u_m}\,d\xi\\ &+ c\int\limits_\Omega |\nabla_\xi U_{m+1}|^2J_{u_m}\,d\xi + c\|\widetilde{F}\|^2_{0,\Omega}\\ &+ \varepsilon_1(\|\widetilde{G}_t\|^2_{0,S} + \varphi(A)\|u_m\|^2_{3,\Omega}\|\widetilde{G}\|^2_{0,S})\\ &+ c(\varepsilon_1)\|U_{m+1}\|^2_{1,\Omega},\end{aligned} \tag{3.52}$$

where $\varepsilon_1 \in (0,1)$.

Integrating with respect to time and using the Korn inequality we obtain from (3.52)

$$\begin{aligned}&\|U_{m+1,t}\|^2_{0,\Omega} + \|U_{m+1}\|^2_{1,\Omega}\\ &\quad\le [\varepsilon_2\|\widetilde{G}\|^2_{0,S} + c(\varepsilon_2)(\varepsilon_3\|U_{m+1,\xi}\|^2_{0,\Omega} + c(\varepsilon_3)\|U_{m+1}\|^2_{0,\Omega})\\ &\qquad + c\|U_{m+1}\|^2_{1,2,2,\Omega^t} + c\|\widetilde{F}\|^2_{0,\Omega^t}\\ &\qquad + \varepsilon_1\varphi(A)(\|\widetilde{G}_t\|^2_{0,S^t} + \sup_t\|\widetilde{G}\|^2_{0,S})]e^A + c\|U_{m+1}\|^2_{0,\Omega},\end{aligned} \tag{3.53}$$

where we used the facts that

$$\int_S \widetilde{G}\cdot U_{m+1}J_{u_m}\,d\xi \le \varepsilon_2\|\widetilde{G}\|_{0,S}^2 + c(\varepsilon_2)\|U_{m+1}\|_{0,S}^2,$$

$$\|U_{m+1}\|_{0,S}^2 \le \varepsilon_3\|U_{m+1\xi}\|_{0,\Omega}^2 + c(\varepsilon_3)\|U_{m+1}\|_{0,\Omega}^2,$$

$$\int_0^t \|u_m\|_{3,\Omega}^2\|\widetilde{G}\|_{0,S}^2\,dt \le \sup_t\|\widetilde{G}\|_{0,S}^2\int_0^t\|u_m\|_{3,\Omega}^2\,dt \le A\sup_t\|\widetilde{G}\|_{0,S}^2.$$

Using (3.49) in (3.53) implies

$$\|U_{m+1}\|_{1,\Omega}^2 + \|U_{m+1,t}\|_{0,\Omega^t}^2 + \|U_{m+1}\|_{1,2,2,\Omega^t}^2 \le \varphi(A)[\varepsilon(\|\widetilde{G}_t\|_{0,S^t}^2 + \sup_t\|\widetilde{G}\|_{0,S}^2) + \|\widetilde{F}\|_{0,\Omega^t}^2] + \varphi(A)t(\|U_m\|_{2,2,2,\Omega^t}^2 + \sup_t\|H_m\|_{1,\Omega}^2). \tag{3.54}$$

Now from the regularity result for the parabolic problem

$$\begin{aligned} &\eta_m U_{m+1,t} - \operatorname{div}_{u_m}\mathbb{D}_{u_m}(U_{m+1}) = \widetilde{F} && \text{in } \Omega^T,\\ &\mathbb{D}_{u_m}(U_{m+1})\cdot\overline{n}_{u_m} = \widetilde{G} && \text{on } S^T,\\ &U_{m+1}|_{t=0} = 0 && \text{in } \Omega,\end{aligned}$$

we obtain (see Theorem 4.1 and Remark 4.2)

$$\|U_{m+1}\|_{2,2,2,\Omega^t}^2 \le c(\|\widetilde{F}\|_{0,\Omega^t}^2 + \|\widetilde{G}\|_{1/2,2,2,S^t}^2) + c\|U_{m+1}\|_{0,\Omega^t}^2. \tag{3.56}$$

Now collecting (3.49), (3.54) and (3.56) together, we get

$$\|U_{m+1}\|_{1,\Omega}^2 + \|U_{m+1,t}\|_{0,\Omega^t}^2 + \|U_{m+1}\|_{2,2,2,\Omega^t}^2 \le \varphi(A)[\varepsilon\sup_t\|\widetilde{G}\|_{0,S}^2 + \varepsilon\|\widetilde{G}_t\|_{0,S^t}^2 + \|\widetilde{F}\|_{0,\Omega^t}^2 + \|\widetilde{G}\|_{1/2,2,2,S^t}^2] + \varphi(A)t(\|U_m\|_{2,2,2,\Omega^t}^2 + \sup_t\|H_m\|_{1,\Omega}^2). \tag{3.57}$$

Using the form of $\widetilde{F}$ and $\widetilde{G}$ we estimate the terms in the first bracket on the r.h.s. of (3.57):

$$\begin{aligned} &\|\widetilde{G}\|_{0,S}^2 \le \varphi(A)\Big[t\int_0^t\|U_m\|_{2,\Omega}^2\,dt + \sup_t\|H_m\|_{1,\Omega}^2\Big],\\ &\|\widetilde{G}_t\|_{0,S^t}^2 \le \varphi(A)\Big[\int_0^t\|U_m\|_{2,\Omega}^2\,dt + \int_0^t\|H_{mt}\|_{1,\Omega}^2\,dt\Big],\\ &\|\widetilde{G}\|_{1/2,2,2,S^t}^2 + \|\widetilde{F}\|_{0,\Omega^t}^2 \le \varphi(A)t\Big[\int_0^t\|U_m\|_{2,\Omega}^2\,dt + \sup_t\|H_m\|_{1,\Omega}^2\Big].\end{aligned} \tag{3.58}$$

Using the equation (3.47) we have the estimate

$$\int_0^t \|H_{mt}\|_{1,\Omega}^2\,dt \le \varphi(A)(t\sup_t\|H_m\|_{1,\Omega}^2 + \|U_m\|_{2,2,2,\Omega^t}^2). \tag{3.59}$$

From (3.57)–(3.59) it follows that

$$\begin{aligned}\|U_{m+1}\|_{1,\Omega}^2 + \|U_{m+1,t}\|_{0,\Omega^t}^2 + \|U_{m+1}\|_{2,2,2,\Omega^t}^2 \\ \le \varphi(A)(c(\varepsilon)t+\varepsilon)(\|U_m\|_{2,2,2,\Omega^t}^2 + \sup_t\|H_m\|_{1,\Omega}^2).\end{aligned} \tag{3.60}$$

Integrating (3.44) we respect to time yields

$$\begin{aligned} &H_m(\xi,t) \\ &= -\exp\Big[-\int_0^t \mathrm{div}_{u_m} u_m\,d\tau\Big]\int_0^t\Big[\eta_{m-1}(\mathrm{div}_{u_m} u_m - \mathrm{div}_{u_{m-1}} u_{m-1}) \\ &\quad\times\exp\Big(\int_0^{t'}\mathrm{div}_{u_m} u_m\,dt''\Big)\Big]\,dt',\end{aligned} \tag{3.61}$$

hence one has

$$\sup_t\|H_m\|_{1,\Omega}^2 \le \varphi(A)t\|U_m\|_{2,2,2,\Omega^t}^2. \tag{3.62}$$

Using (3.62) in (3.60) yields

$$\begin{aligned}\|U_{m+1}\|_{1,\Omega}^2 + \|U_{m+1,t}\|_{0,\Omega^t}^2 + \|U_{m+1}\|_{2,2,2,\Omega^t}^2 \\ \le \varphi(A)(c(\varepsilon)t+\varepsilon)\|U_m\|_{2,2,2,\Omega^t}^2.\end{aligned} \tag{3.63}$$

Therefore for t so small that

$$(c(\varepsilon)t+\varepsilon)\varphi(A) < 1 \tag{3.64}$$

we have convergence of the sequence $\{u_m,\eta_m\}$ to a solution. Assume that (3.64) holds for $t \le T_{**}$. This concludes the proof.

From Lemmas 3.9 and 3.10 we have

THEOREM 3.11. *Let the assumptions of Lemmas* 3.9 *and* 3.10 *be satisfied. Then there exists T^{**} sufficiently small such that for $T \le T^{**}$ there exists a solution to problem* (1.1) *such that*

$$u \in L_\infty(0,T;H^1(\Omega)) \cap L_2(0,T;H^3(\Omega)),$$
$$u_t \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^1(\Omega))$$

and

$$\|u\|_{1,2,\infty,\Omega^T} + \|u\|_{3,2,2,\Omega^T} + \|u_t\|_{0,2,\infty,\Omega^T} + \|u_t\|_{1,2,2,\Omega^T} \le A, \tag{3.65}$$

where A is defined in Lemma 3.9. *Moreover*, $\eta, 1/\eta \in L_\infty(\Omega^T) \cap L_\infty(0,T;H^2(\Omega))$, $\eta_t, (1/\eta)_t \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^2(\Omega))$, $\eta_{tt}, (1/\eta)_{tt}$

$\in L_2(\Omega^T)$, *and*

$$\|\chi\|_{2,2,\infty,\Omega^T} + \|\chi_t\|_{0,2,\infty,\Omega^T} + \|\chi_t\|_{2,2,2,\Omega^T} + \|\chi_{tt}\|_{0,\Omega^T} \le \varphi(A), \tag{3.66}$$

where χ replaces either η or $1/\eta$ and φ is some positive function.

P r o o f. We only have to show the last statement and the estimate (3.66). They follow from the expression for η,

$$\eta(\xi,t) = \varrho_0(\xi)\exp\Big(-\int_0^t \operatorname{div}_u u(\xi,\tau)\,d\tau\Big). \tag{3.67}$$

The most difficult part is to estimate η_{tt}. Taking the second derivative of η with respect to time we obtain

$$\eta_{tt} = \varrho_0 \exp\Big(-\int_0^t \operatorname{div}_u u(\xi,\tau)\,d\tau\Big)(-(\operatorname{div}_u u)_t + (\operatorname{div}_u u)^2).$$

Since the first two factors are bounded we only consider the last bracket.

Qualitatively,

$$\operatorname{div}_u u = f_1\Big(\int_0^t u_\xi\,d\tau\Big)u_\xi^2,$$

where f_1 is a smooth function and $f_1(\int_0^t u_\xi\,d\tau)$ is bounded. Next in view of (3.65),

$$|u_\xi^2|_{2,\Omega^t} \le \Big(\int_0^t |u_\xi|_{\infty,\Omega}^2\,dt\Big)^{1/2} \sup_t |u_\xi|_{2,\Omega} \le A^2.$$

Similarly,

$$(\operatorname{div}_u u)_t = f_2\Big(\int_0^t u_\xi\,d\tau\Big)u_{\xi t} + f_3\Big(\int_0^t u_\xi\,d\tau\Big)u_\xi^2,$$

where f_2, f_3 are smooth functions and the same considerations as above can be applied. This concludes the proof.

4. Appendix. In this section we show the regularity of solutions to problem (1.12). First we consider the problem

$$\begin{aligned} &\eta u_t - \operatorname{div}\mathbb{D}(u) = F &&\text{in } \Omega^T,\\ &\overline{n}\cdot\mathbb{D}(u) = G &&\text{on } S^T,\\ &u|_{t=0} = v_0 &&\text{in } \Omega. \end{aligned} \tag{4.1}$$

We examine (4.1) using the following weak formulation:

$$\int_\Omega \eta u_t\varphi\,dx + \int_\Omega \mathbb{D}'(u)\cdot\mathbb{D}'(\varphi)\,dx = \int_\Omega F\cdot\varphi\,dx + \int_S G\varphi\,ds. \tag{4.2}$$

To examine regularity we only have to consider the integral

$$K(\varphi) := \int_{\Omega} \mathbb{D}'(u) \cdot \mathbb{D}'(\varphi)\, dx. \tag{4.3}$$

Set $\varphi = \zeta\varphi_1$, where ζ is a smooth function with a support in $\widetilde{\Omega} \subset \Omega$ and φ_1 is a test function. Then we get

$$K(\zeta\varphi_1) = \int_{\widetilde{\Omega}} [\mathbb{D}'(u) \cdot \mathbb{D}'(\zeta)\varphi_1 + \mathbb{D}'(u') \cdot \mathbb{D}'(\varphi_1) - u\mathbb{D}'(\zeta) \cdot \mathbb{D}'(\varphi_1)]\, dx, \tag{4.4}$$

where $u' = u\zeta$.

In further considerations we choose $\widetilde{\Omega}$ such that $\widetilde{\Omega} \cap S \neq \emptyset$. Therefore we apply the transformation $\Phi : \widetilde{\Omega} \to \widehat{\Omega}$ which straightens locally the boundary of Ω. Hence (4.4) takes the form

$$K(\widehat{\zeta}\widehat{\varphi}_1) = \int_{\widehat{\Omega}} [\mathbb{D}'_{\Phi}(\widehat{u}) \cdot \mathbb{D}'_{\Phi}(\widehat{\zeta})\widehat{\varphi}_1 + \mathbb{D}'_{\Phi}(\widetilde{u}) \cdot \mathbb{D}'_{\Phi}(\widehat{\varphi}_1) - \widehat{u}\mathbb{D}'_{\Phi}(\widehat{\zeta}) \cdot \mathbb{D}'_{\Phi}(\widehat{\varphi}_1)] J_{\Phi}\, dz, \tag{4.5}$$

where $\widetilde{\Omega} \ni x \to \Phi(x) = z \in \widehat{\Omega}$, $\widehat{u} = u \circ \Phi^{-1}$, $\widetilde{u} = \widehat{u}\widehat{\zeta}$, $\mathbb{D}_{\Phi}$ is such that ∇_x in $\mathbb{D}$ are replaced by $\nabla_x \Phi(x)|_{x=\Phi^{-1}(z)} \cdot \nabla_z$ and J_{Φ} is the Jacobian determinant of the transformation $z = \Phi(x)$. We also need the fact that $\widehat{\Omega} = \{z \in \mathbb{R}^3 : |z_i| < d,\ i = 1, 2,\ 0 < z_3 < d\}$, $\widehat{S} = \Phi(\widetilde{S}) = \{z \in \mathbb{R}^3 : |z_i| < d,\ i = 1, 2,\ z_3 = 0\}$, and $\widetilde{S} = \widetilde{\Omega} \cap S$. Since the integrand in (4.5) vanishes on $\partial\widehat{\Omega} \setminus \widehat{S}$ it can be extended by zero on $\mathbb{R}^3_+ = \{z \in \mathbb{R}^3 : z_3 > 0\}$. Therefore, we assume $\widehat{\varphi}_1 = \delta_h^{-1}\delta_h\widetilde{u}$, where $\delta_h u(z) = \frac{1}{h}(u(z' + h, z_3) - u(z))$, $z' = (z_1, z_2)$, corresponds only to the tangent directions, which will also be denoted by τ. Then from (4.5) under the assumption that S and hence Φ are smooth, we have the estimate

$$K(\widehat{\zeta}\delta_h^{-1}\delta_h\widetilde{u}) \geq \frac{c_0}{2}\|\delta_h\widetilde{u}\|^2_{1,\widehat{\Omega}} - \varepsilon\|\mathbb{D}'_{\Phi}(\delta_h\widehat{u})\|^2_{0,\widehat{\Omega}} - c\|\widehat{u}\|^2_{1,\widehat{\Omega}}, \tag{4.6}$$

where $\varepsilon \in (0, 1)$.

Now we consider the first term in (4.2):

$$\begin{aligned} \int_{\Omega} \eta u_t \varphi\, dx &= \int_{\widetilde{\Omega}} \eta u'_t \varphi_1\, dx = \int_{\widehat{\Omega}} \widehat{\eta}\widetilde{u}_t \widehat{\varphi}_1 J_{\Phi}\, dz \\ &= \int_{\widehat{\Omega}} \delta_h\widehat{\eta} J_{\Phi}\widetilde{u}_t\delta_h\widehat{u}\, dz + \int_{\widehat{\Omega}} \widehat{\eta}\delta_h J_{\Phi}\widetilde{u}_t\delta_h\widehat{u}\, dz + \int_{\widehat{\Omega}} \widehat{\eta}\delta_h\widetilde{u}_t\delta_h\widehat{u} J_{\Phi}\, dz, \end{aligned} \tag{4.7}$$

where the last term is equal to

$$\frac{1}{2}\int_{\widehat{\Omega}} \widehat{\eta}\frac{d}{dt}|\delta_h\widetilde{u}|^2 J_{\Phi}\, dz = \frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}} \widehat{\eta}|\delta_h\widetilde{u}|^2 J_{\Phi}\, dz - \frac{1}{2}\int_{\widehat{\Omega}} \widehat{\eta}_t|\delta_h\widetilde{u}|^2 J_{\Phi}\, dz,$$

and the first two terms on the r.h.s. of (4.7) are bounded by

$$\varepsilon\|\delta_h\widehat{u}\|^2_{0,\widehat{\Omega}}+c(\varepsilon)\|\widehat{\eta}\|^2_{2,\Omega}\|\widehat{u}_t\|^2_{1,\widehat{\Omega}},$$

where $\varepsilon\in(0,1)$.

Finally we consider the terms on the r.h.s. of (4.2):

$$\int\limits_{\Omega}F\cdot\varphi\,dx=\int\limits_{\Omega}\widetilde{F}\delta_{-h}\delta_h\widetilde{u}J_\Phi\,dz,$$

which is bounded by $\varepsilon\|\delta_h\widetilde{u}\|^2_{1,\widehat{\Omega}}+c(\varepsilon)\|\widetilde{F}\|^2_{0,\widehat{\Omega}}$, and

$$\int\limits_{S}G\cdot\varphi\,ds=\int\limits_{\widehat{S}}\widetilde{G}\delta_{-h}\delta_h\widetilde{u}J_\Phi\,dz_S,$$

which is bounded by $\varepsilon\|\delta_h\widetilde{u}\|^2_{1,\widehat{\Omega}}+c(\varepsilon)\|\widetilde{G}\|^2_{1/2,\widehat{S}}$. In view of the above considerations we deduce from (4.2) the inequality

$$\begin{aligned}(4.8)\quad \frac12\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}|\delta_h\widetilde{u}|^2J_\Phi\,dz+\frac{c_0}{2}\|\delta_h\widetilde{u}\|^2_{1,\widehat{\Omega}}&\\ \le c\|\widehat{\eta}\|^2_{2,\widehat{\Omega}}\|\widetilde{u}_t\|^2_{1,\widehat{\Omega}}+c\|\widehat{\eta}_t\|^2_{1,\widehat{\Omega}}\|\delta_h\widetilde{u}\|^2_{0,\widehat{\Omega}}+\varepsilon\|\delta_h\widetilde{u}\|^2_{1,\widehat{\Omega}}&\\ +c\|\widehat{u}\|^2_{1,\widehat{\Omega}}+c(\|\widetilde{F}\|^2_{0,\widetilde{\Omega}}+\|\widetilde{G}\|^2_{1/2,\widehat{S}}).&\end{aligned}$$

Integrating (4.8) with respect to time, going back to the old variables, summing over all neighbourhoods of the partition of unity, using the fact that ε is sufficiently small and passing with h to 0 we get

$$\begin{aligned}(4.9)\quad \int\limits_{\Omega}\eta u_\tau^2\,dx+\mu\|u_\tau\|^2_{1,2,2,\Omega^t}&\\ \le\int\limits_{\Omega}\varrho_0v_{0\tau}^2\,dx+\sup_t\|\eta\|^2_{2,\Omega}\|u_t\|^2_{1,2,2,\Omega^t}&\\ +\sup_t\|\eta_t\|^2_{1,\Omega}\|u\|^2_{1,2,2,\Omega^t}&\\ +c(\|u\|^2_{1,2,2,\Omega^t}+\|F\|^2_{0,\Omega^t}+\|G\|^2_{1/2,2,2,S^t}),&\end{aligned}$$

where u_τ denotes the tangent derivatives to the boundary.

To calculate the normal derivatives we use the equation

$$(4.10)\qquad \operatorname{div}\mathbb{D}(u)=F-\eta u_t$$

to get

$$(4.11)\qquad \|u_n\|^2_{1,\Omega}\le c(\|F\|^2_{0,\Omega}+\|u_\tau\|^2_{1,\Omega})+c\|\eta\|^2_{1,\Omega}\|u_t\|^2_{1,\Omega}.$$

Summarizing the above considerations we see that $u \in L_2(0,T;H^2(\Omega))$ and

$$\int\limits_\Omega \eta u_x^2\,dx + c_0\|u\|^2_{2,2,2,\Omega^t} \le \int\limits_\Omega \varrho_0 v_{0x}^2\,dx + \|\eta\|^2_{2,2,\infty,\Omega^t}\|u_t\|^2_{1,2,2,\Omega^t} + \|\eta_t\|^2_{1,2,\infty,\Omega^t}\|u\|^2_{1,2,2,\Omega^t} + c(\|u\|^2_{1,2,2,\Omega^t} + \|F\|^2_{0,\Omega^t} + \|G\|^2_{1/2,2,2,S^t}), \tag{4.12}$$

where $t \le T$, and the r.h.s. of (4.12) is bounded in terms of the estimates for the weak solutions (see Lemma 3.7).

Now we show the $H^3(\Omega)$ regularity of u under the assumption that S is smooth. To this end we consider the problem (4.1) directly. To obtain the estimate we have to consider the problem locally. We shall restrict our considerations to neighbourhoods close to the boundary only. Under the above assumptions we write problem (4.1) locally in the form

$$\begin{gathered} \widehat{\eta}\widetilde{u}_t - \operatorname{div}_\Phi \mathbb{D}_\Phi(\widetilde{u}) = -\operatorname{div}_\Phi \mathbb{B}_\Phi(\widehat{u},\widehat{\zeta}) - \mathbb{D}_\Phi(\widehat{u})\cdot\nabla_\Phi\widehat{\zeta} + \widetilde{F} \equiv \widehat{F}_1 + \widetilde{F}, \\ \mathbb{D}_\Phi(\widetilde{u})\widehat{n} = \mathbb{B}_\Phi(\widehat{u},\widehat{\zeta})\cdot\widehat{n} + \widetilde{G} \equiv \widehat{G}_1 + \widetilde{G}, \qquad \text{where} \\ \mathbb{B}_{\Phi ij}(\widehat{u},\widehat{\zeta}) = (\widehat{u}_i\nabla_{\Phi j}\widehat{\zeta} + \widehat{u}_j\nabla_{\Phi_i}\widehat{\zeta}) + (\nu-\mu)\delta_{ij}\widehat{u}\cdot\nabla_\Phi\widehat{\zeta}. \end{gathered} \tag{4.13}$$

Now we apply the Friedrichs mollifier operator j_δ to $(4.13)_1$. Hence we get

$$\begin{aligned} j_\delta(\widehat{\eta}\widetilde{u}_t) - \operatorname{div}_\Phi \mathbb{D}_\Phi(j_\delta\widetilde{u}) &= j_\delta \operatorname{div}_\Phi \mathbb{D}_\Phi(\widetilde{u}) - \operatorname{div}_\Phi \mathbb{D}_\Phi(j_\delta\widetilde{u}) \\ &\quad - j_\delta(\operatorname{div}_\Phi \mathbb{B}_\Phi(\widehat{u},\widehat{\zeta})) - j_\delta\big(\mathbb{D}_\Phi\widehat{u}\nabla_\Phi\widehat{\zeta}\big) + j_\delta\widetilde{F}. \end{aligned} \tag{4.14}$$

Next, we differentiate (4.14) two times with respect to τ, multiply by $\partial_\tau^2 j_\delta\widetilde{u}J_\Phi$ and integrate over $\widehat{\Omega}$ to get

$$\begin{aligned} &\int\limits_{\widehat{\Omega}} \partial_\tau^2 j_\delta(\widehat{\eta}\widetilde{u}_t)\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi\,dz - \int\limits_{\widehat{\Omega}} \partial_\tau^2 \operatorname{div}_\Phi \mathbb{D}_\Phi j_\delta(\widetilde{u})\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi\,dz \\ &\quad = \int\limits_{\widehat{\Omega}} \partial_\tau^2\big[j_\delta(\operatorname{div}_\Phi \mathbb{D}_\Phi(\widetilde{u})) - \operatorname{div}_\Phi \mathbb{D}_\Phi j_\delta(\widetilde{u})\big]\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi\,dz \\ &\qquad - \int\limits_{\widehat{\Omega}} \partial_\tau^2[j_\delta(\operatorname{div}_\Phi \mathbb{B}_\Phi(\widehat{u},\widehat{\zeta})) + j_\delta(\mathbb{D}_\Phi(\widehat{u})\nabla_\Phi\zeta^2)]\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi\,dz \\ &\qquad + \int\limits_{\widehat{\Omega}} \partial_\tau^2 j_\delta(\widetilde{F})\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi\,dz. \end{aligned} \tag{4.15}$$

Now we examine the particular terms in (4.15). The first term in (4.15) takes the form

$$\int\limits_{\widehat{\Omega}} \partial_\tau^2(\widehat{\eta} j_\delta(\widetilde{u}_t))\partial_\tau^2 j_\delta(\widetilde{u}) J_\Phi\, dz$$
$$+ \int\limits_{\widehat{\Omega}} \partial_\tau^2(j_\delta(\widehat{\eta}\widetilde{u}_t) - (\widehat{\eta} j_\delta(\widetilde{u}_t))\partial_\tau^2 j_\delta(\widetilde{u}) J_\Phi\, dz \equiv I_1 + I_2,$$

where

$$\begin{aligned} I_1 = &\int\limits_{\widehat{\Omega}} [\partial_\tau^2 \widehat{\eta} j_\delta(\widetilde{u}_t) + 2\partial_\tau \widetilde{\eta}\partial_\tau j_\delta(\widetilde{u}_t) + \widehat{\eta}\partial_\tau^2 j_\delta(\widetilde{u}_t)]\partial_\tau^2 j_\delta(\widetilde{u}) J_\Phi\, dz \\ &\equiv I_{11} + I_{12} + I_{13}, \end{aligned}$$

and

$$I_{11} \le \varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2 + c(\varepsilon)\|\widehat{\eta}\|_{2,\widehat{\Omega}}^2 \|j_\delta(\widetilde{u}_t)\|_{1,\widehat{\Omega}}^2,$$
$$I_{12} \le \varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2 + c(\varepsilon)\|\widehat{\eta}\|_{2,\widehat{\Omega}}^2 \|j_\delta(\widetilde{u}_t)\|_{1,\widehat{\Omega}}^2.$$

Finally, we have

$$\begin{aligned} I_{13} &= \frac{1}{2}\int\limits_{\widehat{\Omega}} \widehat{\eta}\frac{d}{dt}|\partial_\tau^2 j_\delta(\widetilde{u})|^2 J_\Phi\, dz \\ &= \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \widehat{\eta}|\partial_\tau^2 j_\delta(\widetilde{u})|^2 J_\Phi\, dz - \frac{1}{2}\int\limits_{\widehat{\Omega}} \widehat{\eta}_t|\partial_\tau^2 j_\delta(\widetilde{u})|^2 J_\Phi\, dz \\ &\equiv I_{14} + I_{15}, \end{aligned}$$

where

$$I_{15} \le \varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2 + c(\varepsilon)\|\widehat{\eta}_t\|_{1,\widehat{\Omega}}^2 \|\partial_\tau^2 j_\delta(\widetilde{u})\|_{0,\widehat{\Omega}}^2.$$

Next we examine I_2,

$$\begin{aligned} I_2 = &\int\limits_{\widehat{\Omega}} \partial_\tau^2\Big[\int w_\delta(z-y)(\widehat{\eta}(y) - \widehat{\eta}(z))\widetilde{u}_t(y)\, dy\Big]\partial_\tau^2 j_\delta(\widetilde{u}) J_\Phi\, dz \\ = &-\int\limits_{\widehat{\Omega}} \partial_\tau\Big[\int w_\delta(z-y)(\widehat{\eta}(y) - \widehat{\eta}(z))\widetilde{u}_t(y)\, dy\Big] \\ &\times (\partial_\tau^3 j_\delta(\widetilde{u}) J_\Phi + \partial_\tau^2 j_\delta(\widetilde{u})\partial_\tau J_\Phi)\, dz, \end{aligned}$$

where w_δ is the smooth kernel of the mollifier operator j_δ. Continuing we have

$$|I_2| \le \varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2 + c(\varepsilon)\|\widehat{\eta}\|_{2,\widehat{\Omega}}^2 \|\widetilde{u}_t\|_{1,\widehat{\Omega}}^2.$$

Now we examine the second term on the l.h.s. of (4.15). It takes the form

$$\begin{aligned} I_3 = &-\int\limits_{\widehat{\Omega}} [\partial_\tau^2(\mathrm{div}_\Phi\, \mathbb{D}_\Phi)(j_\delta(\widetilde{u})) + 2\partial_\tau(\mathrm{div}_\Phi\, \mathbb{D}_\Phi)(\partial_\tau j_\delta(\widetilde{u})) \\ &+ \mathrm{div}_\Phi\, \mathbb{D}_\Phi(\partial_\tau^2 j_\delta(\widetilde{u}))]\partial_\tau^2 j_\delta(\widetilde{u}) J_\Phi\, dz \equiv I_{31} + I_{32} + I_{33}, \end{aligned}$$

where using the smoothness of S we get

$$|I_{31}| \le \varepsilon \|j_\delta(\widetilde{u})\|^2_{2,\widehat{\Omega}} + c(\varepsilon)\|\partial^2_\tau j_\delta(\widetilde{u})\|^2_{0,\widehat{\Omega}},$$

$$|I_{32}| \le \varepsilon \|\partial_\tau j_\delta(\widetilde{u})\|^2_{2,\widehat{\Omega}} + c(\varepsilon)\|\partial^2_\tau j_\delta(\widetilde{u})\|^2_{0,\widehat{\Omega}},$$

and

$$\begin{aligned} I_{33} = &-\int_{\widehat{S}} \partial^2_\tau(\widehat{n}\mathbb{D}_\Phi)(j_\delta(\widetilde{u}))\partial^2_\tau j_\delta(\widetilde{u}) J_\Phi\, dz_S \\ &+\int_{\widehat{S}} [2\partial_\tau(\widehat{n}\mathbb{D}_\Phi)(\partial_\tau j_\delta(\widetilde{u})) + \partial^2_\tau(\widehat{n}\cdot\mathbb{D}_\Phi)(j_\delta(\widetilde{u}))]\partial^2_\tau j_\delta(\widetilde{u}) J_\Phi\, dz_S \\ &+\int_{\widehat{S}} |\mathbb{D}'_\Phi(\partial^2_\tau j_\delta(\widetilde{u}))|^2 J_\Phi dz_S \equiv I_4 + I_5 + I_6, \end{aligned}$$

where to examine I_4 we rewrite the boundary condition $(4.13)_2$ by applying the Friedrichs mollifier in the form

$$\widehat{n}\cdot\mathbb{D}_\Phi(j_\delta(\widetilde{u})) = \widehat{n}\cdot\mathbb{D}_\Phi(j_\delta(\widetilde{u})) - j_\delta(\widehat{n}\cdot\mathbb{D}_\Phi(\widetilde{u})) + j_\delta(\mathbb{B}_\Phi(\widehat{u},\widehat{\zeta})\cdot\widehat{n}) + j_\delta(\widehat{G}).$$

Therefore, I_4 takes the form

$$\begin{aligned} I_4 = &-\int_{\widehat{S}} \partial^2_\tau[\widehat{n}\cdot\mathbb{D}_\Phi(j_\delta(\widetilde{u})) - j_\delta(\widehat{n}\cdot\mathbb{D}_\Phi(\widetilde{u}))]\partial^2_\tau j_\delta(\widetilde{u}) J_\Phi\, dz_S \\ &-\int_{\widehat{S}} \partial^2_\tau j_\delta(\mathbb{B}_\Phi(\widehat{u},\widehat{\zeta})\cdot\widehat{n})\cdot\partial^2_\tau j_\delta(\widetilde{u}) J_\Phi\, dz_S \\ &-\int_{\widehat{S}} \partial^2_\tau(j_\delta(\widehat{G}))\partial^2_\tau(j_\delta(\widetilde{u})) J_\Phi\, dz_S \equiv I_{41} + I_{42} + I_{43}, \end{aligned}$$

where the expression I_{41} qualitatively has the form

$$\begin{aligned} I_{41} = &-\int_{\widehat{S}} \partial^2_\tau \int w_\delta(z-y)(\nabla\Phi(z) - \nabla\Phi(y))\nabla_y\widetilde{u}(y)\, dy \partial^2_\tau j_\delta(\widetilde{u}) J_\Phi\, dz_S \\ = &-c_0\int_{\widehat{S}} \partial^{3/2}_\tau \int w_\delta(z-y)(\nabla\Phi(z) - \nabla\Phi(y))\nabla_y\widetilde{u}(y)\, dy \\ &\times \partial^{1/2}_\tau(\partial^2_\tau j_\delta(\widetilde{u}) J_\Phi) dz_S \equiv K, \end{aligned}$$

where c_0 is the constant from integration by parts with the $1/2$ derivative (to use such derivatives we have to apply the Fourier transform).

Continuing, we have

$$\begin{aligned} K = &-c_0\int_{\widehat{S}} \Big[\partial^{1/2}_\tau \int \partial_\tau w_\delta(z-y)(\nabla\Phi(z) - \nabla\Phi(y))\nabla_y\widetilde{u}(y)\, dy \\ &+ \partial^{1/2}_\tau \nabla^2\Phi(z)\cdot\int w_\delta(z-y)\nabla_y\widetilde{u}(y)\, dy\Big] \\ &\times \partial^{1/2}_\tau(\partial^2_\tau j_\delta(\widetilde{u}) J_\Phi)\, dz_S \equiv K_1 + K_2, \end{aligned}$$

where

$$K_2 = -c_0\int_{\widehat{S}}\Big[\nabla^2\Phi(z)\int w_\delta(z-y)\partial_\tau^{1/2}\nabla_y\widetilde{u}\,dy + \partial_\tau^{1/2}\nabla^2\Phi(z)\cdot\int w_\delta(z-y)\nabla_y\widetilde{u}(y)\,dy\Big]\cdot\partial_\tau^{1/2}(\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi)\,dz_S,$$

and

$$|K_2|\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+c(\varepsilon)\|\widetilde{u}\|^2_{2,\widehat{\Omega}}.$$

Next

$$\begin{aligned}K_1 = &-c_0\int_{\widehat{S}}\Big[\int\partial_\tau w_\delta(z-y)\partial_\tau^{1/2}\nabla\Phi(z)\nabla_y\widetilde{u}(y)\,dy\\&+\int\partial_\tau w_\delta(z-y)\partial_\tau^{1/2}\nabla\Phi(y)\nabla_y\widetilde{u}(y)\,dy\\&+\int\partial_\tau w_\delta(z-y)(\nabla\Phi(z)-\nabla\Phi(y))\partial_\tau^{1/2}\nabla_y\widetilde{u}(y)\,dy\Big]\\&\times\partial_\tau^{1/2}(\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi)\,dz_S\\ \equiv{}& K_{11}+K_{12}+K_{13},\end{aligned}$$

where

$$|K_{11}|\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+\varepsilon\|\partial_\tau\partial_z j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+c\|j_\delta(\widetilde{u})\|^2_{2,\widehat{\Omega}},$$

and

$$\begin{aligned}K_{12} = &-c_0\int_{\widehat{S}}\Big[\int\partial_\tau w_\delta(z-y)(\partial_\tau^{1/2}\nabla\Phi(y)-\partial_\tau^{1/2}\nabla\Phi(z))\nabla_y\widetilde{u}(y)\,dy\\&+\partial_\tau^{1/2}\nabla\Phi(z)\int\partial_\tau w_\delta(z-y)\nabla_y\widetilde{u}(y)\,dy\Big]\partial_\tau^{1/2}(\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi)\,dz_S\\ \equiv{}& K_{121}+K_{122},\end{aligned}$$

where

$$\begin{aligned}|K_{121}|&\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+c\|\widetilde{u}\|^2_{2,\widehat{\Omega}},\\ |K_{122}|&\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+\varepsilon\|\partial_\tau\partial_z j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+c\|j_\delta(\widetilde{u})\|^2_{2,\widehat{\Omega}}.\end{aligned}$$

Finally,

$$|K_{13}|\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+c\|\widetilde{u}\|^2_{2,\widehat{\Omega}}.$$

Integrating by parts with the 1/2 derivative we have

$$\begin{aligned}|I_{42}|&\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+c\|\widetilde{u}\|^2_{2,\widehat{\Omega}},\\ |I_{43}|&\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,\widehat{\Omega}}+c\|j_\delta(\widetilde{u})\|^2_{2,\widehat{\Omega}}+c\|j_\delta(\widehat{G})\|^2_{3/2,\widehat{S}}.\end{aligned}$$

The first term on the r.h.s. of (4.15) can be written qualitatively in the form

$$\begin{aligned}I_7 = &-\int\limits_{\widehat{\Omega}} \partial_\tau\Big[\int w_\delta(z-y)\nabla_y\Phi(y)\nabla_y(\nabla_y\Phi(y)\widetilde{u}(y))\\&-\nabla_z\Phi(z)\nabla_z(\Phi(z)\nabla_z\int w_\delta(z-y)\widetilde{u}(y)\,dy)\Big]\\&\times\partial_\tau(\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi)\,dz\\=&\int\limits_{\widehat{\Omega}}\partial_\tau\Big[\int w_\delta(z-y)[(\nabla_z\Phi(z))^2-(\nabla_y\Phi(y))^2]\nabla_y^2\widetilde{u}(y)\,dy\\&+\int w_\delta(z-y)(\nabla_z\Phi(z)\nabla_z^2\Phi(z)\\&-\nabla_y\Phi(y)\nabla_y^2\Phi(y))\nabla_y\widetilde{u}(y)\,dy\Big]\partial_\tau(\partial_\tau^2 j_\delta(\widetilde{u})J_\Phi)\,dz\\\equiv&\, I_{71}+I_{72},\end{aligned}$$

and

$$|I_{71}|+|I_{72}|\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2+c(\varepsilon)\|\widetilde{u}\|_{2,\widehat{\Omega}}^2.$$

The second term on the r.h.s. of (4.15) is bounded by

$$\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2+c(\varepsilon)\|\widehat{u}\|_{2,\widehat{\Omega}}^2$$

and the last term by

$$\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2+c(\varepsilon)\|\widetilde{F}\|_{1,\widehat{\Omega}}^2.$$

Finally, we have to estimate I_5. We get

$$|I_5|\le\varepsilon\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2+\varepsilon\|\partial_\tau j_\delta(\widetilde{u})\|_{2,\widehat{\Omega}}^2+c\|\widetilde{u}\|_{2,\widehat{\Omega}}^2.$$

Summarizing the above considerations and using the Korn inequality we obtain

$$\begin{aligned}(4.16)\quad&\frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}}\widehat{\eta}|\partial_\tau^2 j_\delta(\widetilde{u})|^2 J_\Phi\,dz+\frac{c_0}{2}\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,\widehat{\Omega}}^2\\&\le c(\|\widehat{\eta}\|_{2,\widehat{\Omega}}^2\|j_\delta(\widetilde{u}_t)\|_{1,\widehat{\Omega}}^2+\|\widehat{\eta}_t\|_{1,\widehat{\Omega}}^2\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{0,\widehat{\Omega}}^2+\|\widehat{\eta}\|_{2,\widehat{\Omega}}^2\|\widetilde{u}_t\|_{1,\widehat{\Omega}}^2)\\&\quad+c\|\widehat{u}\|_{2,\widehat{\Omega}}^2+c\|\widetilde{F}\|_{1,\widehat{\Omega}}^2+\varepsilon\|\partial_n^2\partial_\tau j_\delta\widetilde{u}\|_{0,\widehat{\Omega}}^2+c\|\widetilde{G}\|_{3/2,\widehat{S}}^2.\end{aligned}$$

Integrating (4.16) with respect to time yields

$$\begin{aligned}(4.17)\quad&\int\limits_{\widehat{\Omega}}\widehat{\eta}|\partial_\tau^2 j_\delta(\widetilde{u})|^2 J_\Phi\,dz+c_0\|\partial_\tau^2 j_\delta(\widetilde{u})\|_{1,2,2,\widehat{\Omega}^t}^2\\&\quad\le c(\sup_t\|\widehat{\eta}\|_{2,\widehat{\Omega}}^2+\sup_t\|\widehat{\eta}_t\|_{1,\widehat{\Omega}}^2+1)(\|\widehat{u}\|_{2,2,2,\widehat{\Omega}^t}^2+\|\widehat{u}_t\|_{1,2,2,\widehat{\Omega}^t}^2)\end{aligned}$$

$$+ c\|\widetilde{F}\|^2_{1,2,2,\widehat{\Omega}^t} + \varepsilon\|\partial_n^2\partial_\tau j_\delta(\widetilde{u})\|^2_{0,\widehat{\Omega}^t} + \int\limits_{\widehat{\Omega}} \widehat{\eta}(0)|\partial_\tau^2\widetilde{u}(0)|^2 J_\Phi\, dz$$
$$+ c\|\widetilde{G}\|^2_{3/2,2,2,\widehat{S}^t}.$$

From (4.14) we have

$$\left\|\partial_\tau\partial_n^2\int w_\delta(z-y)\left(1-\frac{1}{\Phi_{3,z}^2}(\Phi_{3,z}^2-\Phi_{3,y}^2)\right)\widetilde{u}(y)\,dy\right\|^2_{0,\widehat{\Omega}^t} \le c\sup_t\|\widehat{\eta}\|^2_{2,\widehat{\Omega}}\|\widetilde{u}_t\|^2_{1,2,2,\widehat{\Omega}^t} + c_1\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,2,2,\widehat{\Omega}^t} + c\|\widehat{u}\|^2_{2,2,2,\widehat{\Omega}^t} + c\|\widetilde{F}\|^2_{1,2,2,\widehat{\Omega}^t}. \tag{4.18}$$

From (4.17) and (4.18) we obtain

$$\begin{aligned}
&\frac{2c_1}{c_0}\int\limits_{\widehat{\Omega}}\widehat{\eta}|\partial_\tau^2 j_\delta(\widetilde{u})|^2 J_\Phi\,dz + c_1\|\partial_\tau^2 j_\delta(\widetilde{u})\|^2_{1,2,2,\widehat{\Omega}^t}\\
&\quad+\left\|\partial_\tau\partial_n^2\int w_\delta(z-y)\left(1-\frac{1}{\Phi_{3,z}^2}\left(\Phi_{3,z}^2-\Phi_{3,y}^2\right)-\frac{2\varepsilon c_1}{c_0}\right)\widetilde{u}(y)\,dy\right\|^2_{0,\widehat{\Omega}^t}\\
&\le c(\sup_t\|\widehat{\eta}\|^2_{2,\widehat{\Omega}}+\sup_t\|\widehat{\eta}_t\|^2_{1,\widehat{\Omega}}+1)(\|\widehat{u}\|^2_{2,2,2,\widehat{\Omega}^t}+\|\widehat{u}_t\|^2_{1,2,2,\widehat{\Omega}^t})\\
&\quad+c\|\widehat{u}\|^2_{2,2,2,\widehat{\Omega}^t}+c\|\widetilde{F}\|^2_{1,2,2,\widehat{\Omega}^t}\\
&\quad+\int\limits_{\widehat{\Omega}}\eta(0)|\partial_\tau\widetilde{u}(0)|^2 J_\Phi\,dz + c\|\widetilde{G}\|_{3/2,2,2,\widehat{S}^t}\\
&\quad+\varepsilon\|\partial_n^2\partial_\tau j_\delta(\widetilde{u})\|^2_{0,\widehat{\Omega}^t}\equiv cX_1+\varepsilon\|\partial_n^2\partial_\tau j_\delta(\widetilde{u})\|^2_{0,\widehat{\Omega}}.
\end{aligned}\tag{4.19}$$

Using the fact that $\Phi_{3,x}^2$ is close to one and $\Phi_{3,x}^2-\Phi_{3,y}^2$ is close to zero for $x,y\in\widehat{\Omega}$ and for $\widehat{\Omega}$ sufficiently small, from (4.19) after passing with δ to 0 we obtain

$$\int\limits_{\widehat{\Omega}}\widehat{\eta}|\partial_\tau^2\widetilde{u}|^2 J_\Phi\,dz + \|\partial_\tau^2\widetilde{u}\|^2_{1,2,2,\widehat{\Omega}^t} + \|\partial_\tau\partial_n^2\widetilde{u}\|^2_{0,\widehat{\Omega}^t} \le cX_1. \tag{4.20}$$

Finally, from (4.13) we have

$$\|\partial_n^3\widetilde{u}\|^2_{0,\widehat{\Omega}^t} \le c\|\partial_n^2\partial_\tau\widehat{u}\|^2_{0,\widehat{\Omega}^t} + c\|\partial_n\partial_\tau^2\widetilde{u}\|^2_{0,\widehat{\Omega}^t} + c\|\widehat{u}\|^2_{2,2,2,\widehat{\Omega}^t} + c\sup_t\|\widehat{\eta}\|^2_{2,\widehat{\Omega}}\|\widetilde{u}_t\|^2_{1,2,2,\widehat{\Omega}^t} + c\|\widetilde{F}\|^2_{1,2,2,\widehat{\Omega}^t}. \tag{4.21}$$

Summarizing the above considerations and summing over all neighbourhoods of the partition of unity we see that $u\in L_2(0,T;H^3(\Omega))$ and

$$
\begin{aligned}
(4.22)\quad & \|u\|^2_{3,2,2,\Omega^t} \\
& \le c(\sup_t \|\eta\|^2_{2,\Omega} + \sup_t \|\eta_t\|^2_{1,\Omega} + 1)(\|u\|^2_{2,2,2,\Omega^t} + \|u_t\|^2_{1,2,2,\Omega^t}) \\
& \quad + c\|F\|^2_{1,2,2,\Omega^t} + c\|G\|^2_{3/2,2,2,S^t} + \int_\Omega \eta(0)|\partial_\tau^2 u(0)|^2\,dx,
\end{aligned}
$$

where we used the fact that Φ is smooth.

Now we choose a sequence Φ_m converging to Φ in $H^3(\widehat{\Omega})$. Then we show that the corresponding sequence u_m of (4.13) converges to u which corresponds to Φ. Therefore, we consider

$$
\begin{aligned}
(4.23)\quad & \widehat{\eta}\widetilde{u}_{mt} - \operatorname{div}_{\Phi_m} \mathbb{D}_{\Phi_m}(\widetilde{u}_m) \\
& = -\operatorname{div}_{\Phi_m} \mathbb{B}_{\Phi_m}(\widehat{u}_m, \widehat{\zeta}) - \mathbb{D}_{\Phi_m}(\widehat{u}_m)\cdot\nabla_{\Phi_m}\widehat{\zeta} + \widetilde{F}.
\end{aligned}
$$

Differentiating (4.23) with respect to τ, applying the difference δ_h which corresponds to the tangent directions too, multiplying the result by $\delta_h\widetilde{u}_{m\tau}$ and integrating over $\widehat{\Omega}$ yields, after passing with h to 0, the estimate

$$
\begin{aligned}
(4.24)\quad & \int_{\widehat{\Omega}} \widehat{\eta}|\partial_\tau^2\widetilde{u}_m|^2 J_\Phi\,dz + c_0\|\partial_\tau^2\widetilde{u}_m\|^2_{1,2,2,\widehat{\Omega}^t} \\
& \le c(\sup_t \|\widehat{\eta}\|^2_{2,\widehat{\Omega}} + \sup_t \|\widehat{\eta}_t\|^2_{1,\widehat{\Omega}} + 1) \\
& \quad \times (\|\widehat{u}_m\|^2_{2,2,2,\widehat{\Omega}^t} + \|\widehat{u}_{mt}\|^2_{1,2,2,\widehat{\Omega}^t}) + c\|\widetilde{F}\|^2_{1,2,2,\widehat{\Omega}^t} \\
& \quad + \int_{\widehat{\Omega}} \widehat{\eta}(0)|\partial_\tau^2\widetilde{u}(0)|^2 J_\Phi\,dz,
\end{aligned}
$$

where $c = c(\|\Phi_m\|_{3,\widehat{\Omega}})$. Now from (4.13) we have

$$
\begin{aligned}
(4.25)\quad \|\partial_n^2\partial_\tau\widetilde{u}_m\|^2_{0,\widehat{\Omega}^t} & \le c\|\partial_n\partial_\tau^2\widetilde{u}_m\|^2_{0,\widehat{\Omega}^t} + c\|\widehat{u}_m\|^2_{2,2,2,\widehat{\Omega}^t} \\
& \quad + c\sup_t \|\widehat{\eta}\|^2_{2,\widehat{\Omega}}\|\widetilde{u}_{mt}\|^2_{1,2,2,\widehat{\Omega}^t} + c\|\widetilde{F}\|^2_{1,2,2,\widehat{\Omega}^t}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.26)\quad \|\partial_n^3\widetilde{u}_m\|^2_{0,\widehat{\Omega}^t} & \le c(\|\partial_\tau\partial_n^2\widetilde{u}_m\|^2_{0,\widehat{\Omega}^t} + \|\partial_\tau^2\partial_n\widehat{u}_m\|^2_{0,\widehat{\Omega}^t} + \|\widehat{u}_m\|^2_{2,2,2,\widehat{\Omega}^t}) \\
& \quad + c\sup_t \|\widehat{\eta}\|^2_{2,\widehat{\Omega}}\|\widetilde{u}_{mt}\|^2_{1,2,2,\widehat{\Omega}^t} + c\|\widetilde{F}\|^2_{1,2,2,\widehat{\Omega}^t}.
\end{aligned}
$$

From (4.24)–(4.26) we get the estimate (4.22) for u_m under the assumption that $\Phi_m \in H^3$. Continuing, we use the estimates for the weak solutions and the estimate in H^2. Hence we finally get the estimate

$$
\begin{aligned}
(4.27)\quad \|u_m\|^2_{3,2,2,\Omega^t} & \le c(\|\Phi_m\|_{3,\Omega}, \sup_t \|\eta\|_{2,\Omega}, \sup_t \|\eta_t\|_{1,\Omega}) \\
& \quad \times [\|F\|^2_{1,2,2,\Omega^t} + \|u(0)\|^2_{2,\Omega}].
\end{aligned}
$$

To show convergence we consider the following problem for the difference $U_m = u_m - u_{m-1}$:

$$
\begin{aligned}
&\widehat{\eta}\widetilde{U}_{mt} - \operatorname{div}_{\Phi_m} \mathbb{D}_{\Phi_m}(\widetilde{U}_m)\\
&\qquad\qquad = (\operatorname{div}_{\Phi_m} \mathbb{D}_{\Phi_m} - \operatorname{div}_{\Phi_{m-1}} \mathbb{D}_{\Phi_{m-1}})(\widetilde{u}_{m-1})\\
&\qquad\qquad\quad - (\operatorname{div}_{\Phi_m} \mathbb{B}_{\Phi_m}(\widehat{u}_m, \widehat{\zeta}) - \operatorname{div}_{\Phi_{m-1}} \mathbb{B}_{\Phi_{m-1}}(\widehat{u}_{m-1}, \widehat{\zeta}))\\
&\qquad\qquad\quad - \mathbb{D}_{\Phi_m}(\widehat{U}_m) \cdot \nabla_{\Phi_m}(\widehat{\zeta}) - (\mathbb{D}_{\Phi_m}(\widehat{u}_{m-1}) \cdot \nabla_{\Phi_m}(\widehat{\zeta})\\
&\qquad\qquad\quad - \mathbb{D}_{\Phi_{m-1}}(\widetilde{u}_{m-1}) \cdot \nabla_{\Phi_{m-1}}(\widehat{\zeta})),\\
&\mathbb{D}_{\Phi_m}(\widetilde{U}_m) \cdot \widehat{n} = - (\mathbb{D}_{\Phi_m}(\widetilde{u}_{m-1}) \cdot \widehat{n} - \mathbb{D}_{\Phi_{m-1}}(\widetilde{u}_{m-1}) \cdot \widehat{n})\\
&\qquad\qquad\quad + \mathbb{B}_{\Phi_m}(\widehat{u}_m, \widehat{\zeta}) \cdot \widehat{n} - \mathbb{B}_{\Phi_{m-1}}(\widehat{u}_{m-1}, \widehat{\zeta}) \cdot \widehat{n},\\
&\widetilde{U}_m|_{t=0} = 0.
\end{aligned}
\tag{4.28}
$$

To obtain an estimate for the difference we differentiate $(4.28)_1$ with respect to τ, multiply by $\widetilde{U}_{m\tau}$ and integrate over $\widehat{\Omega}$ to get

$$
\int_{\widehat{\Omega}} \widehat{\eta}|\widetilde{U}_{m\tau}|^2 J_\Phi\, dz + \|\widetilde{U}_{m\tau}\|^2_{1,2,2,\widehat{\Omega}^t} \le c_1\|\Phi_m - \Phi_{m-1}\|^2_{2,\widehat{\Omega}},
\tag{4.29}
$$

where $c_1 = c_1(\|\Phi_m\|_{3,\widehat{\Omega}}, \|\Phi_{m-1}\|_{3,\widehat{\Omega}}, \|\widehat{u}_{m-1}\|_{2,2,2,\widehat{\Omega}^t})$.

Calculating $\partial_n^2\widetilde{U}_m$ from $(4.28)_1$ and using the estimate for weak solutions to (4.28) and (4.29) we finally get

$$
\|\widetilde{U}_m\|_{2,2,2,\widehat{\Omega}^t} \le c\|\Phi_m - \Phi_{m-1}\|_{2,\widehat{\Omega}}.
\tag{4.30}
$$

Therefore the sequence $\{u_m\}$ is bounded in $L_2(0,T;H^3(\Omega))$ and converges strongly in $L_2(0,T;\ H^2(\Omega))$ to a limit $u \in L_2(0,T;H^3(\Omega))$.

Now we can pass to the limit in (4.23) and finally we come to equations for the limit function u.

Summarizing the above considerations we have proved

THEOREM 4.1. *Assume that* $\eta \in L_\infty(0,T;H^2(\Omega))$, $\eta_t \in L_\infty(0,T;H^1(\Omega))$, $F \in L_2(0,T;H^1(\Omega))$, $G \in L_2(0,T;\ H^{3/2}(S))$, $S \in H^{5/2}$ *and* $u_0 \in H^1(\Omega)$. *Then there exists a solution to problem* (4.1) *such that* $u \in L_2(0,T;\ H^3(\Omega))$ *and*

$$
\begin{aligned}
\|u\|_{3,2,2,\Omega^t} &\le \alpha(\sup_t \|\eta\|_{2,\Omega}, \sup_t \|\eta_t\|_{1,\Omega}, \|S\|_{5/2})\\
&\quad \times [\|F\|_{1,2,2,\Omega^t} + \|G\|_{3/2,2,2,S^t} + \|u_0\|_{2,\Omega}], \qquad t \le T,
\end{aligned}
\tag{4.31}
$$

where α is an increasing positive function.

REMARK 4.2. To prove the existence of solutions to problem (1.12) we write it in the form

$$\begin{aligned}
&\eta u_t - \operatorname{div}\mathbb{D}(u) = \operatorname{div}_w \mathbb{D}_w(u) - \operatorname{div}\mathbb{D}(u) + F,\\
&\overline{n}\cdot\mathbb{D}(u) = \overline{n}\cdot\mathbb{D}(u) - \overline{n}_w\mathbb{D}_w(u) + G,\\
&u|_{t=0} = v_0.
\end{aligned}\tag{4.32}$$

Now using Theorem 4.1 and the method of successive approximations we have the existence of solutions to problem (3.1) for sufficiently small T in the same space as in Theorem 4.1 and under the additional assumption that $w \in L_2(0,T;H^3(\Omega))$.

Moreover, we have the estimate

$$\begin{aligned}
\|u\|_{3,2,2,\Omega^t} \le \beta(\|w\|_{3,2,2,\Omega^t}, \sup_t \|\eta\|_{2,\Omega}, \sup_t \|\eta_t\|_{1,\Omega}, \|S\|_{5/2})&\\
\times [\|F\|_{1,2,2,\Omega^t} + \|G\|_{3/2,2,2,S^t} + \|v_0\|_{1,\Omega}], \quad t \le T,&
\end{aligned}\tag{4.33}$$

where β is an increasing positive function.

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