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I. K. ARGYROS (Lawton, Okla.)

# A NEW KANTOROVICH-TYPE THEOREM FOR NEWTON'S METHOD

*Abstract.* A new Kantorovich-type convergence theorem for Newton's method is established for approximating a locally unique solution of an equation $F(x) = 0$ defined on a Banach space. It is assumed that the operator $F$ is twice Fréchet differentiable, and that $F'$, $F''$ satisfy Lipschitz conditions. Our convergence condition differs from earlier ones and therefore it has theoretical and practical value.

**I. Introduction.** In this study we are concerned with the problem of approximating a locally unique solution $x^*$ of the equation

$$F(x) = 0 \tag{1}$$

where $F$ is a twice Fréchet differentiable operator defined on a convex subset $D$ of a Banach space $E_1$ with values in a Banach space $E_2$.

Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \ge 0), \qquad x_0 \in D, \tag{2}$$

has been used extensively by many authors (see [1]–[6] and the references there) to generate a sequence $\{x_n\}_{n\ge 0}$ converging to $x^*$. In particular the following conditions have been used:

CONDITION A (Kantorovich [6]). Let $F : D \subseteq E_1 \to E_2$ be Fréchet differentiable in $D$, $F'(x_0)^{-1} \in L(E_2, E_1)$ for some $x_0 \in D$, where $L(E_2, E_1)$ is the set of bounded linear operators from $E_2$ into $E_1$, and assume

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \le l\|x - y\| \quad \text{for all } x, y \in D, \tag{3}$$

$$\|F'(x_0)^{-1}F(x_0)\| \le a \tag{4}$$

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and

$$(5) \qquad 2al \leq 1.$$

Under condition A, one can obtain error estimates, existence and uniqueness regions of solutions, and know whether $x_0$ is a convergent initial guess, i.e., Newton's method (2) starting from $x_0$ converges to $x^*$. But sometimes when we want to determine whether the Newton iteration (2) starting from $x_0$ converges, Condition A fails.

Example 1.1. Let $E_1 = E_2 = \mathbb{R}$, $D = [\sqrt{2}-1, \sqrt{2}+1]$, $x_0 = \sqrt{2}$ and define the real polynomial $F$ on $D$ by

$$(6) \qquad F(x) = \frac{1}{6}x^3 - \alpha, \qquad \alpha = \frac{2^{3/2}}{6} + .23.$$

Using (3), (4), (6) and the above choices we get $a = .23$ and $l = 2.4142136$. Condition (5) is not satisfied since

$$2al = 1.1105383 > 1.$$

Therefore under condition A we cannot determine whether Newton's method (2) starting from $x_0 = \sqrt{2}$ converges.

That is why in this study we introduce a new condition and a new theorem under which we will see that Newton's method starting from $x_0 = \sqrt{2}$ in Example 1.1 converges.

From now on we assume:

Condition B. Let $F : D \subseteq E_1 \to E_2$ be twice Fréchet differentiable in $D$, with $F'(x) \in L(E_1, E_2)$, $F''(x) \in L(E_1, L(E_1, E_2))$ $(x \in D)$, $F'(x_0)^{-1}$ exists at some $x_0 \in D$, and assume

$$(7) \qquad 0 < \|F'(x_0)^{-1}F(x_0)\| \leq a, \qquad \|F'(x_0)^{-1}F''(x_0)\| \leq b,$$

$$(8) \qquad \|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq c\|x - x_0\|, \qquad c > 0,$$

$$(9) \qquad \|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| \leq d\|x - x_0\| \qquad \text{for all } x \in D,$$

and

$$(10) \qquad 2ka \leq 1,$$

where either

$$(11) \qquad k = \max\{c, b + 2ad\},$$

or, if the function

$$(12) \qquad f(t) = t^3 - 2bt^2 - (2d - b^2)t + 2d(b + ad)$$

has two positive zeros $k^1$, $k^2$ such that

$$(13) \qquad [b, b + 2ad] \subseteq [k^1, k^2],$$

then $k \geq c$ and

$$k \in [b, b + 2ad]. \tag{14}$$

**2. Convergence analysis.** We need the lemma:

LEMMA 2.1. *Let $a, k$ be given positive constants. Define the real polynomial $p$ on $[0, \infty)$ by*

$$p(t) = \frac{k}{2}t^2 - t + a \tag{15}$$

*and the iteration $\{t_n\}_{n\geq 0}$ by*

$$t_0 = 0, \tag{16}$$

$$t_{n+1} = t_n - \frac{p(t_n)}{p'(t_n)}. \tag{17}$$

*Assume*

$$2ka \leq 1. \tag{18}$$

*Then the equation*

$$p(t) = 0 \tag{19}$$

*has two positive roots $r_1$, $r_2$ with $r_1 \leq r_2$ and the iteration $\{t_n\}_{n\geq 0}$ generated by* (16)–(17) *is such that $t_0 < t_1 < \ldots < t_n < t_{n+1} < \ldots < r_1$ with* $\lim_{n\to\infty} t_n = r_1$.

Proof. Using (15) and (18) we deduce that equation $p(t) = 0$ has two positive roots

$$r_1 = \frac{1 - \sqrt{1 - 2ka}}{k} \quad \text{and} \quad r_2 = \frac{1 + \sqrt{1 - 2ka}}{k} \tag{20}$$

with $r_1 \leq r_2$. Moreover the function $t - p(t)/p'(t)$ increases on $[0, r_1]$, since $p'(t) < 0$, $p''(t) > 0$ and $p(t) > 0$ on $[0, r_1]$. Furthermore if $t_n \in [0, r_1]$ for all integer values smaller than or equal to $n$, then we obtain

$$t_n \leq t_n - \frac{p(t_n)}{p'(t_n)} = t_{n+1} \quad \text{and} \quad t_{n+1} = t_n - \frac{p(t_n)}{p'(t_n)} \leq r_1 - \frac{p(r_1)}{p'(r_1)} = r_1. \ \blacksquare$$

We set $\overline{U}(x_0, s) = \{x \in E_2 \mid \|x - x_0\| \leq s\}$ and $U(x_0, s) = \{x \in E_1 \mid \|x - x_0\| < s\}$.

LEMMA 2.2. *The following estimates are true for $x \in U(x_0, 1/c)$:*

$$\|F'(x)^{-1}F'(x_0)\| \leq (1 - c\|x - x_0\|)^{-1} \tag{21}$$

*and*

$$\|F'(x_0)^{-1}F''(x)\| \leq b + d\|x - x_0\|. \tag{22}$$

Proof. If $x \in U(x_0, 1/c)$, using (7), the estimate

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le c\|x - x_0\| < 1,$$

and the Banach lemma on invertible operators [6], the operator $F'(x)$ has a continuous inverse on $U(x_0, 1/c)$ and

$$\|F'(x)^{-1}F'(x_0)\| \le (1 - c\|x - x_0\|)^{-1}.$$

Moreover by (6) and (11) we get

$$\begin{aligned}\|F'(x_0)^{-1}F''(x)\| &\le \|F'(x_0)^{-1}F''(x_0)\| + \|F'(x_0)^{-1}(F''(x) - F''(x_0))\| \\ &\le b + d\|x - x_0\|. \ \blacksquare\end{aligned}$$

We can now prove the following semilocal result concerning the convergence of Newton's method (2).

THEOREM 2.3. *Let $F$ be the operator defined in* (1). *Let $p$ be the polynomial defined in* (15). *Assume that $U(x_0, 1/c) \subseteq D$ and Condition B holds. Then Newton's iteration $\{x_n\}_{n\ge0}$ generated by* (2) *is well defined, remains in $\overline{U}(x_0, r_1)$ for all $n \ge 0$, and converges to a solution $x^* \in \overline{U}(x_0, r_1)$ of the equation $F(x) = 0$, which is unique in $U(x_0, r_2)$ if $r_1 < r_2$. If $r_1 = r_2$ the solution $x^*$ is unique in $\overline{U}(x_0, r_1)$. Moreover the following estimates hold for all $n \ge 0$*:

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \tag{23}$$

*and*

$$\|x_n - x^*\| \le r_1 - t_n = (r_1/r_2)^{2^n}(r_2 - t_n) \tag{24}$$

*where $r_1$ and $r_2$ are the roots of the quadratic equation $p(t) = 0$ given by* (20).

Proof. Using induction on $n$ we first show estimate (23). The approximation $x_1$ is defined and

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \le a = t_1 - t_0 < r_1.$$

It follows that $x_1 \in \overline{U}(x_0, r_1)$ and (23) holds for $n = 0$.

Assume that (23) holds for all integer values $i \le n$. Using (2) we can write in turn

$$\begin{aligned} F'(x_0)^{-1}F(x_{i+1}) &= F'(x_0)^{-1}[F(x_{i+1}) - F(x_i) - F'(x_i)(x_{i+1} - x_i)] \\ &= F'(x_0)^{-1}\Big\{\int_0^1 [F''[x_i + t(x_{i+1} - x_i)] \\ &\quad - F''(x_0)](1 - t)\,dt\,(x_{i+1} - x_i)^2 \\ &\quad + \frac{1}{2}F''(x_0)(x_{i+1} - x_i)^2\Big\}. \end{aligned} \tag{25}$$

Using the induction hypothesis we have

$$\|x_{i+1}-x_0\| \le \sum_{j=1}^{i+1} \|x_j - x_{j-1}\| \le \sum_{j=1}^{i+1}(t_i - t_{i-1}) = t_{i+1}-t_0 = t_{i+1} < r_1$$

and

$$\|x_i + t(x_{i+1}-x_i) - x_0\| \le t_i + t(t_{i+1}-t_i) < r_1.$$

Hence, by (7), (9), (15), (22), (23) and (25) we get

$$\begin{aligned}
(26)\qquad &\|F'(x_0)^{-1}F(x_{i+1})\| \\
&\le \frac{1}{2}\left[b + d\|x_i - x_0\| + \frac{d}{3}\|x_{i+1}-x_i\|\right]\|x_{i+1}-x_i\|^2 \\
&\le \frac{1}{2}\left[b + dt_i + \frac{d}{3}(t_{i+1}-t_i)\right](t_{i+1}-t_i)^2 \\
&\le \frac{1}{2}\left[b + \frac{2}{3}dt_i + \frac{dt_{i+1}}{3}\right](t_{i+1}-t_i)^2 \\
&\le \frac{1}{2}\left[b + \frac{2}{3}dr_1 + \frac{dr_1}{3}\right](t_{i+1}-t_i)^2 \\
&\le \frac{k}{2}(t_{i+1}-t_i)^2 \le p(t_{i+1}).
\end{aligned}$$

By (2), (17), (21) and (26) we obtain

$$\|x_{i+2}-x_{i+1}\| \le -\frac{p(t_{i+1})}{p'(t_{i+1})} = t_{i+2}-t_{i+1},$$

which shows (23) for all $n \ge 0$.

By Lemma 2.1 and estimate (23) it follows that $\{x_n\}_{n\ge 0}$ is a Cauchy sequence in the Banach space $E_1$ and so it converges to some limit $x^* \in \overline{U}(x_0,r_1)$ (since $\overline{U}(x_0,r_1)$ is a closed set). By (2) and the continuity of $F$, we get $F(x^*)=0$. To show uniqueness let $y \in U(x_0,r_2)$ be such that $F(y)=0$. Using (2) we obtain

$$\begin{aligned}
(27)\qquad y - x_{n+1} = {} & -[F'(x_n)^{-1}F'(x_0)]\Big\{\int_0^1 F'(x_0)^{-1}(F''(x_n+t(y-x_n)) \\
& - F''(x_0))(1-t)\,dt\,(y-x_n)^2 \\
& + \int_0^1 F'(x_0)^{-1}F''(x_0)(1-t)\,dt\,(y-x_n)^2\Big\}.
\end{aligned}$$

As in (25), (26) we get $\|y-x_0\| \le r_1 - t_0$ if $y \in \overline{U}(x_0,r_1)$, and $\|y-x_0\| = \lambda(r_2-t_0)$, $0<\lambda<1$, if $y \in U(x_0,r_2)$. That is, as in (25), by (27) we have $\|y-x_n\| \le r_1 - t_n$ if $y \in \overline{U}(x_0,r_1)$ $(n \ge 0)$, and $\|y-x_n\| \le \lambda^{2^n}(r_2-t_n)$ if

$y \in U(x_0, r_2)$ $(n \geq 0)$. From the above estimates and $F(x^*) = 0$ it follows that $x^* = \lim_{n\to\infty} x_n = y$ in either case.

Finally estimates (24) follow by using standard majorization techniques, (17) and (23) ([2], [3], [6]). ■

## 3. Applications and concluding remarks

REMARK 3.1. Let us apply Theorem 2.3 to Example 1.1. By (7)–(9), (11) we get $a = .23$, $b = \sqrt{2}$, $c = \sqrt{2} + .5$, $d = 1$, and $k = 1.9142136$. Then condition (10) becomes

$$2ka = .8805383 < 1,$$

which is true. Hence equation (6) has a solution $x^* \in U(\sqrt{2}, 1)$. Moreover Newton's method (2) starting from $x_0 = \sqrt{2}$ converges quadratically to $x^*$. We also remark that as we noted in Example 1.1, Condition A fails to determine whether Newton's method converges in this case. We found $x^* = 1.614507$.

REMARK 3.2. The convergence of Newton's method (2) can be established independently using Conditions A and B. In practice we can use both of them to determine the smallest region where the solution is located and the largest one where this solution is unique. Let us make such a comparison between Conditions A and B. Consider the polynomial $q$ given by

$$q(t) = \frac{l}{2}t^2 - t + a$$

with roots denoted by $r_3$ and $r_4$ $(r_3 \leq r_4)$. Then since $c \leq l$ we find from (15) that $p(r_3) \leq 0$ and $p(r_4) \leq 0$. Hence we get $r_1 \leq r_3 \leq r_4 \leq r_2$ and $r_3 \leq 1/c$. Note also that our theorem uses simply a quadratic polynomial $p$ and condition (10) instead of a cubic polynomial and condition (27) in [3], [5] (which are more difficult to handle in general).

REMARK 3.3. We can extend the result obtained in Theorem 2.3 to include the Hölder case. Assume, instead of (9) in Condition B, that $F$ satisfies

$$\|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| \leq d_0\|x - x_0\|^q \quad \text{for all } x \in D,\ q \in [0,1] \text{ and some } d_0 \geq 0. \tag{28}$$

For $q = 0$, we obtain $\|F'(x_0)^{-1}F''(x)\| \leq d_0 + b$, and we are in the situation of the Kantorovich theorem [6, Theorem XVIII.1.6]. If $q = 1$ in (28) we get (9). Moreover if $q \in (0,1)$, then $F''$ is $q$-Hölder continuous on $D$. Let $a$, $b$, $c$ be as before. Assume there exists $k_0 \geq c$ such that $b + d_0 r_1^q \geq k_0$, where $r_1$ is given by (20), and condition (10) holds with $k$ replaced by $k_0$.

With the above changes the conclusions of Theorem 2.3 hold for the Hölder case.

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Ioannis K. Argyros
Cameron University
Department of Mathematics
Lawton, Oklahoma 73505
U.S.A.
E-mail: Ioannisa@cameron.edu


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