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ON GLOBAL MOTION OF A COMPRESSIBLE BAROTROPIC VISCOUS FLUID WITH BOUNDARY SLIP CONDITION

Abstract. Global-in-time existence of solutions for equations of viscous compressible barotropic fluid in a bounded domain $\Omega \subset \mathbb{R}^3$ with the boundary slip condition is proved. The solution is close to an equilibrium solution. The proof is based on the energy method. Moreover, in the L_2-approach the result is sharp (the regularity of the solution cannot be decreased) because the velocity belongs to $H^{2+\alpha,1+\alpha/2}(\Omega \times \mathbb{R}_+)$ and the density belongs to $H^{1+\alpha,1/2+\alpha/2}(\Omega \times \mathbb{R}_+)$, $\alpha \in (1/2, 1)$.

1. Introduction. In this paper we prove the existence of global-in-time solutions to the following problem (see [5, 10]):

$$
\begin{aligned}
&\varrho(v_t + v \cdot \nabla v) - \operatorname{div} \mathbb{T}(v,p) = \varrho f && \text{in } \Omega^T = \Omega \times (0,T),\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \Omega^T,\\
&\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0 && \text{in } \Omega,\\
&\overline{n} \cdot \mathbb{T}(v,p) \cdot \overline{\tau}_\alpha + \gamma v \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T = S \times (0,T),\\
&v \cdot \overline{n} = 0 && \text{on } S^T,
\end{aligned}
\tag{1.1}
$$

where $v = v(x,t)$ is the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $f = f(x,t)$ the external force field, $\overline{n}$ the unit outward vector normal to S, $\overline{\tau}_\alpha$, $\alpha = 1,2$, tangent vectors to S, $0 < \gamma$ the constant slip coefficient, and $S = \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain. By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$
\mathbb{T}(v,p) = \{\mu(\partial_{x_i} v_j + \partial_{x_j} v_i) + (\nu - \mu)\delta_{ij} \operatorname{div} v - \delta_{ij} p\}_{i,j=1,2,3},
\tag{1.2}
$$

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where μ, ν are constant viscosity coefficients satisfying the thermodynamic restrictions

$$\nu > \tfrac{1}{3}\mu > 0. \tag{1.3}$$

Moreover, we consider the barotropic motion,

$$p = p(\varrho). \tag{1.4}$$

From $(1.1)_{2,5}$ it follows that the total mass of the fluid in Ω is conserved,

$$\int_\Omega \varrho\,dx = M = \int_\Omega \varrho_0\,dx. \tag{1.5}$$

In this paper we prove existence of global-in-time solutions which are close to the equilibrium solution

$$v_e = 0, \qquad \varrho_e = \frac{M}{|\Omega|}, \tag{1.6}$$

where $|\Omega| = \operatorname{vol}\Omega$.

The proof is based on a local existence result from [3] and on the prolongation technique from [12]. To recall the result from [3] we have to introduce the Lagrangian coordinates which are initial data to the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x,t), \qquad x|_{t=0} = \xi \in \Omega. \tag{1.7}$$

Integrating (1.7) we obtain a relation between the Eulerian x and the Lagrangian ξ coordinates:

$$x = \xi + \int_0^t u(\xi,\tau)\,d\tau \equiv x_u(\xi,t) \equiv x(\xi,t), \tag{1.8}$$

where $u(\xi,t) = v(x(\xi,t),t)$. Moreover, we introduce $\eta(\xi,t) = \varrho(x(\xi,t),t)$, $q(\xi,t) = p(\eta(\xi,t))$, and $g(\xi,t) = f(x(\xi,t),t)$.

THEOREM 1.1. *Assume that $g \in H^{\alpha,\alpha/2}(\Omega^T)$, $v_0 \in H^{1+\alpha}(\Omega)$, $\varrho_0 \in H^{1+\alpha}(\Omega)$, $1/\varrho_0 \in L_\infty(\Omega)$, $S \in C^{2+\alpha}$, $\alpha \in (1/2,1)$. Then there exists a time $T > 0$ such that for $t \le T$ there exists a solution (u,η) to problem (1.1) with $u \in H^{2+\alpha,1+\alpha/2}(\Omega^t)$, $\eta \in H^{1+\alpha,1/2+\alpha/2}(\Omega^t)$, and the following estimate holds*:

$$\begin{aligned} \|u\|_{2+\alpha,\Omega^t} + \|\eta\|_{1+\alpha,\Omega^t} &\le \varphi_1(T, \|S\|_{C^{2+\alpha}}, \gamma(T))[\|v_0\|_{1+\alpha,\Omega} \\ &\quad + \|\varrho_0\|_{1+\alpha,\Omega} + \|g\|_{\alpha,\Omega^t}], \qquad t \le T, \\ |1/\eta|_{\infty,\Omega^t} &\le \varphi_2(T, \|S\|_{C^{2+\alpha}}, \gamma(T))|1/\varrho_0|_{\infty,\Omega}, \end{aligned} \tag{1.9}$$

where $\gamma(t) = \|v_0\|_{1+\alpha,\Omega} + \|\varrho_0\|_{1+\alpha,\Omega} + \|g\|_{\alpha,\Omega^t}$ and φ_1, φ_2 are increasing positive functions of their arguments.

To prove global existence we have to control the variation of the solution in a neighbourhood of the equilibrium solution. For this purpose we introduce

$$\varrho_\sigma = \varrho - \varrho_e, \quad p_\sigma = p - p_e, \tag{1.10}$$

where $p_e = p(\varrho_e)$. Then (1.1) implies

$$\begin{aligned}
&\varrho(v_t + v\cdot\nabla v) - \operatorname{div}\mathbb{T}(v,p_\sigma) = \varrho f && \text{in } \Omega^T,\\
&\varrho_{\sigma t} + v\cdot\nabla\varrho_\sigma + \varrho\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha=1,2, && \text{on } S^T,\\
&v\cdot\overline{n} = 0 && \text{on } S^T,\\
&v|_{t=0} = v_0, \quad \varrho_\sigma|_{t=0} = \varrho_{\sigma 0} \equiv \varrho_0 - \varrho_e && \text{in } \Omega,
\end{aligned} \tag{1.11}$$

where

$$\mathbb{D}(v) = \{\mu(v_{ix_j} + v_{jx_i}) + (\nu-\mu)\delta_{ij}\operatorname{div} v\}_{i,j=1,2,3}.$$

Sometimes instead of $(1.11)_2$ we use the following equation for the pressure:

$$p_{\sigma t} + v\cdot\nabla p_\sigma + p_\varrho\varrho\operatorname{div} v = 0. \tag{1.12}$$

Applying Theorem 1.1 to problem (1.11) yields

THEOREM 1.2. *Let the assumptions of Theorem* 1.1 *hold. Let* $\varrho_{\sigma 0} \in H^{1+\alpha}(\Omega)$. *Then for solutions of the problem* (1.11) *we have the estimate*

$$\begin{aligned}
\|u\|_{2+\alpha,\Omega^t} + \|\eta_\sigma\|_{1+\alpha,\Omega^t} \le \varphi_3(T,\|S\|_{C^{2+\alpha}},\gamma(T))[\|v_0\|_{1+\alpha,\Omega}\\
+ \|\varrho_{\sigma 0}\|_{1+\alpha,\Omega} + \|g\|_{\alpha,\Omega^t}], \quad t\le T,
\end{aligned} \tag{1.13}$$

where φ_3 *is an increasing positive function of its arguments and* $\gamma(T)$ *is defined in Theorem* 1.1.

Since the local existence of solutions to problem (1.1) was proved in spaces such that $u \in H^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\eta \in H^{1+\alpha,1/2+\alpha/2}(\Omega^T)$, we have to formulate problem (1.11) and (1.12) in Lagrangian coordinates. Hence we have

$$\begin{aligned}
&\eta u_t - \operatorname{div}_u \mathbb{T}_u(u,q_\sigma) = \eta g && \text{in } \Omega^T,\\
&q_{\sigma t} + p_\eta\eta\operatorname{div}_u u = 0 && \text{in } \Omega^T,\\
&\overline{n}_u\cdot\mathbb{D}_u(u)\cdot\overline{\tau}_{u\alpha} + \gamma u\cdot\overline{\tau}_{u\alpha} = 0, \quad \alpha=1,2, && \text{on } S^T,\\
&u\cdot\overline{n}_u = 0 && \text{on } S^T,\\
&u|_{t=0} = v_0, \quad q_\sigma|_{t=0} = q_{\sigma 0} \equiv q_0 - p_e && \text{in } \Omega,
\end{aligned} \tag{1.14}$$

where $q_\sigma = q - p_e$, and

$$\begin{aligned}
&\eta_t + \eta\operatorname{div}_u u = 0 && \text{in } \Omega^T,\\
&\eta|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned} \tag{1.15}$$

where $\nabla_u = \xi_{ix}\partial_{\xi_i}$, $\mathbb{T}_u(u,q_\sigma) = -q_\sigma I + \mathbb{D}_u(u)$, I is the unit matrix and the operators $\mathbb{D}_u$, div_u are obtained from $\mathbb{D}$ and div by replacing ∇ by ∇_u. Finally $\overline{n}_u(\xi,t) = \overline{n}(x(\xi,t))$ and $\overline{\tau}_{u,\alpha}(\xi,t) = \overline{\tau}_\alpha(x(\xi,t))$, $\alpha=1,2$.

Moreover, in view of the local solvability the Jacobi matrix with determinant $J_{x(\xi,t)}$ of the transformation $x = x(\xi,t)$, with elements $a_{ij} = \delta_{ij} + \int_0^t \partial_{\xi_j} u_i(\xi,\tau)\,d\tau$, is invertible.

We have to underline that the use of the Lagrangian coordinates is not natural for problem (1.1) because the domain Ω is fixed. However in [3] and also in this paper the Lagrangian coordinates have to be used because otherwise the nonlinear term $v\cdot\nabla\varrho$ in the continuity equation cannot be estimated in the proofs for v and ϱ such that $v \in H^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\varrho \in H^{1+\alpha,1/2+\alpha/2}(\Omega^T)$, $\alpha \in (1/2,1)$.

Moreover, we have not been able to show the invariance of the norm of $H^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\alpha \in (1/2,1)$, under the transformation (1.8), where $u \in H^{2+\alpha,1+\alpha/2}(\Omega^T)$. More precisely we examine the following problem. Let the transformation (1.8) be generated by $w \in H^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\alpha \in (1/2,1)$. Then we have $x = x_w(\xi,t)$. Let $v = v(x,t)$ and $u(\xi,t) = v(x_w(\xi,t),t)$. We want both u, v to belong to $H^{2+\alpha,1+\alpha/2}(\Omega^T)$. We can easily show that

$$c_1\|v\|_{L_2(0,T;H^{2+\alpha}(\Omega))} \le \|u\|_{L_2(0,T;H^{2+\alpha}(\Omega))} \le c_2\|v\|_{L_2(0,T;H^{2+\alpha}(\Omega))}, \tag{1.16}$$

where c_1, c_2 are positive, nonvanishing and depend on $\|w\|_{2+\alpha,\Omega^T}$.

However, we are not able to show (1.16) for v and u in $H^{2+\alpha,1+\alpha/2}(\Omega^T)$ under the assumption that c_1, c_2 depend on $\|w\|_{2+\alpha,\Omega^T}$ only. We can only show this if w is more regular but this is not appropriate for us.

Now we formulate the main result.

Theorem 1.3. *Assume that the equilibrium solution is defined by* (1.6) *and the total mass of the fluid is determined by* (1.5). *Assume the thermodynamic restrictions* (1.3) *and that the barotropic motion with state equation* (1.4) *is considered. Assume that the bounded domain* Ω *is not rotationally symmetric and* $S \in C^{2+\alpha}$, $\varrho_0, v_0 \in H^{1+\alpha}(\Omega)$, $1/\varrho_0 \in L_\infty(\Omega)$, $g \in H^{\alpha,\alpha/2}(\Omega^T)$, $p \in C^2$, $\alpha \in (1/2,1)$, $g = g(\xi,t) = f(x_u(\xi,t),t)$ *and* $x = x_u(\xi,t)$ *defined by* (1.8) *determines the transformation between the Eulerian and Lagrangian coordinates. Assume that* $\|v_0\|_{1+\alpha,\Omega}$, $\|\varrho_0 - \varrho_e\|_{1+\alpha,\Omega}$, $\|g\|_{\alpha,\Omega^t}$, $t \in \mathbb{R}_+$ *are sufficiently small*, *where* $\varrho_e = M/|\Omega|$. *Then there exists a global solution to problem* (1.1) *such that*

$$u \in H^{2+\alpha,1+\alpha/2}(\Omega^t), \qquad \eta \in H^{1+\alpha,1/2+\alpha/2}(\Omega^t),$$

$t \in \mathbb{R}_+$, *where* $u(\xi,t) = v(x_u(\xi,t),t)$, $\eta(\xi,t) = \varrho(x_u(\xi,t),t)$ *and* u *determines the transformation* (1.8).

Finally, we want to add some historical remarks and comments concerning global-in-time existence for viscous compressible fluids. The first results concerning global-in-time existence for the Cauchy problem for equations of viscous compressible and heat conducting fluids were obtained by Matsumura and Nishida [6, 7]. The corresponding result for an initial boundary value problem for the same equations was also shown by Matsumura and Nishida [8, 9]. They considered only a half space and they assumed that the initial density, velocity and temperature are from H^3 and the external force has a gradient form. They assumed the Dirichlet boundary conditions for velocity and the Neumann boundary conditions for temperature. Valli [15] improved the results for the barotropic case showing global-in-time existence for an arbitrary bounded domain $\Omega \subset \mathbb{R}^3$, general small external force field and for initial density and velocity from $H^2(\Omega)$. The result of [15] was extended to general viscous compressible heat-conducting fluid equations by Valli–Zajączkowski [16]. The global-in-time existence results from [6–9, 15, 16] follow from some a priori estimate whose proof depends heavily on the L_2-approach. However, the results are not sharp in the L_2-framework, where sharpness means that the existence result cannot be proved with less regular data. In the L_p-framework global-in-time existence of solutions for equations of viscous compressible heat-conducting fluids with Dirichlet boundary conditions was proved by Ströhmer in [13, 14]. However his results are not sharp because he imposed too much regularity on data functions.

Our result is sharp for the L_2-approach and this is the reason why the fractional derivative spaces have been used. However this forced us to use the Lagrangian coordinates which are not appropriate for problems in fixed domains. Unfortunately in general our result is not sharp because then the L_p-approach is necessary.

Our result can be extended to the Dirichlet and Neumann problems and also to equations of viscous compressible heat-conducting fluids.

2. Notation and auxiliary results. First we introduce a partition of unity $(\{\widetilde{\Omega}_i\}, \{\zeta_i\})$, $\Omega = \bigcup_i \widetilde{\Omega}_i$. Let $\widetilde{\Omega}$ be one of the $\widetilde{\Omega}_i$'s and $\zeta(\xi) = \zeta_i(\xi)$ the corresponding function. If $\widetilde{\Omega}$ is an interior subdomain, then let $\widetilde{\omega}$ be such that $\overline{\widetilde{\omega}} \subset \widetilde{\Omega}$ and $\zeta(\xi) = 1$ for $\xi \in \widetilde{\omega}$. Otherwise we assume that $\overline{\widetilde{\Omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \subset \overline{\widetilde{\Omega}}$. Let $\beta \in \overline{\widetilde{\omega}} \cap S \subset \overline{\widetilde{\Omega}} \cap S$, $\widetilde{S} \equiv S \cap \widetilde{\Omega}$. Introduce local coordinates $\{y\}$ connected with $\{\xi\}$ by

$$y_k = \alpha_{kl}(\xi_l - \beta_l), \qquad \alpha_{3k} = n_k(\beta), \qquad k = 1, 2, 3, \tag{2.1}$$

where α_{kl} is a constant orthogonal matrix, such that $\widetilde{S}$ is determined by $y_3 = F(y_1, y_2)$, $F \in H^{3/2+\alpha}$ and

$$\widetilde{\Omega} = \{y : |y_i| < d,\ i = 1, 2,\ F(y') < y_3 < F(y') + d,\ y' = (y_1, y_2)\}.$$

Next we introduce functions u' and q' by

$$u'_i(y) = \alpha_{ij} u_j(\xi)|_{\xi=\xi(y)}, \qquad q'(y) = q(\xi)|_{\xi=\xi(y)}, \tag{2.2}$$

where $\xi = \xi(y)$ is the inverse transformation to (2.1). Further we introduce new variables by

$$z_i = y_i, \quad i = 1,2, \quad z_3 = y_3 - \widetilde{F}(y), \quad y \in \widetilde{\Omega}, \tag{2.3}$$

which will be denoted by $z = \Phi(y)$, where $\widetilde{F}$ is an extension of F to $\widetilde{\Omega}$ with $\widetilde{F} \in H^{2+\alpha}(\widetilde{\Omega})$. Let $\widehat{\Omega} = \Phi(\widetilde{\Omega}) = \{z : |z_i| < d,\ i = 1,2,\ 0 < z_3 < d\}$ and $\widehat{S} = \Phi(\widetilde{S})$. Define

$$\widehat{u}(z) = u'(y)|_{y=\Phi^{-1}(z)}, \quad \widehat{q}(z) = q'(y)|_{y=\Phi^{-1}(z)}. \tag{2.4}$$

Introduce $\widehat{\nabla}_k = \xi_{lx_k} z_{i\xi_l} \nabla_{z_i}|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi) = \Phi(\psi_0(\xi))$ and $y = \psi_0(\xi)$ is described by (2.1). Introduce also the notation

$$\widetilde{u}(\xi) = u(\xi)\zeta(\xi), \quad \widetilde{q}_\sigma(\xi) = q_\sigma(\xi)\zeta(\xi), \quad \xi \in \widetilde{\Omega},\ \widetilde{\Omega} \cap S = \emptyset, \tag{2.5}$$

$$\widetilde{u}(z) = \widehat{u}(z)\widehat{\zeta}(z), \quad \widetilde{q}_\sigma(z) = \widehat{q}_\sigma(z)\widehat{\zeta}(z), \quad z \in \widehat{\Omega} = \Phi(\widetilde{\Omega}),\ \overline{\widehat{\Omega}} \cap S \neq \emptyset, \tag{2.6}$$

where $\widehat{\zeta}(z) = \zeta(\xi)|_{\xi=\chi^{-1}(z)}$.

Under the above notation problem (1.14) has the following form in an interior subdomain:

$$\begin{aligned}
&\eta\widetilde{u}_{it} - \nabla_j T_{ij}(\widetilde{u}, \widetilde{q}_\sigma) = \eta\widetilde{g}_i - \nabla_{uj} B_{uij}(u,\zeta) - T_{uij}(u, q_\sigma)\nabla_{uj}\zeta \\
&\qquad - (\nabla_j T_{ij}(\widetilde{u}, \widetilde{q}_\sigma) - \nabla_{uj} T_{uij}(\widetilde{u}, \widetilde{q}_\sigma)) \equiv \eta\widetilde{g}_i + k_{1i}, \quad i = 1,2,3, \\
&\widetilde{q}_{\sigma t} + p_1 \nabla \cdot \widetilde{u} = p_1 u \cdot \nabla_u \zeta + (p_1 - \Psi(\eta))\nabla_u \cdot u\zeta \\
&\qquad\qquad + p_1(\nabla \cdot \widetilde{u} - \nabla_u \cdot \widetilde{u}) \equiv k_2,
\end{aligned} \tag{2.7}$$

where $\Psi(\eta) = p_\eta(\eta)\eta$, $p_1 = \Psi(\varrho_e)$, and in a boundary subdomain:

$$\begin{aligned}
&\widehat{\eta}\,\widetilde{u}_{it} - \nabla_j T_{ij}(\widetilde{u}, \widetilde{q}_\sigma) = \widehat{\eta}\,\widetilde{g}_i - \widehat{\nabla}_j \widehat{B}_{ij}(\widehat{u}, \widehat{\zeta}) - \widehat{T}_{ij}(\widehat{u}, \widehat{q}_\sigma)\widehat{\nabla}_j\widehat{\zeta} \\
&\qquad - (\nabla_j T_{ij}(\widetilde{u}, \widetilde{q}_\sigma) - \widehat{\nabla}_j \widehat{T}_{ij}(\widetilde{u}, \widetilde{q}_\sigma)) \equiv \widehat{\eta}\,\widetilde{g}_i + k_{3i}, \quad i = 1,2,3, \\
&\widetilde{q}_{\sigma t} + p_1 \nabla \cdot \widetilde{u} = p_1 \widehat{u} \cdot \widehat{\nabla}\widehat{\zeta} + (p_1 - \Psi(\widehat{\eta}))\widehat{\nabla} \cdot \widetilde{u}\widehat{\zeta} \\
&\qquad\qquad + p_1(\nabla \cdot \widetilde{u} - \widehat{\nabla} \cdot \widetilde{u}) \equiv k_4, \\
&\widehat{n} \cdot \widehat{\mathbb{D}}(\widetilde{u}) \cdot \widehat{\tau}_\alpha = \widehat{n} \cdot \widehat{\mathbb{B}}(\widehat{u}, \widehat{\zeta}) \cdot \widehat{\tau}_\alpha - \gamma\widehat{u} \cdot \widehat{\tau}_\alpha \equiv k_{5\alpha}, \quad \alpha = 1,2, \\
&\widehat{n} \cdot \widetilde{u} = 0,
\end{aligned} \tag{2.8}$$

where the summation convention is assumed,

$$B_{uij}(u,\zeta) = \mu(u_i \nabla_{uj}\zeta + u_j \nabla_{ui}\zeta) + (\nu - \mu)\delta_{ij} u \cdot \nabla_u \zeta, \tag{2.9}$$

and $\widehat{\mathbb{T}}$, $\widehat{\mathbb{B}}$ indicate that the operator ∇_u is replaced by $\widehat{\nabla}$. Finally the dot $\cdot$ denotes the scalar product in $\mathbb{R}^3$.

Let $\psi(y) \equiv y_3 - \widetilde{F}(y) = 0$. Then $\widehat{n} = \nabla\psi/|\nabla\psi|$, $\overline{\tau}_1 = (0,1,0) \times \widehat{n}$, $\overline{\tau}_2 = (1,0,0) \times \widehat{n}$. Moreover, in the next considerations we denote z_1, z_2 by τ, and z_3 by n. From [11] we recall the Korn inequality. For vectors

$u, v \in H^1(\Omega)$ we introduce

$$E(u,v) = \int_\Omega (\partial_{x_i} u_j + \partial_{x_j} u_i)(\partial_{x_i} v_j + \partial_{x_j} v_i)\, dx. \tag{2.10}$$

The vectors which satisfy $E(u,u) = 0$ have the form

$$u = A + B \times x, \tag{2.11}$$

where A, B are arbitrary constant vectors.

We define $\widetilde{H}(\Omega) = \{u : E(u,u) < \infty,\ u \cdot \overline{n} = 0 \text{ on } S\}$. If Ω is a domain obtained by rotation about the vector B we set $H(\Omega) = \{u \in \widetilde{H}(\Omega) : \int_\Omega u \cdot B \times x\, dx = 0\}$. Otherwise we set $H(\Omega) = \widetilde{H}(\Omega)$.

Lemma 2.1 (see [11]). *Let $S \in C^2$. Then for $u \in H(\Omega)$,*

$$\|u\|_{1,\Omega}^2 \le c_0 E(u,u). \tag{2.12}$$

Finally, we introduce the notation. By $H^{k+\alpha,k/2+\alpha/2}(\Omega^T)$, $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (1/2, 1)$, we denote the Hilbert space with the norm

$$\begin{aligned}\|u\|^2_{H^{k+\alpha,k/2+\alpha/2}(\Omega^T)} = & \sum_{|\beta|+2i\le k} \|\partial_x^\beta \partial_t^i u\|^2_{L_2(\Omega^T)} \\ & + \sum_{|\beta|=k} \int_0^T \int_\Omega \int_\Omega \frac{|\partial_x^\beta u(x,t) - \partial_{x'}^\beta u(x',t)|^2}{|x-x'|^{3+2\alpha}}\, dx\, dx'\, dt \\ & + \int_\Omega \int_0^T \int_0^T \frac{|\partial_t^{[k/2]} u(x,t) - \partial_{t'}^{[k/2]} u(x,t')|^2}{|t-t'|^{1+\alpha+k-2[k/2]}}\, dx\, dt\, dt'.\end{aligned}$$

Similarly we introduce the spaces $H^{k+\alpha}(\Omega)$, $H^{k+\alpha,k/2+\alpha/2}(S^T)$. Moreover, we use the notation

$$\begin{aligned}\|u\|_{H^{k+\alpha,k/2+\alpha/2}(\Omega^T)} &= \|u\|_{k+\alpha,\Omega^T}, \\ \|u\|_{H^{k+\alpha}(\Omega)} &= \|u\|_{k+\alpha,\Omega}, \qquad \|u\|_{H^{k+\alpha}(0,T)} = \|u\|_{k+\alpha,(0,T)}, \\ \|u\|_{L_p(\Omega)} &= |u|_{p,\Omega}, \qquad p \in [1,\infty], \\ \|u\|_{H^s(\Omega^T)} &= \|u\|_{s,\Omega^T}, \qquad s \in \mathbb{N} \cup \{0\}.\end{aligned}$$

The spaces $H^{k+\alpha,k/2+\alpha/2}(\Omega^T)$ are Sobolev–Slobodetskiĭ spaces and are denoted also by $W_2^{k+\alpha,k/2+\alpha/2}(\Omega^T)$. They are particular cases of Besov spaces.

First we recall some properties of isotropic Besov spaces which are frequently used in this paper; next we define anisotropic Besov spaces and formulate some imbedding theorems which we need.

Let us introduce the differences

$$\Delta_i(h) u(x) = u(x + h e_i) - u(x),$$

where $x \in \mathbb{R}^n$ and e_i, $i = 1, \ldots, n$, are the standard unit vectors. Then we define inductively the m-difference

$$\Delta_i^m(h)u(x) = \Delta_i(h)(\Delta_i^{m-1}(h)u(x)) = \sum_{j=0}^{m}(-1)^{m-j}c_{jm}u(x+jhe_i),$$

where $c_{jm} = \binom{m}{j} = \frac{m!}{j!(m-j)!}$. Moreover, we introduce the difference

$$\Delta(y)f(x) = f(x+y) - f(x), \qquad x, y \in \mathbb{R}^n,$$

and inductively

$$\Delta^m(y)f(x) = \Delta(y)(\Delta^{m-1}(y)f(x)).$$

Since

$$\Delta(x-y)f(y) = f(x) - f(y)$$

we have

$$\Delta^m(x-y)f(y) = \sum_{i=1}^{n}\Delta^m((x-y)\cdot e_i)f(y) = \sum_{i=1}^{n}\Delta_i^m(h)f(y),$$

where the last equality holds for $(x-y)\cdot e_i = h$.

We define the isotropic Besov spaces by introducing the norm (see [2, Sect. 18])

$$\|u\|_{B_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \sum_{i=1}^{n}\bigg(\int_0^{h_0} dh \int_{\mathbb{R}^n} dx\, \frac{|\Delta_i^m(h)\partial_{x_i}^k u|^p}{h^{1+(l-k)p}}\bigg)^{1/p}, \tag{2.13}$$

where $m > l-k$, $m, k \in \mathbb{N}\cup\{0\}$, $l \in \mathbb{R}_+$ and $l \notin \mathbb{Z}$.

It was shown in [4] that the Besov spaces defined by (2.13) all coincide and have equivalent norms for all m, k satisfying $m > l-k$.

Next we define the L_p-scale of Sobolev–Slobodetskiĭ spaces by introducing the norm

$$\|u\|_{W_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \sum_{i=1}^{n}\bigg(\int_0^{h_0} dh \int_{\mathbb{R}^n} dx\, \frac{|\Delta_i(h)\partial_{x_i}^{[l]} u|^p}{h^{1+p(l-[l])}}\bigg)^{1/p}, \tag{2.14}$$

where $l \notin \mathbb{Z}$, $[l]$ is the integral part of l. We frequently write $l = k+\alpha$, $k \in \mathbb{N}\cup\{0\}$, $\alpha \in (0,1)$, so $k = [l]$ and $\alpha = l-[l]$.

By the Golovkin theorem [4] the norms of the spaces $B_p^l(\mathbb{R}^n)$ and $W_p^l(\mathbb{R}^n)$ are equivalent.

We also define the norms

$$\|u\|_{\widetilde{B}_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \bigg(\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, \frac{|\Delta^m(x-y)\partial_y^k u(y)|^p}{|x-y|^{n+p(l-k)}}\bigg)^{1/p} \tag{2.15}$$

for any $m > l - k$ (here $\partial_y^k u = \sum_{|\alpha|=k} D_y^\alpha u$) and

$$(2.16) \qquad \|u\|_{\widetilde{W}_p^l(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \bigg(\int\limits_{\mathbb{R}^n} dx \int\limits_{\mathbb{R}^n} dy\, \frac{|\Delta(x-y)\partial_y^{[l]} u(y)|^p}{|x-y|^{n+p(l-[l])}}\bigg)^{1/p}.$$

Using Lemma 7.44 of [1] we can show that the spaces $\widetilde{B}_p^l(\mathbb{R}^n)$ and $B_p^l(\mathbb{R}^n)$ (with $h_0 = \infty$) and $\widetilde{W}_p^l(\mathbb{R}^n)$ and $W_p^l(\mathbb{R}^n)$ (with $h_0 = \infty$) all coincide and have equivalent norms.

Therefore all norms (2.13)–(2.16) are equivalent. Now we recall some imbedding theorems for Besov spaces (see [2, Sect. 18]):

$$(2.17) \qquad D^\sigma B_p^l(\mathbb{R}^n) \subset B_q^\varrho(\mathbb{R}^n) \quad \text{for } n/p - n/q + \sigma + \varrho \le l.$$

For $\varkappa = 1 - \frac{1}{l}\big(\frac{n}{p} - \frac{n}{q} + \sigma + \varrho\big) > 0$ we have the interpolation inequality

$$(2.18) \qquad \|\partial_x^\sigma u\|_{B_q^\varrho(\mathbb{R}^n)} \le \varepsilon^{1-\varkappa}\|u\|_{B_p^l(\mathbb{R}^n)} + c\varepsilon^{-\varkappa}\|u\|_{L_{p_0}(\mathbb{R}^n)},$$

with $p_0 \ge 1$, $\varepsilon \in (0,1)$. In the above notation $B_p^l(\mathbb{R}^n)$ with $l \in \mathbb{Z}_+$ is a Sobolev space and $L_p(\mathbb{R}^n) = W_p^0(\mathbb{R}^n)$.

All the above remarks can be applied to spaces of functions defined on a bounded domain $\Omega \subset \mathbb{R}^n$. Moreover by using a partition of unity we can define spaces of traces on the boundary of Ω and formulate the corresponding trace theorems.

Now we introduce some anisotropic Sobolev–Slobodetskiĭ and Besov spaces which are necessary for our considerations. First we define

$$\|u\|_{L_{p_1,p_2}(\Omega^T)} = \bigg(\int\limits_0^T dt\, \bigg(\int\limits_\Omega dx\, |u(x,t)|^{p_1}\bigg)^{p_2/p_1}\bigg)^{1/p_2}$$

and

$$\|u\|_{\overline{L}_{p_1,p_2}(\Omega^T)} = \bigg(\int\limits_\Omega dx\, \bigg(\int\limits_0^T dt\, |u(x,t)|^{p_1}\bigg)^{p_2/p_1}\bigg)^{1/p_2},$$

where $p_i \in [1,\infty]$, $i = 1,2$.

We need imbedding theorems between the above spaces and $W_2^{l,l/2}(\Omega^T)$, $l \in \mathbb{R}_+$, $\Omega \subset \mathbb{R}^n$. We apply the results from [2, Sect. 18]. Since $W_2^{l,l/2}(\Omega^T)$ is isotropic with respect to the power of integration we have, for $2 \le p_1, p_2 < \infty$,

$$(2.19) \quad \partial_x^{\alpha_1}\partial_t^{\alpha_2} W_2^{l,l/2}(\Omega^T) \subset L_{p_1,p_2}(\Omega^T) \text{ if } \frac{n}{2} - \frac{n}{p_1} + \frac{2}{2} - \frac{2}{p_2} + \alpha_1 + 2\alpha_2 \le l,$$

$$(2.20) \quad \partial_x^{\alpha_1}\partial_t^{\alpha_2} W_2^{l,l/2}(\Omega^T) \subset \overline{L}_{p_1,p_2}(\Omega^T) \text{ if } \frac{n}{2} - \frac{n}{p_2} + \frac{2}{2} - \frac{2}{p_1} + \alpha_1 + 2\alpha_2 \le l,$$

$$(2.21)\quad \partial_x^{\alpha_1}\partial_t^{\alpha_2}W_2^{l,l/2}(\Omega^T)\subset L_p(0,T;B_q^\sigma(\Omega))\quad \text{if } \frac{n}{2}-\frac{n}{q}+\frac{2}{2}-\frac{2}{p}+\alpha_1+2\alpha_2+\sigma\le l,$$

and finally

$$(2.22)\quad \partial_x^{\alpha_1}\partial_t^{\alpha_2}W_2^{l,l/2}(\Omega^T)\subset L_p(\Omega;B_q^\sigma(0,T))\quad \text{if } \frac{n}{2}-\frac{n}{p}+\frac{2}{2}-\frac{2}{q}+\alpha_1+2\alpha_2+2\sigma\le l,\qquad 2\le p,q<\infty.$$

Moreover, the corresponding interpolation inequalities hold.

3. Inequality for global existence. First we obtain an energy type inequality:

LEMMA 3.1. *Assume that* (v,ϱ_σ) *is the solution to problem* (1.11) *determined by Theorem* 1.2. *Let the assumptions of Lemma* 2.1 *hold. Then*

$$(3.1)\quad \frac{d}{dt}\int_\Omega\left(\frac{1}{2}\varrho v^2+\frac{p_\sigma^2}{2p_\varrho\varrho}\right)dx+c_0\|v\|_{1,\Omega}^2+\gamma\sum_{\alpha=1}^2\int_S|v\cdot\overline{\tau}_\alpha|^2\,dS \le \psi_1(|\varrho|_{\infty,\Omega^T},|1/\varrho|_{\infty,\Omega^T})\|p_\sigma\|_{1,\Omega}^4+\int_\Omega\varrho f\cdot v\,dx,$$

where ψ_1 *is an increasing positive function.*

Proof. Multiplying $(1.11)_1$ by v, integrating over Ω, using the boundary conditions and the continuity equation we obtain

$$(3.2)\quad \frac{d}{dt}\frac{1}{2}\int_\Omega\varrho v^2\,dx+\int_\Omega\mathbb{D}(v)\cdot\nabla v\,dx -\int_\Omega p_\sigma\operatorname{div}v\,dx+\gamma\sum_{\alpha=1}^2\int_S|v\cdot\overline{\tau}_\alpha|^2\,dS=\int_\Omega\varrho f\cdot v\,dx.$$

In view of the Korn inequality (see Lemma 2.1), using (1.12) in the third term on the l.h.s. of (3.2) yields

$$(3.3)\quad \frac{d}{dt}\frac{1}{2}\int_\Omega\varrho v^2\,dx+c_0\|v\|_{1,\Omega}^2+\int_\Omega\frac{p_\sigma}{p_\varrho\varrho}(p_{\sigma t}+v\cdot\nabla p_\sigma)\,dx +\gamma\sum_{\alpha=1}^2\int_S|v\cdot\overline{\tau}_\alpha|^2\,dS\le\int_\Omega\varrho f\cdot v\,dx,$$

where c_0 is the constant from the Korn inequality.

Continuing calculations in the third term on the l.h.s. of (3.3) we get

$$\begin{aligned}\int_\Omega \frac{1}{2p_\varrho \varrho}(p^2_{\sigma,t}+v\cdot\nabla p^2_\sigma)\,dx &= \int_\Omega \left[\varrho\left(\frac{p^2_\sigma}{2p_\varrho\varrho^2}\right)_t + \varrho v\cdot\nabla\left(\frac{p^2_\sigma}{2p_\varrho\varrho^2}\right)\right]dx\\ &\quad - \int_\Omega \frac{1}{2}\varrho p^2_\sigma\left(\frac{1}{p_\varrho\varrho^2}\right)_p (p_{\sigma t}+v\cdot\nabla p_\sigma)\,dx\\ &= \frac{d}{dt}\int_\Omega \frac{p^2_\sigma}{2p_\varrho\varrho}\,dx + N_1,\end{aligned}$$

where

$$N_1 = \frac{1}{2}\int_\Omega p_\varrho\varrho^2\left(\frac{1}{p_\varrho\varrho^2}\right)_p p^2_\sigma \operatorname{div} v\,dx,$$

so

$$|N_1| \le \varepsilon\|\operatorname{div} v\|^2_{0,\Omega} + \varphi_1(1/\varepsilon, |\varrho|_{\infty,\Omega^T}, |1/\varrho|_{\infty,\Omega^T})\|p_\sigma\|^4_{1,\Omega},$$

where φ_1 is an increasing positive function. Using the above considerations in (3.3) we obtain (3.1). This concludes the proof.

Now we show

LEMMA 3.2. *For the local solution determined by Theorems* 1.1, 1.2 *we have*

$$\begin{aligned}(3.4)\qquad \varphi(t,\Omega) + \|u\|^2_{2+\alpha,\Omega^t} + \|q_\sigma\|^2_{1+\alpha,\Omega^t} &\le c(\|u\|^2_{0,\Omega^t} + \|q_\sigma\|^2_{0,\Omega^t})\\ &\quad + c\|g\|^2_{\alpha,\Omega^t} + cX_1^2 + \varphi(0,\Omega),\end{aligned}$$

where $t \le T$, T *is the time of local existence*, *and* $\varphi(t,\Omega)$ *and* X_1 *are defined by* (3.62) *and* (3.64), *respectively.*

Proof. First we obtain the inequality for interior subdomains. Assume that $\xi_0 \in \widetilde{\Omega}$, where $\widetilde{\Omega}$ is an interior subdomain, and $\zeta \in C_0^\infty(\mathbb{R}^3)$ is the corresponding function from the partition of unity such that $\zeta(\xi) = 1$ for $\xi \in B_\lambda(\xi_0) = \{\xi \in \mathbb{R}^3 : |\xi - \xi_0| < \lambda\}$ and $\zeta(\xi) = 0$ for $\xi \in \mathbb{R}^3 \setminus B_{2\lambda}(\xi_0)$. Denote by $\Delta^s(z)f(x)$ the sth finite difference of f such that

$$\Delta^s(z)f(x) = \sum_{k=0}^{s} c_s^k(-1)^{s-k}f(x+kz),$$

where $c_s^k = \binom{s}{k} = \frac{s!}{k!(s-k)!}$.

In this proof, functions φ_j, ψ_j, $j \in \mathbb{N}\cup\{0\}$, are increasing continuous positive functions of their arguments. Applying $\Delta^s(z)$ to $(2.7)_1$, multiplying the result by $\Delta^s(z)\widetilde{u}_i$, integrating over $B_{2\lambda}(\xi_0)$ and integrating with respect to z over $\mathbb{R}^3$ with the weight $1/|z|^{3+2(1+\alpha)}$ we obtain

$$\begin{aligned}(3.5)\quad &\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\eta\Delta^s(z)\widetilde{u}_{it}\Delta^s(z)\widetilde{u}_i}{|z|^{3+2(1+\alpha)}}\\
&\quad - \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\nabla_{\xi_j}(\Delta^s(z)T_{ij}(\widetilde{u},\widetilde{q}_\sigma))\Delta^s(z)\widetilde{u}_i}{|z|^{3+2(1+\alpha)}}\\
&= -\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{(\Delta^s(z)(\eta\widetilde{u}_{it}) - \eta\Delta^s(z)\widetilde{u}_{it})\Delta^s(z)\widetilde{u}_i}{|z|^{3+2(1+\alpha)}}\\
&\quad + \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\Delta^s(z)(\eta\widetilde{g}_i + k_{1i})\Delta^s(z)\widetilde{u}_i}{|z|^{3+2(1+\alpha)}} \equiv I_1 + I_2.\end{aligned}$$

Using the continuity equation $(1.15)_1$ in the first term and integrating by parts in the second with the use of the boundary condition $\widetilde{u}|_{\partial B_{2\lambda}(\xi_0)} = 0$ we obtain

$$\begin{aligned}(3.6)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\eta|\Delta^s(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}}\\
&\quad + \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\mathbb{D}(\Delta^s(z)\widetilde{u})\cdot\nabla\Delta^s(z)\widetilde{u}}{|z|^{3+2(1+\alpha)}}\\
&\quad - \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\Delta^s(z)\widetilde{q}_\sigma \operatorname{div}(\Delta^s(z)\widetilde{u})}{|z|^{3+2(1+\alpha)}}\\
&= I_1 + I_2 - \frac{1}{2}\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\eta \operatorname{div}_u u|\Delta^s(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}}\\
&\equiv \sum_{i=1}^3 I_i.\end{aligned}$$

To examine the last term on the l.h.s. of (3.6) we use $(2.7)_2$. Applying $\Delta^s(z)$ to $(2.7)_2$, multiplying the result by $\Delta^s(z)\widetilde{q}_\sigma$, integrating over $B_{2\lambda}(\xi_0)$ and integrating with respect to z with the weight $1/|z|^{3+2(1+\alpha)}$ we obtain

$$\begin{aligned}(3.7)\quad &\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{|\Delta^s(z)\widetilde{q}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\\
&\quad + p_1\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\Delta^s(z)\widetilde{q}_\sigma \operatorname{div}\Delta^s(z)\widetilde{u}}{|z|^{3+2(1+\alpha)}}\\
&= \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi\, \frac{\Delta^s(z)k_2\Delta^s(z)\widetilde{q}_\sigma}{|z|^{3+2(1+\alpha)}} \equiv I_4.\end{aligned}$$

Multiplying (3.7) by $1/p_1$, adding to (3.6) and using the Korn inequality we get

$$\tag{3.8} \frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \left(\frac{\eta|\Delta^s(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}} + \frac{1}{p_1}\cdot\frac{|\Delta^s(z)\widetilde{q}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\right) + \frac{1}{c_0}\int\limits_{\mathbb{R}^3} dz \frac{\|\Delta^s(z)\widetilde{u}\|^2_{1,B_{2\lambda}(\xi_0)}}{|z|^{3+2(1+\alpha)}} \le \sum_{i=1}^3 I_i + \frac{1}{p_1}I_4.$$

Now we estimate the terms on the r.h.s. of (3.8). Since $\alpha \in (1/2, 1)$ and s must be such that $s > 1+\alpha$ (see [4]) we assume that $s = 2$. First we consider

$$\begin{aligned} I_2 &= \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{\Delta^2(z)(\eta\widetilde{g}_i + k_{1i})\cdot\Delta^2(z)\widetilde{u}_i}{|z|^{3+2(1+\alpha)}} \\ &= -\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{\Delta(z)(\eta\widetilde{g}_i + k_{1i})\cdot\Delta^3(z)\widetilde{u}_i}{|z|^{3+2(1+\alpha)}} \\ &\le \varepsilon_1 \int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta^3(z)\widetilde{u}|^2}{|z|^{3+2(2+\alpha)}} + c(\varepsilon_1)\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)(\eta\widetilde{g} + k_1)|^2}{|z|^{3+2\alpha}} \\ &\equiv I_2^1 + I_2^2, \end{aligned}$$

where

$$\begin{aligned} I_2^2 \le c\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{1}{|z|^{3+2\alpha}}(&|\Delta(z)(\eta\widetilde{g})|^2 + |\Delta(z)\nabla_u\mathbb{B}_u(u,\zeta)|^2 \\ &+ |\Delta(z)\mathbb{T}_u(u,q_\sigma)\nabla_u\zeta|^2 + |\Delta(z)(\nabla\mathbb{T}(\widetilde{u},\widetilde{q}_\sigma) - \nabla_u\mathbb{T}_u(\widetilde{u},\widetilde{q}_\sigma))|^2) \equiv \sum_{i=1}^4 I_5^i. \end{aligned}$$

Next we estimate

$$\begin{aligned} I_5^1 &\le c\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\eta\widetilde{g}|^2 + |\eta\Delta(z)\widetilde{g}|^2}{|z|^{3+2\alpha}} \\ &\le c\bigg(\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\eta|^{2p}}{|z|^{3+2p(\alpha+\varepsilon/2)}}\bigg)^{1/p}\bigg(\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{|\widetilde{g}(\xi,t)|^{2p'}}{|z|^{3-p'\varepsilon}}\bigg)^{1/p'} \\ &\quad + c|\eta|_{\infty,\Omega}\int\limits_{\mathbb{R}^3} dz \int\limits_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\widetilde{g}(\xi,t)|^2}{|z|^{3+2\alpha}}, \end{aligned}$$

for all p, p' with $1/p + 1/p' = 1$ and $\varepsilon > 0$. Using the imbeddings $B_2^{1+\alpha}(\Omega) \subset B_{2p}^{\alpha+\varepsilon/2}(\Omega)$, $\Omega \subset \mathbb{R}^3$, valid if $3/2 - 3/(2p) + \alpha + \varepsilon/2 < 1 + \alpha$, and $B_2^\alpha(\Omega) \subset L_{2p'}(\Omega)$, $\Omega \subset \mathbb{R}^3$, valid if $3/2 - 3/(2p') < \alpha$, which both hold for suitable p,

ε because $3/2+\varepsilon/2<1+\alpha$ can be satisfied for $\alpha>1/2$, we obtain

$$I_5^1 \le c\|\eta\|^2_{1+\alpha,\Omega}\|\widetilde{g}\|^2_{\alpha,\Omega},$$

where we used the fact that $B_2^{1+\alpha}(\Omega)\subset L_\infty(\Omega)$ for $\alpha>1/2$.

Introducing the quantity $\alpha(t)=\int_0^t u_\xi(\xi,\tau)\,d\tau$ we see that

$$\xi_x - I = \alpha\psi_0(\alpha), \quad \xi_x^2 - I = \alpha\psi_1(\alpha),$$
$$\xi_x\nabla_\xi\xi_x = \nabla_\xi\alpha\psi_2(\alpha) + \alpha\nabla_\xi\alpha\psi_3(\alpha),$$

where ξ_x, I are matrices, I is the unit matrix,

$$|\alpha(t)|^2 \le T\int_0^t |u_\xi|^2\,d\tau \le cT\int_0^t \|u\|^2_{2+\alpha,\Omega}\,d\tau,$$

and

$$a \equiv cT^{1/2}\Big(\int_0^T \|u\|^2_{2+\alpha,\Omega}\,d\tau\Big)^{1/2}$$

so

$$\begin{aligned} &|\xi_x - I|^2, |\xi_x^2 - I|^2 \le a^2\psi_4(a), \\ &|\xi_x\nabla_\xi\xi_x|^2 \le \psi_5(a)\int_0^T |u_{\xi\xi}|^2\,d\tau. \end{aligned} \tag{3.8$'$}$$

Using the Hölder inequality and the imbeddings $B_2^\alpha(\Omega)\subset L_{2p}(\Omega)$, $B_2^{1+\alpha}(\Omega)\subset B^\alpha_{2p'}(\Omega)$, $\Omega\subset\mathbb{R}^3$, holding for $3/2-3/(2p)\le\alpha$, $3/2-3/(2p')+\alpha\le 1+\alpha$, $1/p+1/p'=1$, we have

$$\begin{aligned}
I_5^2 &= c\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} \frac{|\Delta(z)\nabla_u\mathbb{B}_u(u\varphi)|^2}{|z|^{3+2\alpha}}\,d\xi \\
&\le c\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \\
&\quad\times \frac{1}{|z|^{3+2\alpha}}[|\Delta(z)\nabla u|^2 + \xi_x^4|\nabla\zeta|^2 + |\nabla u|^2(|\Delta(z)\xi_x^2|^2|\nabla\zeta|^2 \\
&\quad + \xi_x^2|\Delta(z)\nabla\zeta|^2) + |\Delta(z)u|^2(|\xi_x\cdot\nabla\xi_x|^2|\nabla\zeta|^2 + \xi_x^2|\nabla^2\zeta|^2) \\
&\quad + |u|^2(|\Delta(z)(\xi_x\cdot\nabla\xi_x)|^2|\nabla\zeta|^2 + |\xi_x\cdot\nabla\xi_x|^2|\Delta(z)\nabla\zeta|^2 \\
&\quad + |\Delta(z)\xi_x^2|^2|\nabla^2\zeta|^2 + \xi_x^4|\Delta(z)\nabla^2\zeta|^2)] \\
&\le \psi_6(a)\Big[\|u\|^2_{1+\alpha,B_{2\lambda}(\xi_0)} + (\|u_\xi\|^2_{B^\alpha_{2p'}(B_{2\lambda}(\xi_0))} \\
&\quad + \|u\|^2_{B^\alpha_{2p'}(B_{2\lambda}(\xi_0))})\int_0^t \|u_{\xi\xi}\|^2_{L_{2p}(B_{2\lambda}(\xi_0))}\,d\tau\Big] \\
&\quad + \psi_7(a)(|\nabla u|^2_{\infty,B_{2\lambda}(\xi_0)} + |u|^2_{\infty,B_{2\lambda}(\xi_0)})
\end{aligned}$$

$$\times\Big[\int_0^t \|u\|^2_{2+\alpha,B_{2\lambda}(\xi_0)}\,d\tau+\int_0^t \|u_\xi\|^2_{B^\alpha_{2p'}(B_{2\lambda}(\xi_0))}\,d\tau\int_0^t \|u_{\xi\xi}\|^2_{L_{2p}(B_{2\lambda}(\xi_0))}\,d\tau\Big]$$

$$\le c\varphi_0(a)\|u\|^2_{1+\alpha,\Omega}+\varphi_1(a)\|u\|^2_{2+\alpha,\Omega}\int_0^t \|u\|^2_{2+\alpha,\Omega}\,d\tau.$$

Similarly, we have

$$I_5^3\le\varphi_2(a)(\|u\|^2_{1+\alpha,\Omega}+\|q_\sigma\|^2_{\alpha,\Omega}).$$

Since qualitatively

$$\nabla\mathbb{T}(\widetilde{u},\widetilde{q}_\sigma)-\nabla_u\mathbb{T}_u(\widetilde{u},\widetilde{q}_\sigma)=\varphi_3(\alpha)\int_0^t u_{\xi\xi}(\tau)\,d\tau\,\widetilde{u}_\xi+\alpha\varphi_4(\alpha)(\widetilde{u}_{\xi\xi}+\widetilde{q}_{\sigma\xi}),$$

where $\varphi_i(0)\ne 0$, $i=3,4$, we obtain

$$I_5^4\le\varphi_5(a)\int_0^t \|u\|^2_{2+\alpha,\Omega}\,d\tau\,(\|\widetilde{u}\|^2_{2+\alpha,\Omega}+\|\widetilde{q}_\sigma\|^2_{1+\alpha,\Omega}).$$

Summarizing the above considerations we have

$$|I_2|\le\varepsilon_1\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}}+\varphi_6(1/\varepsilon_1,a)(\|u\|^2_{1+\alpha,\widetilde{\Omega}}+\|q_\sigma\|^2_{\alpha,\widetilde{\Omega}})$$
$$+\varphi_7(1/\varepsilon_1,a)\int_0^t \|u\|^2_{2+\alpha,\Omega}\,d\tau\,(\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}}+\|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}),\tag{3.9}$$

where $\varepsilon_1\in(0,1)$.

Now we calculate

$$I_1=-\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\,\frac{(\Delta^2(z)\eta\widetilde{u}_t+2\Delta(z)\eta\Delta(z)\widetilde{u}_t)\Delta^2(z)\widetilde{u}}{|z|^{3+2(1+\alpha)}}$$
$$=-\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\,\frac{(\Delta^2(z)\eta_\sigma\widetilde{u}_t+2\Delta(z)\eta_\sigma\Delta(z)\widetilde{u}_t)\Delta^2(z)\widetilde{u}}{|z|^{3+2(1+\alpha)}}.$$

We have

$$|I_1|\le c\Big(\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\,\frac{|\Delta^2(z)\eta_\sigma|^2}{|z|^{3+2(1+\alpha)}}\Big)^{1/2}\Big(\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\,\frac{|\widetilde{u}_t|^p}{|z|^{3-\varepsilon p}}\Big)^{1/p}$$
$$\times\Big(\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\,\frac{|\Delta^2(z)\widetilde{u}|^{p'}}{|z|^{3+p'(1+\alpha)+\varepsilon p'}}\Big)^{1/p'}$$
$$+c\Big(\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\,\frac{|\Delta(z)\eta_\sigma|^{p_1}}{|z|^{3+(p_1/2)(1+\alpha)}}\Big)^{1/p_1}$$

$$\times\bigg(\int\limits_{\mathbb{R}^3}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\frac{|\Delta(z)\widetilde{u}_t|^{p_2}}{|z|^{3+(p_2/2)(1+\alpha)}}\bigg)^{1/p_2}$$
$$\times\bigg(\int\limits_{\mathbb{R}^3}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\frac{|\Delta^2(z)\widetilde{u}|^{p_3}}{|z|^{3+p_3(1+\alpha)}}\bigg)^{1/p_3}\equiv I_1^1+I_1^2,$$

for all p, p_i with $1/p+1/p'=1/2$ and $1/p_1+1/p_2+1/p_3=1$.

Using the imbeddings $B_2^\alpha(\Omega)\subset L_p(\Omega)$, $B_2^{2+\alpha}(\Omega)\subset B_{p'}^{1+\alpha+\varepsilon}(\Omega)$, valid if $3/2-3/p\le\alpha$, $3/2-3/p'+1+\alpha+\varepsilon\le 2+\alpha$, which both hold for suitable p, ε because $3/2+\varepsilon\le 1+\alpha$ can be satisfied for $\alpha>1/2$, we obtain

$$\begin{aligned}I_1^1&\le c\|\eta_\sigma\|_{1+\alpha,\Omega}\|\widetilde{u}_t\|_{\alpha,\widetilde{\Omega}}\|\widetilde{u}\|_{2+\alpha,\widetilde{\Omega}}\\&\le\varepsilon_2\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}}+c(\varepsilon_2)\|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\widetilde{u}_t\|^2_{\alpha,\widetilde{\Omega}}.\end{aligned}$$

Using the imbeddings $B_2^{1+\alpha}(\Omega)\subset B_{p_1}^{(1+\alpha)/2}(\Omega)$, $B_2^\alpha(\Omega)\subset B_{p_2}^{(1+\alpha)/2}(\Omega)$, $B_2^{2+\alpha}(\Omega)\subset B_{p_3}^{1+\alpha}(\Omega)$ valid for $3/2-3/p_1+(1+\alpha)/2\le 1+\alpha$, $3/2-3/p_2+(1+\alpha)/2\le\alpha$, $3/2-3/p_3+1+\alpha\le 2+\alpha$, which all hold for suitable p_i because $3/2\le 1+\alpha$, we get the same bound as for I_1^1,

$$I_1^2\le\varepsilon_2\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}}+c(\varepsilon_2)\|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\widetilde{u}_t\|^2_{\alpha,\widetilde{\Omega}}.$$

Next we have

$$|I_3|\le\varepsilon_3\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}}+c(\varepsilon_3)\|u\|^2_{2+\alpha,\Omega}\|\widetilde{u}\|^2_{1+\alpha,\widetilde{\Omega}}.$$

Finally, we have

$$\begin{aligned}|I_4|\le{}&\varepsilon_4\int\limits_{\mathbb{R}^3}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\frac{|\Delta^2(z)\widetilde{q}_\sigma|^2}{|z|^{3+2(1+\alpha)}}+\varphi_8(1/\varepsilon_4,a)\Big[\|u\|^2_{1+\alpha,\widetilde{\Omega}}\\&+\|u\|^2_{2+\alpha,\widetilde{\Omega}}\int_0^t\|u\|^2_{2+\alpha,\Omega}\,d\tau+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\|u\|^2_{2+\alpha,\widetilde{\Omega}}\\&+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\|u\|^2_{2+\alpha,\widetilde{\Omega}}\int_0^t\|u\|^2_{2+\alpha,\Omega}\,d\tau+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\int_0^t\|u\|^2_{2+\alpha,\Omega}\,d\tau\Big].\end{aligned}$$

Summarizing the above considerations we obtain

$$\begin{aligned}(3.10)\qquad&\frac{d}{dt}\int\limits_{\mathbb{R}^3}dz\int\limits_{B_{2\lambda}(\xi_0)}d\xi\,\bigg(\frac{\eta}{2}\cdot\frac{|\Delta^2(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}}+\frac{1}{2p_1}\cdot\frac{|\Delta^2(z)\widetilde{q}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\bigg)\\&\quad+\frac{1}{c_0}\int\limits_{\mathbb{R}^3}dz\,\frac{\|\Delta^2(z)\widetilde{u}\|^2_{1,B_{2\lambda}(\xi_0)}}{|z|^{3+2(1+\alpha)}}\le\varepsilon_5(\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}}+\|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})\end{aligned}$$

$$+\varphi_9(1/\varepsilon_5,a)(\|u\|^2_{1+\alpha,\widetilde\Omega}+\|q_\sigma\|^2_{\alpha,\widetilde\Omega})$$

$$+\varphi_{10}(1/\varepsilon_5,a)\int_0^t\|u\|^2_{2+\alpha,\Omega}\,d\tau\,(\|u\|^2_{2+\alpha,\widetilde\Omega}+\|q_\sigma\|^2_{1+\alpha,\widetilde\Omega}+\|q_\sigma\|^2_{1+\alpha,\widetilde\Omega}\|u\|^2_{2+\alpha,\widetilde\Omega})$$

$$+\varphi_{11}(1/\varepsilon_5,a)(\|q_\sigma\|^2_{1+\alpha,\widetilde\Omega}\|\widetilde u_t\|^2_{\alpha,\widetilde\Omega}+\|u\|^2_{2+\alpha,\widetilde\Omega}\|\widetilde u\|^2_{1+\alpha,\widetilde\Omega}$$

$$+\|q_\sigma\|^2_{1+\alpha,\widetilde\Omega}\|u\|^2_{2+\alpha,\widetilde\Omega}+\|\widetilde g\|^2_{\alpha,\widetilde\Omega}).$$

Now we want to obtain an energy type estimate for the local solutions satisfying (2.7). Multiplying $(2.7)_1$ by $\widetilde u_i$, summing over i, integrating over $\widetilde\Omega$ and using the boundary condition

$$\widetilde u|_{\partial\widetilde\Omega}=0, \tag{3.11}$$

and employing the continuity equation, we obtain

$$\int_{\widetilde\Omega}\eta\frac12\frac{d}{dt}\widetilde u^2\,d\xi+\int_{\widetilde\Omega}\mathbb D(\widetilde u)\nabla\widetilde u\,d\xi-\int_{\widetilde\Omega}\widetilde q_\sigma\nabla\cdot\widetilde u\,d\xi=\int_{\widetilde\Omega}(\eta\widetilde g+k_1)\widetilde u\,d\xi. \tag{3.12}$$

Using the continuity equation $(1.15)_1$ and the Korn inequality we get

$$\begin{aligned}\frac{d}{dt}\int_{\widetilde\Omega}\frac12\eta\widetilde u^2\,d\xi+\frac1{c_0}\|\widetilde u\|^2_{1,\widetilde\Omega}&-\int_{\widetilde\Omega}\widetilde q_\sigma\nabla\cdot\widetilde u\,d\xi\\ &\le-\frac12\int_{\widetilde\Omega}\eta\operatorname{div}_u u\widetilde u^2\,d\xi+\int_{\widetilde\Omega}(\eta\widetilde g+k_1)\widetilde u\,d\xi.\end{aligned} \tag{3.13}$$

Multiplying $(2.7)_2$ by $\widetilde q_\sigma$ and integrating over $\widetilde\Omega$ yields

$$\frac12\frac{d}{dt}\int_{\widetilde\Omega}\widetilde q_\sigma^2\,d\xi+p_1\int_{\widetilde\Omega}\widetilde q_\sigma\nabla\cdot\widetilde u\,d\xi=\int_{\widetilde\Omega}k_2\widetilde q_\sigma\,d\xi. \tag{3.14}$$

From (3.13) and (3.14) we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\widetilde\Omega}\frac12\bigg(\eta\widetilde u^2+\frac1{p_1}\widetilde q_\sigma^2\bigg)\,d\xi&+\frac1{c_0}\|\widetilde u\|^2_{1,\widetilde\Omega}\\ &\le\frac1{p_1}\int_{\widetilde\Omega}k_2\widetilde q_\sigma\,d\xi-\frac12\int_{\widetilde\Omega}\eta\operatorname{div}_u u\widetilde u^2\,d\xi+\int_{\widetilde\Omega}(\eta\widetilde g+k_1)\widetilde u\,d\xi\\ &\equiv J_1+J_2+J_3,\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}|J_1|&\le\varepsilon\|\widetilde q_\sigma\|^2_{0,\widetilde\Omega}\\ &\quad+\varphi_{12}(1/\varepsilon,a)\Big(\|u\|^2_{0,\widetilde\Omega}+\|q_\sigma\|^2_{1,\widetilde\Omega}\|u\|^2_{2,\widetilde\Omega}+\int_0^t\|u\|^2_{2+\alpha,\Omega}\,d\tau\,\|\widetilde u\|^2_{1,\widetilde\Omega}\Big),\end{aligned}$$

$$|J_2| \le \varepsilon \|\widetilde{u}\|^2_{1,\widetilde{\Omega}} + \varphi_{13}(1/\varepsilon, a)\|u\|^2_{1,\widetilde{\Omega}}\|\widetilde{u}\|^2_{1,\widetilde{\Omega}},$$

$$|J_3| \le \varepsilon(\|\widetilde{u}\|^2_{1,\widetilde{\Omega}} + \|q_\sigma\|^2_{0,\widetilde{\Omega}}) + \varphi_{14}(1/\varepsilon, a)\Big[\|u\|^2_{1,\widetilde{\Omega}} + \int_0^t \|u\|^2_{2,\widetilde{\Omega}}\, d\tau\, (\|\widetilde{u}\|^2_{2,\widetilde{\Omega}} + \|\widetilde{q}_\sigma\|^2_{1,\widetilde{\Omega}})\Big] + c\|\widetilde{g}\|^2_{0,\widetilde{\Omega}}.$$

Using the above estimates in (3.15) gives

$$\begin{aligned}(3.16)\quad & \frac{d}{dt}\int_{\widetilde{\Omega}} \bigg(\eta\widetilde{u}^2 + \frac{1}{p_1}\widetilde{q}_\sigma^2\bigg) d\xi + \frac{1}{c_0}\|\widetilde{u}\|^2_{1,\widetilde{\Omega}} \\ &\le \varepsilon\|q_\sigma\|^2_{0,\widetilde{\Omega}} + \varphi_{15}(1/\varepsilon, a)\Big[\|\widetilde{g}\|^2_{0,\widetilde{\Omega}} + \|u\|^2_{1,\widetilde{\Omega}} + \|u\|^2_{1,\widetilde{\Omega}}\|\widetilde{u}\|^2_{1,\widetilde{\Omega}} \\ &\quad + \|q_\sigma\|^2_{1,\widetilde{\Omega}}\|u\|^2_{2,\widetilde{\Omega}} + \int_0^t \|u\|^2_{2+\alpha,\Omega}\, dt\, (\|u\|^2_{2,\widetilde{\Omega}} + \|q_\sigma\|^2_{1,\widetilde{\Omega}})\Big].\end{aligned}$$

From (3.10) and (3.16) after using the equivalence of norms for fractional spaces (see [4]) we obtain

$$\begin{aligned}(3.17)\quad & \frac{d}{dt}\int_{\widetilde{\Omega}} \bigg(\eta\widetilde{u}^2 + \frac{1}{p_1}\widetilde{q}_\sigma^2\bigg) d\xi \\ &+ \frac{d}{dt}\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \bigg(\eta\frac{|\Delta^2(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}} + \frac{1}{p_1}\cdot\frac{|\Delta^2(z)\widetilde{q}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\bigg) + \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}} \\ &\le \varepsilon\|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} + c(\|u\|^2_{1+\alpha,\widetilde{\Omega}} + \|q_\sigma\|^2_{\alpha,\widetilde{\Omega}}) \\ &\quad + c[\|\widetilde{g}\|^2_{\alpha,\widetilde{\Omega}} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\|\widetilde{u}_t\|^2_{\alpha,\widetilde{\Omega}} + \|u\|^2_{2+\alpha,\widetilde{\Omega}}\|\widetilde{u}\|^2_{1+\alpha,\widetilde{\Omega}} \\ &\quad + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\|u\|^2_{2+\alpha,\widetilde{\Omega}}] \\ &\quad + c\int_0^t \|u\|^2_{2+\alpha,\Omega}\, d\tau\, [\|u\|^2_{2+\alpha,\widetilde{\Omega}} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\|u\|^2_{2+\alpha,\widetilde{\Omega}}].\end{aligned}$$

To examine the second term on the r.h.s. of (3.17) we use the interpolation inequality

$$\begin{aligned}(3.18)\quad \|u\|^2_{1+\alpha,\widetilde{\Omega}} + \|q_\sigma\|^2_{\alpha,\widetilde{\Omega}} &\le \varepsilon(\|u\|^2_{2+\alpha,\widetilde{\Omega}} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}) \\ &\quad + c(\varepsilon)(\|u\|^2_{0,\widetilde{\Omega}} + \|q_\sigma\|^2_{0,\widetilde{\Omega}}).\end{aligned}$$

From $(2.7)_1$ we have

$$\begin{aligned}(3.19)\quad \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} &\le c(\|\widetilde{u}_t\|^2_{\alpha,\widetilde{\Omega}} + \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}} + \|\widetilde{g}\|^2_{\alpha,\widetilde{\Omega}}) \\ &\quad + c\int_0^t \|u\|^2_{2+\alpha,\Omega}\, d\tau\, (\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}} + \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}) \\ &\quad + \varepsilon(\|u\|^2_{2+\alpha,\widetilde{\Omega}} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}) + c(\varepsilon)(\|u\|^2_{0,\widetilde{\Omega}} + \|q_\sigma\|^2_{0,\widetilde{\Omega}}).\end{aligned}$$

From (3.17)–(3.19) we obtain

$$
\begin{aligned}
(3.20)\qquad & \frac{d}{dt}\int_{\widetilde{\Omega}}\Big(\eta\widetilde{u}^2+\frac{1}{p_1}\widetilde{q}_\sigma^2\Big)\,d\xi\\
&+\frac{d}{dt}\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\Big(\frac{\eta|\Delta^2(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}}+\frac{1}{p_1}\cdot\frac{|\Delta^2(z)\widetilde{q}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\Big)\\
&+\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}}+\|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\\
&\le\varepsilon\|\widetilde{u}_t\|^2_{\alpha,\widetilde{\Omega}}+c\|\widetilde{g}\|^2_{\alpha,\widetilde{\Omega}}+\varepsilon(\|u\|^2_{2+\alpha,\widetilde{\Omega}}+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})c(\varepsilon)(\|u\|^2_{0,\widetilde{\Omega}}+\|q_\sigma\|^2_{0,\widetilde{\Omega}})\\
&+c[\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\|\widetilde{u}_t\|^2_{\alpha,\widetilde{\Omega}}+\|u\|^2_{2+\alpha,\widetilde{\Omega}}\|\widetilde{u}\|^2_{1+\alpha,\widetilde{\Omega}}+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\|u\|^2_{2+\alpha,\widetilde{\Omega}}]\\
&+c\int_0^t\|u\|^2_{2+\alpha,\Omega}\,d\tau\,[\|u\|^2_{2+\alpha,\widetilde{\Omega}}+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}\|u\|^2_{2+\alpha,\widetilde{\Omega}}].
\end{aligned}
$$

To examine the norm $\|\widetilde{u}_t\|_{\alpha,\widetilde{\Omega}}$ from the r.h.s. of (3.20) we use the interpolation inequality (see [12])

$$
(3.21)\qquad \int_0^t\|\omega_t\|^2_{\alpha,\widetilde{\Omega}}\,dt\le\varepsilon\int_0^t\|w\|^2_{2+\alpha,\widetilde{\Omega}}\,dt+c(\varepsilon)\int_{\widetilde{\Omega}}\|w_t\|^2_{\alpha/2,(0,t)}\,d\xi.
$$

To estimate the last term on the r.h.s. of (3.21) we consider the time differences

$$
\Delta_t^k(h)f(\xi,t)=\sum_{j=0}^k c_k^j(-1)^{k-j}f(\xi,t+jh),\qquad k>\alpha.
$$

Applying $\Delta_t^k(h)$ to $(2.7)_1$ we obtain

$$
(3.22)\qquad \eta\Delta_t^k\widetilde{u}_{it}-\nabla_jT_{ij}(\Delta_t^k\widetilde{u},\Delta_t^k\widetilde{q}_\sigma)=\eta\Delta_t^k\widetilde{u}_{it}-\Delta_t^k(\eta\widetilde{u}_{it})+\Delta_t^k(\eta\widetilde{g}_i+k_{1i}).
$$

From (3.22) we have

$$
\begin{aligned}
(3.23)\qquad & \int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-kh_0}\psi(t)\,dt\int_{\widetilde{\Omega}}\eta|\Delta_t^k\widetilde{u}_t|^2\,d\xi\\
&-\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-kh_0}\psi(t)\,dt\int_{\widetilde{\Omega}}\nabla\mathbb{T}(\Delta_t^k\widetilde{u},\Delta_t^k\widetilde{q}_\sigma)\Delta_t^k\widetilde{u}_t\,d\xi\\
&=\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-kh_0}\psi(t)\,dt\int_{\widetilde{\Omega}}[(\eta\Delta_t^k\widetilde{u}_t-\Delta_t^k(\eta\widetilde{u}_t))\Delta_t^k\widetilde{u}_t\\
&\quad+\Delta_t^k(\eta\widetilde{g}+k_1)\Delta_t^k\widetilde{u}_t]\,d\xi\equiv I_1+I_2,
\end{aligned}
$$

where $\psi(t)$ is a smooth function vanishing for $t\le t_0$ and equal to 1 for

$t \ge 2t_0$. Denote the second term on the l.h.s. of (3.23) by I_3. Writing the tensor $\mathbb{T}(u, q_\sigma)$ in the form

$$\mathbb{T}(u, q_\sigma) = \mathbb{D}(u) - q_\sigma I, \tag{3.24}$$

where I is the unit matrix, we obtain

$$\begin{aligned} I_3 = &- \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{T-kh_0} \psi(t)\,dt \int_{\widetilde{\Omega}} \nabla \mathbb{D}(\Delta_t^k \widetilde{u}) \cdot \Delta_t^k \widetilde{u}_t \,d\xi \\ &+ \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{T-kh_0} \psi(t)\,dt \int_{\widetilde{\Omega}} \nabla \Delta_t^k \widetilde{q}_\sigma \Delta_t^k \widetilde{u}_t \,d\xi \equiv I_4 + I_5. \end{aligned}$$

Integrating by parts in I_4 and using the fact that the boundary term vanishes we obtain

$$I_4 = \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{T-kh_0} \psi(t)\,dt \int_{\widetilde{\Omega}} \mathbb{D}'(\Delta_t^k \widetilde{u}) \cdot \mathbb{D}'(\Delta_t^k \widetilde{u}_t)\,d\xi,$$

where $\mathbb{D}'(v) \cdot \mathbb{D}'(w) = \frac{\mu}{2}(\partial_{x_i} v_j + \partial_{x_j} v_i)(\partial_{x_i} w_j + \partial_{x_j} w_i) + (\nu - \mu) \operatorname{div} v \operatorname{div} w$. Continuing we have

$$I_4 = \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{T-kh_0} \psi(t)\,dt \, \frac{1}{2}\frac{d}{dt} \int_{\widetilde{\Omega}} \left(\frac{\mu}{2} |S(\Delta_t^k \widetilde{u})|^2 + (\nu - \mu) |\operatorname{div} \Delta_t^k \widetilde{u}|^2 \right) d\xi,$$

where

$$S(v) = \{\partial_{x_i} v_j + \partial_{x_j} v_i\}_{i,j=1,2,3}.$$

Continuing,

$$\begin{aligned} I_4 = &\int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widetilde{\Omega}} \left(\frac{\mu}{4} |S(\Delta_t^k \widetilde{u})|^2 + \frac{\nu - \mu}{2} |\operatorname{div} \Delta_t^k \widetilde{u}|^2 \right)\bigg|_{t = T - kt_0} d\xi \\ &- \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{t_0}^{2t_0} \psi'(t)\,dt \int_{\widetilde{\Omega}} \left(\frac{\mu}{4} |S(\Delta_t^k \widetilde{u})|^2 + \frac{\nu - \mu}{2} |\operatorname{div} \Delta_t^k \widetilde{u}|^2 \right) d\xi. \end{aligned}$$

Now we estimate I_5. In view of the Hölder and Young inequalities we have

$$\begin{aligned} |I_5| \le \varepsilon &\int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{T-kh_0} \psi(t)\,dt \int_{\widetilde{\Omega}} |\Delta_t^k \widetilde{u}_t|^2 d\xi \\ &+ c(\varepsilon) \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{T-kh_0} \psi(t)\,dt \int_{\widetilde{\Omega}} |\Delta_t^k \nabla \widetilde{q}_\sigma|^2 d\xi \equiv I_5^1 + I_5^2, \end{aligned}$$

where to estimate I_5^2 we calculate $\widetilde{q}_\sigma$ from $(1.14)_2$ in the form

$$\widetilde{q}_\sigma = \widetilde{q}_{\sigma 0} + \int_0^t \Psi(\eta)\zeta \operatorname{div}_u u \, d\tau. \tag{3.25}$$

It is enough to consider the highest order term. Therefore assuming $k = 1$ we have the estimate

$$\begin{aligned}|I_5^2| &\le c \int_{\widetilde{\Omega}} d\xi \int_0^T dt \int_0^T dt' \left(\frac{|\int_t^{t'} |\nabla q_\sigma| \cdot |\operatorname{div} u| \, d\tau|^2}{|t-t'|^{1+\alpha}} + \frac{|\int_t^{t'} |\nabla \operatorname{div} u| \, d\tau|^2}{|t-t'|^{1+\alpha}} \right)\\ &\equiv I_5^3 + I_5^4,\end{aligned}$$

where

$$I_5^3 \le c \int_{\widetilde{\Omega}} d\xi \int_0^T dt \int_0^T dt' \, \frac{\sup_t |\nabla q_\sigma|^2 |t-t'|^{2/p'} (\int_{t'}^t |\operatorname{div} u|^p \, d\tau)^{2/p}}{|t-t'|^{1+\alpha}} \equiv I_5^5$$

for all p, p' with $1/p + 1/p' = 1$. Hence for $2/p' > \alpha$ we have

$$\begin{aligned}I_5^5 &\le c \int_{\widetilde{\Omega}} d\xi \, \sup_t |\nabla q_\sigma|^2 \Big(\int_0^T |\operatorname{div} u|^p \, d\tau \Big)^{2/p}\\ &\le c \sup_t \Big(\int_{\widetilde{\Omega}} |\nabla q_\sigma|^{2q_1} \, d\xi \Big)^{1/q_1} \Big(\int_{\widetilde{\Omega}} d\xi \Big(\int_0^T |\operatorname{div} u|^p \, d\tau \Big)^{2q_2/p} \Big)^{1/p}\end{aligned}$$

if $1/q_1 + 1/q_2 = 1$. Using the imbeddings $B_2^\alpha(\Omega) \subset L_{2q_1}(\Omega)$ valid if $3/2 - 3/(2q_1) \le \alpha$, and $D_\xi^1 B_2^{2+\alpha,1+\alpha/2}(\Omega^T) \subset L_{2q_2,p}(\Omega^T)$ valid if $3/2 - 3/(2q_2) + 2/2 - 2/p \le 1+\alpha$, which both hold for suitable q_1, q_2 because $1/2+\varepsilon \le 1+\alpha$, where we used $2/p' = \alpha + \varepsilon$, $\varepsilon > 0$, we obtain

$$I_5^5 \le c \sup_t \|q_\sigma\|_{1+\alpha,\widetilde{\Omega}}^2 \|u\|_{2+\alpha,\widetilde{\Omega}^T}^2.$$

Next we consider

$$I_5^4 \le c \int_{\widetilde{\Omega}} d\xi \int_0^T dt \int_0^T dt' \, \frac{|t-t'|^{2/p'} (\int_{t'}^t |\nabla \operatorname{div} u|^p \, d\tau)^{2/p}}{|t-t'|^{1+\alpha}} \equiv I_5^6$$

for all p, p' with $1/p + 1/p' = 1$. Take $2/p' > \alpha + \varepsilon$, $\varepsilon > 0$. We have

$$I_5^6 \le c \int_{\widetilde{\Omega}} d\xi \Big(\int_0^T |\nabla \operatorname{div} u|^p \, d\tau \Big)^{2/p} \le \varepsilon_1 \|u\|_{2+\alpha,\widetilde{\Omega}^T}^2 + c(\varepsilon_1) \|u\|_{0,\widetilde{\Omega}^T}^2,$$

because $3/2 - 3/2 + 2/2 - 2/p < \alpha$ means $1 - 2/p < \alpha$ or equivalently $2/p' - 1 < \alpha$, and this holds for suitable p because $\alpha + \varepsilon - 1 < \alpha$, so the above interpolation inequality can be applied.

In the above estimates the constants c depend on T, the time of local existence.

Now we have to estimate I_1 and I_2. Since $k=1$ we have

$$\begin{aligned}|I_1| &\le \int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-h_0}\psi(t)\,dt\int_{\widetilde\Omega}|\Delta_t(\eta)\widetilde u_t|\cdot|\Delta_t\widetilde u_t|\,d\xi\\ &\le \varepsilon\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-h_0}\psi(t)\,dt\int_{\widetilde\Omega}d\xi\,|\Delta_t\widetilde u_t|^2\\ &\quad+c(\varepsilon)\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-h_0}\psi(t)\,dt\int_{\widetilde\Omega}d\xi\,|\Delta_t(\eta)\widetilde u_t|^2.\end{aligned}$$

Denoting the second expression on the r.h.s. by I_1^1 we have

$$I_1^1\le c\int_{\widetilde\Omega}d\xi\int_0^T dt\int_0^T dt'\,\frac{|\Delta(t'-t)q_\sigma(t)|^2|\widetilde u_t|^2}{|t-t'|^{1+\alpha}}\equiv I_1^2.$$

Using (3.25) we obtain

$$\begin{aligned}I_1^2&\le c\int_{\widetilde\Omega}d\xi\int_0^T dt\int_0^T dt'\,\frac{|\int_{t'}^t|\operatorname{div}u|\,d\tau|^2}{|t-t'|^{1+\alpha}}|\widetilde u_t|^2\\ &\le c\int_{\widetilde\Omega}d\xi\int_0^T dt\int_0^T dt'\,\frac{|t-t'|^{2/p'}(\int_0^T|\operatorname{div}u|^p\,d\tau)^{2/p}|\widetilde u_t|^2}{|t-t'|^{1+\alpha}}\equiv I_1^3.\end{aligned}$$

If $2/p'>\alpha$ we obtain

$$\begin{aligned}I_1^3&\le c\int_{\widetilde\Omega}d\xi\int_0^T dt\,|\widetilde u_t|^2\Big(\int_0^T|\operatorname{div}u|^p\,d\tau\Big)^{2/p}\\ &\le c\Big(\int_{\widetilde\Omega}d\xi\,\Big(\int_0^T dt\,|\widetilde u_t|^2\Big)^{\lambda_1}\Big)^{1/\lambda_1}\Big(\int_{\widetilde\Omega}d\xi\Big(\int_0^T|\operatorname{div}u|^p\,d\tau\Big)^{2\lambda_2/p}\Big)^{1/\lambda_2}\equiv I_1^4\end{aligned}$$

for any λ_1, λ_2 with $1/\lambda_1+1/\lambda_2=1$. We use the imbeddings

$$B_2^{\alpha,\alpha/2}(\Omega^T)\subset L_{2\lambda_1,2}(\Omega^T)\quad\text{if }\frac32-\frac{3}{2\lambda_1}+\frac22-\frac22\le\alpha,$$

$$D_\xi^1B_2^{2+\alpha,1+\alpha/2}(\Omega^T)\subset L_{2\lambda_2,p}(\Omega^T)\quad\text{if }\frac32-\frac{3}{2\lambda_2}+\frac22-\frac2p+1\le2+\alpha,\tag{3.26}$$

which both hold for suitable λ_1, λ_2 because $\frac32-\frac{3}{2\lambda_1}\le\alpha$, $\frac32-\frac{2}{2\lambda_2}-\frac2p\le\alpha$ can be satisfied since $\frac32-\frac2p\le2\alpha$, that is, $-\frac12+\frac{2}{p'}\le\alpha$, can hold. From $(3.26)_1$ we have $\lambda_1\le3/2$ for $\alpha>1/2$. Choosing $\lambda_1=3/2$ we have $\lambda_2=3$, so $(3.26)_2$ holds for $p\le4$. Choosing $p=4$ we have $p'=4/3$ so the condition

$2/p' > \alpha$ holds for $\alpha \in (1/2, 1)$. Finally, we have

$$I_1^4 \le c\|\widetilde{u}_t\|^2_{\alpha,\widetilde{\Omega}^T}\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}.$$

Now we examine I_2. First we obtain

$$\begin{aligned}I_2 &\le \varepsilon \int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{T-h_0} \psi(t)\,dt \int\limits_{\widetilde{\Omega}} d\xi\, |\Delta_t \widetilde{u}_t|^2 + c(\varepsilon)(1+\sup_t \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})\|\widetilde{g}\|^2_{\alpha,\widetilde{\Omega}^T}\\ &\quad + c(\varepsilon)\int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{T-h_0} \psi(t)\,dt \int\limits_{\widetilde{\Omega}} d\xi\,|\Delta_t k_1|^2.\end{aligned}$$

Denoting the last term by I_2^1 and using the form of k_1 from $(2.7)_1$ we get

$$\begin{aligned}I_2^1 &\le c\int\limits_0^{h_0} \frac{dh}{h^{1+\alpha}} \int\limits_0^{T-h_0} \psi(t)\,dt \int\limits_{\widetilde{\Omega}} d\xi\,\Big|\Delta_t(\xi_x u\zeta_\xi) + \Delta_t(\xi_x q_\sigma \zeta_\xi + \xi_x u_\xi \zeta_\xi)\\ &\quad + \Delta_t\Big[(I-\xi_x)(\widetilde{u}_{\xi\xi} + \widetilde{q}_{\sigma\xi})\xi_x + \xi_x \int\limits_0^t u_{\xi\xi}\,d\tau\,(\widetilde{u}_\xi + \widetilde{q}_\sigma)\Big]\Big|^2.\end{aligned}$$

Continuing we have

$$\begin{aligned}I_2^1 &\le c\int\limits_{\widetilde{\Omega}} d\xi \int\limits_0^T dt \int\limits_0^T dt'\, \frac{1}{|t-t'|^{1+\alpha}}(|\Delta_t u|^2 + |\Delta_t q_\sigma|^2 + |\Delta_t u_\xi|^2)\\ &\quad + c\int\limits_{\widetilde{\Omega}} d\xi \int\limits_0^T dt \int\limits_0^T dt'\, \frac{1}{|t-t'|^{1+\alpha}}\Big|\int\limits_{t'}^t u_\xi\,d\tau\Big|^2(|u|^2 + |q_\sigma|^2 + |u_\xi|^2\\ &\quad + |\widetilde{u}_{\xi\xi}|^2 + |\widetilde{q}_{\sigma\xi}|^2)\\ &\quad + c\int\limits_{\widetilde{\Omega}} d\xi \int\limits_0^T dt \int\limits_0^T dt'\, \frac{1}{|t-t'|^{1+\alpha}}\Big|\int\limits_0^t u_\xi\,d\tau\Big|^2(|\Delta_t \widetilde{u}_{\xi\xi}|^2 + |\Delta_t \widetilde{q}_{\sigma\xi}|^2)\\ &\quad + c\int\limits_{\widetilde{\Omega}} d\xi \int\limits_0^T dt \int\limits_0^T dt'\, \frac{1}{|t-t'|^{1+\alpha}}\Big|\int\limits_{t'}^t u_\xi\,d\tau\Big|^2\Big|\int\limits_0^t u_{\xi\xi} d\tau\Big|^2(|\widetilde{u}_\xi|^2 + |\widetilde{q}_\sigma|^2)\\ &\quad + c\int\limits_{\widetilde{\Omega}} d\xi \int\limits_0^T dt \int\limits_0^T dt'\, \frac{1}{|t-t'|^{1+\alpha}}\Big|\int\limits_{t'}^t u_{\xi\xi}\,d\tau\Big|^2(|\widetilde{u}_\xi|^2 + |\widetilde{q}_\sigma|^2)\\ &\quad + c\int\limits_{\widetilde{\Omega}} d\xi \int\limits_0^T dt \int\limits_0^T dt'\, \frac{1}{|t-t'|^{1+\alpha}}\Big|\int\limits_0^t u_{\xi\xi}\,d\tau\Big|^2(|\Delta_t \widetilde{u}_\xi|^2 + |\Delta_t \widetilde{q}_\sigma|^2)\\ &\equiv \sum_{i=1}^6 I_6^i.\end{aligned}$$

Continuing we obtain

$$I_6^1 \le \varepsilon(\|u\|^2_{2+\alpha,\widetilde{\Omega}^T} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T}) + c(\varepsilon)(\|u\|^2_{0,\widetilde{\Omega}^T} + \|q_\sigma\|^2_{0,\widetilde{\Omega}^T}).$$

If $2/p' > \alpha$ then

$$\begin{aligned} I_6^2 &\le c\int_\Omega d\xi\Big(\int_0^T |u_\xi|^p\,d\tau\Big)^{2/p}\int_0^T(|u|^2 + |q_\sigma|^2 + |u_\xi|^2 + |\widetilde{u}_{\xi\xi}|^2 + |\widetilde{q}_{\sigma\xi}|^2)\,dt \\ &\le c\Big(\int_\Omega d\xi\Big(\int_0^T |u_\xi|^p\,d\tau\Big)^{2\lambda_1/p}\Big)^{1/\lambda_1} \\ &\quad\times\Big(\int_\Omega d\xi\,\Big|\int_0^T(|u|^2 + |q_\sigma|^2 + |u_\xi|^2 + |\widetilde{u}_{\xi\xi}|^2 + |\widetilde{q}_{\sigma\xi}|^2)\,dt\Big|^{\lambda_2}\Big)^{1/\lambda_2} \\ &\equiv I_6^7, \end{aligned}$$

provided $1/p + 1/p' = 1$ and $1/\lambda_1 + 1/\lambda_2 = 1$.

Restricting our considerations to the last two terms in the second factor of I_6^7, we use the imbeddings

$$\partial_\xi W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \subset \overline{L}_{p,2\lambda_1}(\Omega^T) \quad \text{if } \frac{3}{2} - \frac{3}{2\lambda_1} - \frac{2}{p} \le \alpha,$$

and

$$\partial_\xi^2 W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \subset \overline{L}_{2,2\lambda_2}(\Omega^T) \quad \text{if } \frac{3}{2} - \frac{3}{2\lambda_2} \le \alpha,$$

which are both satisfied for some λ_1, λ_2 because $-1/2 + 2/p' \le 2\alpha$ can hold. We obtain

$$I_6^7 \le c\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}(\|u\|^2_{2+\alpha,\widetilde{\Omega}^T} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T}).$$

Similarly, we have

$$I_6^3 \le c\int_0^T \|u\|^2_{2+\alpha,\widetilde{\Omega}}\,d\tau\,(\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^T} + \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T}).$$

If $2/p' > \alpha$ then

$$\begin{aligned} I_6^4 &\le c\int_{\widetilde{\Omega}} d\xi\,\Big(\int_0^T |u_\xi|^p\,d\tau\Big)^{2/p}\int_0^T |u_{\xi\xi}|^2\,d\tau\int_0^T(|\widetilde{u}_\xi|^2 + |\widetilde{q}_\sigma|^2)\,d\tau \\ &\le c\|u_\xi\|^2_{\overline{L}_{p,2\lambda_1}(\widetilde{\Omega}^T)}\|u_{\xi\xi}\|^2_{\overline{L}_{2,2\lambda_2}(\widetilde{\Omega}^T)}(\|\widetilde{u}_\xi\|^2_{\overline{L}_{2,2\lambda_3}(\widetilde{\Omega}^T)} + \|\widetilde{q}_\sigma\|^2_{\overline{L}_{2,2\lambda_3}(\widetilde{\Omega}^T)}) \\ &\equiv I_6^8 \end{aligned}$$

for all λ_1, λ_2, λ_3 with $1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 = 1$ and $1/p + 1/p' = 1$. Using the imbeddings

$$\begin{aligned}
&\partial_\xi W_2^{2+\alpha,1+\alpha/2}(\widetilde{\Omega}^T) \subset \overline{L}_{p,2\lambda_1}(\widetilde{\Omega}^T) \quad \text{if } \frac{3}{2} - \frac{3}{2\lambda_1} - \frac{2}{p} \le \alpha,\\
&\partial_\xi^2 W_2^{2+\alpha,1+\alpha/2}(\widetilde{\Omega}^T) \subset \overline{L}_{2,2\lambda_2}(\widetilde{\Omega}^T) \quad \text{if } \frac{3}{2} - \frac{3}{2\lambda_2} \le \alpha,\\
&\partial_\xi W_2^{2+\alpha,1+\alpha/2}(\widetilde{\Omega}^T) \subset \overline{L}_{2,2\lambda_3}(\widetilde{\Omega}^T),\\
&W_2^{1+\alpha,1/2+\alpha/2}(\widetilde{\Omega}^T) \subset \overline{L}_{2,2\lambda_3}(\widetilde{\Omega}^T) \quad \text{if } \frac{3}{2} - \frac{3}{2\lambda_3} \le 1+\alpha,
\end{aligned}$$

which all hold for suitable λ_i because $2/p' = 2 - 2/p \le 3\alpha$ can hold, we obtain

$$I_6^8 \le c\|u\|^4_{2+\alpha,\widetilde{\Omega}^T}(\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^T} + \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T}).$$

If $2/p' > \alpha$ we get

$$\begin{aligned}
I_6^5 &\le c\int_{\widetilde{\Omega}} d\xi\Big(\int_0^T |u_{\xi\xi}|^p\,d\tau\Big)^{2/p}\int_0^T(|\widetilde{u}_\xi|^2 + |\widetilde{q}_\sigma|^2)\,d\tau\\
&\le c\|u_{\xi\xi}\|^2_{\overline{L}_{p,2\lambda_1}(\widetilde{\Omega}^T)}(\|\widetilde{u}_\xi\|^2_{\overline{L}_{2,2\lambda_2}(\widetilde{\Omega}^T)} + \|\widetilde{q}_\sigma\|^2_{\overline{L}_{2,2\lambda_2}(\widetilde{\Omega}^T)}) \equiv I_6^9,
\end{aligned}$$

provided $1/p + 1/p' = 1$ and $1/\lambda_1 + 1/\lambda_2 = 1$. Using the imbeddings

$$\begin{aligned}
&\partial_\xi^2 W_2^{2+\alpha,1+\alpha/2}(\widetilde{\Omega}^T) \subset \overline{L}_{p,2\lambda_1}(\widetilde{\Omega}^T) \quad \text{if } \frac{3}{2} - \frac{3}{2\lambda_1} + 2 - \frac{2}{p} \le 1+\alpha,\\
&\partial_\xi W_2^{2+\alpha,1+\alpha/2}(\widetilde{\Omega}^T) \subset \overline{L}_{2,2\lambda_2}(\widetilde{\Omega}^T),\\
&W_2^{1+\alpha,1/2+\alpha/2}(\widetilde{\Omega}^T) \subset \overline{L}_{2,2\lambda_2}(\widetilde{\Omega}^T) \quad \text{if } \frac{3}{2} - \frac{3}{2\lambda_2} \le 1+\alpha,
\end{aligned}$$

which all hold for suitable λ_i, p because $1/2 + 2/p' \le 1 + 2\alpha$ can be satisfied, we obtain

$$I_6^9 \le c\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}(\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^T} + \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T}).$$

Finally, we consider

$$\begin{aligned}
I_6^6 &\le c\int_{\widetilde{\Omega}} d\xi\int_0^T |u_{\xi\xi}|^2\,d\tau\int_0^T\int_0^T \frac{|\Delta_t\widetilde{u}_\xi|^2 + |\Delta_t\widetilde{q}_\sigma|^2}{|t-t'|^{1+\alpha}}\,dt\,dt'\\
&\le c\|u_{\xi\xi}\|^2_{\overline{L}_{2,2\lambda_1}(\widetilde{\Omega}^T)}(\|\widetilde{u}_\xi\|^2_{L_{2\lambda_2}(\widetilde{\Omega};B_2^\alpha(0,T))} + \|\widetilde{q}_\sigma\|^2_{L_{2\lambda_2}(\widetilde{\Omega};B_2^\alpha(0,T))}),
\end{aligned}$$

for all λ_1, λ_2 with $1/\lambda_1 + 1/\lambda_2 = 1$. Using the imbeddings

$$\begin{aligned}
&\partial_\xi^2 W_2^{2+\alpha,1+\alpha/2}(\widetilde{\Omega}^T) \subset \overline{L}_{2,2\lambda_1}(\widetilde{\Omega}^T) \qquad \text{if } \frac{3}{2} - \frac{3}{2\lambda_1} \le \alpha,\\
&\partial_\xi W_2^{2+\alpha,1+\alpha/2}(\widetilde{\Omega}^T) \subset L_{2\lambda_2}(\widetilde{\Omega};B_2^\alpha(0,T)),\\
&W_2^{1+\alpha,1/2+\alpha/2}(\widetilde{\Omega}^T) \subset L_{2\lambda_2}(\widetilde{\Omega};B_2^\alpha(0,T)) \quad \text{if } \frac{3}{2} - \frac{3}{2\lambda_2} \le 1,
\end{aligned}$$

which hold for suitable λ_i because $3/2 \le 1+\alpha$, we obtain
$$I_6^6 \le c\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}(\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^T} + \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T}).$$

Applying the above considerations in (3.23) we obtain

$$\begin{aligned}(3.27)\quad &\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-h_0}\psi(t)dt\int_{\widetilde{\Omega}}\eta|\Delta_t\widetilde{u}_t|^2\,d\xi\\
&\quad+\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_{\widetilde{\Omega}}\left(\frac{\mu}{4}|S(\Delta_t\widetilde{u})|^2+\frac{\nu-\mu}{2}|\operatorname{div}\Delta_t\widetilde{u}|^2\right)d\xi\Big|_{t=T-h_0}\\
&\le\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_{t_0}^{2t_0}\psi'(t)dt\int_{\Omega}\left(\frac{\mu}{4}|S(\Delta_t\widetilde{u})|^2+\frac{\nu-\mu}{2}|\operatorname{div}\Delta_t\widetilde{u}|^2\right)d\xi\\
&\quad+\varepsilon(\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T})+c(\varepsilon)(\|u\|^2_{0,\widetilde{\Omega}^T}+\|q_\sigma\|^2_{0,\widetilde{\Omega}^T})\\
&\quad+c(1+\sup_t\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})\|\widetilde{g}\|_{\alpha,\widetilde{\Omega}^T}\\
&\quad+c(\sup_t\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}+\|u_t\|^2_{\alpha,\widetilde{\Omega}^T})\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}\\
&\quad+c\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}(1+\|u\|^2_{2+\alpha,\widetilde{\Omega}^T})(\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T}).\end{aligned}$$

Using the Gronwall lemma between the second term on the l.h.s. and the first term on the r.h.s. of (3.27) we obtain

$$\begin{aligned}(3.28)\quad &\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-h_0}\psi(t)\,dt\int_{\widetilde{\Omega}}\eta|\Delta_t\widetilde{u}_t|^2\,d\xi\\
&\quad+\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_{\widetilde{\Omega}}\left(\frac{\mu}{4}|S(\Delta_t\widetilde{u})|^2+\frac{\nu-\mu}{2}|\operatorname{div}\Delta_t\widetilde{u}|^2\right)d\xi\Big|_{t=T-h_0}\\
&\le\varepsilon(\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T})+c(\varepsilon)(\|u\|^2_{0,\widetilde{\Omega}^T}+\|q_\sigma\|^2_{0,\widetilde{\Omega}^T})\\
&\quad+c(1+\sup_t\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})\|\widetilde{g}\|^2_{\alpha,\widetilde{\Omega}^T}\\
&\quad+c(\sup_t\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}+\|u_t\|^2_{\alpha,\widetilde{\Omega}^T})\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}\\
&\quad+c\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}(1+\|u\|^2_{2+\alpha,\widetilde{\Omega}^T})(\|u\|^2_{2+\alpha,\widetilde{\Omega}^T}+\|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^T}).\end{aligned}$$

Let us introduce the quantity

$$\begin{aligned}(3.29)\quad \varphi_1(t,\widetilde{\Omega})&=\int_{\widetilde{\Omega}}\left(\eta\widetilde{u}^2+\frac{1}{p_1}\widetilde{q}_\sigma^2\right)d\xi\\
&\quad+\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\left(\eta\frac{|\Delta^2(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}}+\frac{1}{p_1}\frac{|\Delta^2(z)\widetilde{q}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\right),\end{aligned}$$

which satisfies

$$c_1(\|\widetilde{u}\|^2_{1+\alpha,\widetilde{\Omega}} + \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}) \le \varphi_1(t,\widetilde{\Omega}) \le c_2(\|\widetilde{u}\|^2_{1+\alpha,\widetilde{\Omega}} + \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}), \tag{3.30}$$

where c_1, c_2 depend on the bound from the estimate for the local solution.

Then integrating (3.20) with respect to time, using (3.21) and (3.28) we obtain

$$\begin{aligned}
&\varphi_1(t,\widetilde{\Omega}) + \|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|\widetilde{q}_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t} \\
&\le \varepsilon(\|u\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t}) \\
&\quad + c(\varepsilon)(\|u\|^2_{0,\widetilde{\Omega}^t} + \|q_\sigma\|^2_{0,\widetilde{\Omega}^t}) + c(1+\sup_t \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})\|\widetilde{g}\|^2_{\alpha,\widetilde{\Omega}^t} \\
&\quad + c(\sup_t \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}} + \|u_t\|^2_{\alpha,\widetilde{\Omega}^t})\|u\|^2_{2+\alpha,\widetilde{\Omega}^t} \\
&\quad + c\|u\|^2_{2+\alpha,\widetilde{\Omega}^t}(1+\|u\|^2_{2+\alpha,\widetilde{\Omega}^t})(\|u\|^2_{2+\alpha,\widetilde{\Omega}^t} + \|q_\sigma\|^2_{1+\alpha,\widetilde{\Omega}^t}) \\
&\quad + \varphi_1(0,\widetilde{\Omega}), \qquad t \le T,
\end{aligned} \tag{3.31}$$

where T is the time of local existence.

Now, we obtain an estimate in a boundary subdomain. Applying $\Delta^2(\tau)$ to $(2.8)_1$, where $\tau = (z_1, z_2)$, multiplying the result by $\Delta^2(\tau)\widetilde{u}_i$, summing over i, integrating over $\widehat{\Omega}(\xi_0)$ and integrating with respect to τ over $\mathbb{R}^2$ with the weight $1/|\tau|^{2+2(1+\alpha)}$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz\, \frac{\widehat{\eta}\Delta^2(\tau)\widetilde{u}_t \cdot \Delta^2(\tau)\widetilde{u}_i}{|\tau|^{2+2(1+\alpha)}} \\
&\quad - \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz\, \frac{\Delta^2(\tau)\widehat{\nabla}(\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma)) \cdot \Delta^2(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}} \\
&= - \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz\, \frac{(\Delta^2(\tau)(\widehat{\eta}\,\widetilde{u}_t) - \widehat{\eta}\Delta^2(\tau)\widetilde{u}_t) \cdot \Delta^2(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}} \\
&\quad + \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz\, \frac{\Delta^2(\tau)(\widehat{\eta}\,\widetilde{g} + k_3) \cdot \Delta^2(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}} \equiv K_1 + K_2.
\end{aligned} \tag{3.32}$$

The next considerations are similar to the case of interior subdomain. Using the continuity equation in $\widehat{\Omega}$,

$$\widehat{\eta}_t + \widehat{\eta}\,\widehat{\nabla} \cdot \widehat{u} = 0, \tag{3.33}$$

we write the first term on the l.h.s. of (3.32) in the form

$$\frac{d}{dt}\frac{1}{2} \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz\, \frac{\widehat{\eta}|\Delta^2(\tau)\widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}} + K_3,$$

where

$$|K_3| \le \varepsilon \|\widetilde{u}\|^2_{2+\alpha,\widehat{\Omega}} + c(\varepsilon)\|u\|^2_{1+\alpha,\widehat{\Omega}}\|\widetilde{u}\|^2_{2+\alpha,\widehat{\Omega}}.$$

The second term on the l.h.s. of (3.32) takes the form

$$-\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz\, \frac{\widehat{\nabla}\widehat{\mathbb{T}}(\Delta^2(\tau)\widetilde{u}, \Delta^2(\tau)\widetilde{q}_\sigma)\cdot\Delta^2(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}}$$

$$-\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz\, \frac{[\Delta^2(\tau)(\widehat{\nabla}\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma)) - \widehat{\nabla}\widehat{\mathbb{T}}(\Delta^2(\tau)\widetilde{u}, \Delta^2(\tau)\widetilde{q}_\sigma)]\cdot\Delta^2(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}}$$

$$\equiv K_4 + K_5.$$

Integrating by parts in K_4 gives

$$K_4 = -\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}}\, \frac{\widehat{n}\widehat{\mathbb{T}}(\Delta^2(\tau)\widetilde{u}, \Delta^2(\tau)\widetilde{q}_\sigma)\cdot\Delta^2(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}}$$
$$+\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz\, \frac{\widehat{\mathbb{T}}(\Delta^2(\tau)\widetilde{u}, \Delta^2(\tau)\widetilde{q}_\sigma)\cdot\widehat{\nabla}\Delta^2(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}} \equiv K_6 + K_7.$$

Then in view of the boundary condition $(2.8)_3$ we have

$$K_6 = -\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{S}(\xi_0)} dz_{\widehat{S}}\, \frac{\Delta^2(\tau)k_5\cdot\Delta^2(\tau)\widetilde{u}}{|\tau|^{2+2(1+\alpha)}}$$

and

$$|K_6| \le \varepsilon\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz\, \frac{|\nabla\Delta^2(\tau)\widetilde{u}|^2 + |\Delta^2(\tau)\widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}}$$
$$+\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz\, \frac{\varepsilon_1|\nabla\Delta^2(\tau)\widehat{u}|^2 + c(\varepsilon)|\Delta^2(\tau)\widehat{u}|^2}{|\tau|^{2+2(1+\alpha)}},$$

where we recall that ∇ is an operator with respect to z.

Hence repeating the considerations from the internal subdomain we obtain

$$\frac{d}{dt}\int\limits_{\widehat{\Omega}}\left(\widehat{\eta}\,\widetilde{u}^2 + \frac{1}{p_1}\widetilde{q}_\sigma^2\right)dz$$
$$+\frac{d}{dt}\int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}(\xi_0)} dz\left(\widehat{\eta}\frac{|\Delta^2(\tau)\widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{1}{p_1}\frac{|\Delta^2(\tau)\widetilde{q}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\right)$$
$$+\frac{1}{c_0}\left(\|\widetilde{u}\|^2_{1,\widehat{\Omega}} + \int\limits_{\mathbb{R}^2} d\tau\, \frac{\|\Delta^2(\tau)\widetilde{u}\|^2_{1,\widehat{\Omega}}}{|\tau|^{2+2(1+\alpha)}}\right) \le cX, \tag{3.34}$$

where

$$X = (1 + \|\widehat{q}_\sigma\|^2_{1+\alpha,\widehat{\Omega}})(\|\widetilde{u}_t\|^2_{\alpha,\widehat{\Omega}} + \|\widetilde{g}\|^2_{\alpha,\widehat{\Omega}}) + \Big(\|\widehat{q}_\sigma\|^2_{1+\alpha,\widehat{\Omega}} + \|\widehat{u}\|^2_{1+\alpha,\widehat{\Omega}} + \int_0^t \|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}}\,dt'\Big)\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}} + \|q_\sigma\|^2_{1+\alpha,\widehat{\Omega}}\int_0^t \|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}}\,dt'\,(1 + \|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}}) + \varepsilon(\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{1+\alpha,\widehat{\Omega}}) + c(\varepsilon)(\|\widehat{u}\|^2_{0,\widehat{\Omega}} + \|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}}). \tag{3.35}$$

Now, we consider the following artificial problem:

$$\begin{aligned}
&\widehat{\nabla}\,\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma) = \widehat{\eta}\,\widetilde{u}_t + \widehat{\nabla}\widehat{\mathbb{B}}(\widehat{u},\widehat{\zeta}) + \widehat{\mathbb{T}}(\widehat{u},\widehat{q}_\sigma)\widehat{\nabla}\widehat{\zeta} - \widehat{\eta}\,\widetilde{g} && \text{in } \widehat{\Omega},\\
&\widehat{\nabla}\cdot\widetilde{u} = \widehat{\nabla}\cdot\widetilde{u} && \text{in } \widehat{\Omega},\\
&\widehat{\tau}_\alpha\cdot\widehat{\mathbb{T}}(\widetilde{u},\widetilde{q}_\sigma)\cdot\widehat{n} = k_{5\alpha}, \quad \alpha = 1,2, && \text{on } \widehat{S},\\
&\widehat{n}\cdot\widetilde{u} = 0 && \text{on } \widehat{S},\\
&\widetilde{u} = 0 && \text{on } \partial\widehat{\Omega}\setminus\widehat{S}.
\end{aligned} \tag{3.36}$$

From (3.36) we obtain the estimate

$$\int_{\mathbb{R}^2} \frac{\|\Delta(\tau)\widetilde{u}\|^2_{2,\widehat{\Omega}}}{|\tau|^{2+2\alpha}}\,d\tau + \int_{\mathbb{R}^2} \frac{\|\Delta(\tau)\widetilde{q}_\sigma\|^2_{1,\widehat{\Omega}}}{|\tau|^{2+2\alpha}}\,d\tau \le c\int_{\mathbb{R}^2} \frac{\|\Delta(\tau)\nabla\cdot\widetilde{u}\|^2_{1,\widehat{\Omega}}}{|\tau|^{2+2\alpha}} + cX. \tag{3.37}$$

Applying $\frac{\mu+\nu}{p_1}\widehat{\nabla}_i$ to $(2.8)_2$ and adding to $(2.8)_1$ gives

$$\frac{\mu+\nu}{p_1}\widehat{\nabla}_i\widetilde{q}_{\sigma t} + \widehat{\nabla}_i\widetilde{q}_\sigma = \mu(\widehat{\nabla}^2\widetilde{u}_i - \widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u}) - \widehat{\eta}\,\widetilde{u}_{it} + \widehat{\eta}\,\widetilde{g}_i + \frac{\mu+\nu}{p_1}\widehat{\nabla}_i((p_1 - \Psi(\eta))\widehat{\nabla}\cdot\widehat{u}\widehat{\zeta}) + k_{6i}, \tag{3.38}$$

where

$$k_{6i} = -\widehat{\nabla}_j\widehat{B}_{ij}(\widehat{u},\widehat{\zeta}) - \widehat{T}_{ij}(\widehat{u},\widehat{q}_\sigma)\widehat{\nabla}_j\widehat{\zeta} + (\mu+\nu)\widehat{\nabla}_i(\widehat{u}\widehat{\nabla}\widehat{\zeta}),$$

where the summation over the repeated indices is assumed.

Applying $\Delta(\tau)$ to the normal component of (3.38), multiplying the result by $\Delta(\tau)\widehat{\nabla}_3\widetilde{q}_\sigma$, integrating over $\widehat{\Omega}$ and with respect to τ over $\mathbb{R}^2$ with the weight $1/|\tau|^{2+2\alpha}$ we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz\,\frac{\mu+\nu}{p_1}\cdot\frac{|\Delta(\tau)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|\tau|^{2+2\alpha}} + \int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz\,\frac{|\Delta(\tau)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|\tau|^{2+2\alpha}} \le c\int_{\mathbb{R}^2} d\tau \int_{\widehat{\Omega}(\xi_0)} dz\,\frac{|\Delta(\tau)\widetilde{u}_{zz'}|^2}{|\tau|^{2+2\alpha}} + cX, \tag{3.39}$$

where $z' = (z_1, z_2)$.

Writing $(2.8)_1$ in the form

$$\mu\widehat{\nabla}^2\widetilde{u}_i+\nu\widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u}=\widehat{\nabla}_i\widetilde{q}_\sigma+\widehat{\eta}\,\widetilde{u}_{it}-\widehat{\eta}\,\widetilde{g}_i+\widehat{\nabla}_j\widehat{B}_{ij}(\widehat{u},\widehat{\zeta})+\widehat{T}_{ij}(\widehat{u},\widehat{q}_\sigma)\widehat{\nabla}_j\widehat{\zeta},\tag{3.40}$$

we obtain the following estimate for the third component:

$$\int_{\mathbb{R}^2}d\tau\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta(\tau)\widehat{\nabla}_3^2\widetilde{u}_3|^2}{|\tau|^{2+2\alpha}}\le c\int_{\mathbb{R}^2}d\tau\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta(\tau)\widetilde{u}_{zz'}|^2}{|\tau|^{2+2\alpha}}+c\int_{\mathbb{R}^2}d\tau\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta(\tau)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|\tau|^{2+2\alpha}}+cX.\tag{3.41}$$

We have

$$\int_{\mathbb{R}^1_+}dn\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta_3(n)\widetilde{u}_{zz'}|^2}{|n|^{1+2\alpha}}\le c\int_{\mathbb{R}^2}d\tau\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta(\tau)\widetilde{u}_{zz'}|^2}{|\tau|^{2+2\alpha}}+c\int_{\mathbb{R}^2}d\tau\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta(\tau)\widetilde{u}_{zz}|^2}{|\tau|^{2+2\alpha}}.\tag{3.42}$$

In fact, extending u on $\mathbb{R}^n$, applying the Fourier transform and using the inequality

$$|\xi'\xi^\alpha|\le|\xi'|^{1+\alpha}+|\xi||\xi'|^\alpha,$$

which follows from

$$|\xi'\xi^\alpha|=|\xi'|^{\mu_1}|\xi'|^{\mu_2}|\xi|^\alpha\le c(|\xi'|^{\mu_1\lambda_1}+|\xi'|^{\mu_2\lambda_2}|\xi|^{\alpha\lambda_2})$$

for μ_i, λ_i such that $\mu_1+\mu_2=1$, $1/\lambda_1+1/\lambda_2=1$, $\mu_1\lambda_1=1+\alpha$, $\mu_2\lambda_2=\alpha$, $\alpha\lambda_2=1$, so $\lambda_1=1/(1-\alpha)$, $\lambda_2=1/\alpha$, $\mu_1=1-\alpha^2$, $\mu_2=\alpha^2$, we obtain (3.42).

Applying $\Delta_3(n)$ to the third component of (3.38), multiplying the result by $\Delta_3(n)\widehat{\nabla}_3\widetilde{q}_\sigma$, integrating over $\widehat{\Omega}$ and with respect to n over $\mathbb{R}^1_+$ with the weight $1/|n|^{1+2\alpha}$ we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^1_+}dn\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{\mu+\nu}{p_1}\frac{|\Delta_3(n)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|n|^{1+2\alpha}}+\int_{\mathbb{R}^1_+}dn\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta_3(n)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|n|^{1+2\alpha}}\le c\int_{\mathbb{R}^1_+}dn\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta_3(n)\widetilde{u}_{zz'}|^2}{|n|^{1+2\alpha}}+cX.\tag{3.43}$$

Finally, from (3.40) we have

$$\int_{\mathbb{R}^1_+}dn\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta_3(n)\widetilde{u}_{z_3z_3}|^2}{|n|^{1+2\alpha}}\le c\int_{\mathbb{R}^1_+}dn\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta_3(n)\widetilde{u}_{zz'}|^2}{|n|^{1+2\alpha}}+c\int_{\mathbb{R}^1_+}dn\int_{\widehat{\Omega}(\xi_0)}dz\,\frac{|\Delta_3(n)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|n|^{1+2\alpha}}+cX.\tag{3.44}$$

Now we summarize the above considerations. From (3.34) and (3.37) we have

$$
\begin{aligned}
(3.45)\qquad &\frac{d}{dt}\bigg[\int\limits_{\widehat{\Omega}}\bigg(\widehat{\eta}\,\widetilde{u}^2+\frac{1}{p_1}\widetilde{q}_\sigma^2\bigg)\,dz\\
&\quad+\int\limits_{\mathbb{R}^2}d\tau\int\limits_{\widehat{\Omega}}dz\bigg(\widehat{\eta}\,\frac{|\Delta^2(\tau)\widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}}+\frac{1}{p_1}\frac{|\Delta^2(\tau)\widetilde{q}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\bigg)\bigg]\\
&\quad+\|\widetilde{u}\|_{1,\widehat{\Omega}}^2+\int\limits_{\mathbb{R}^2}d\tau\bigg(\frac{\|\Delta^2(\tau)\widetilde{u}\|_{1,\widehat{\Omega}}^2}{|\tau|^{2+2(1+\alpha)}}+\frac{\|\Delta(\tau)\widetilde{u}\|_{2,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}+\frac{\|\Delta(\tau)\widetilde{q}_\sigma\|_{1,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}\bigg)\\
&\le c\int\limits_{\mathbb{R}^2}d\tau\frac{\|\Delta(\tau)\widetilde{u}_{z_3z_3}\|_{0,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}+cX.
\end{aligned}
$$

From (3.34) and (3.39) we obtain

$$
\begin{aligned}
(3.46)\qquad &\frac{d}{dt}\bigg[\int\limits_{\widehat{\Omega}}\bigg(\widehat{\eta}\,\widetilde{u}^2+\frac{1}{p_1}\widetilde{q}_\sigma^2\bigg)dz+\int\limits_{\mathbb{R}^2}d\tau\int\limits_{\widehat{\Omega}}dz\bigg(\widehat{\eta}\frac{|\Delta^2(\tau)\widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}}+\frac{1}{p_1}\frac{|\Delta^2(\tau)\widetilde{q}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\bigg)\\
&\quad+\int\limits_{\mathbb{R}^2}d\tau\int\limits_{\widehat{\Omega}}dz\,\frac{\mu+\nu}{p_1}\frac{|\Delta(\tau)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|\tau|^{3+2\alpha}}\bigg]\\
&\quad+\|\widetilde{u}\|_{1,\widehat{\Omega}}^2+\int\limits_{\mathbb{R}^2}d\tau\,\frac{\|\Delta^2(\tau)\widetilde{u}\|_{1,\widehat{\Omega}}^2}{|\tau|^{2+2(1+\alpha)}}+\int\limits_{\mathbb{R}^2}d\tau\int\limits_{\widehat{\Omega}}dz\,\frac{|\Delta(\tau)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|\tau|^{2+2\alpha}}\le cX.
\end{aligned}
$$

Denote the term under the derivative sign by $\varphi_2(t,\widehat{\Omega})$ and the other terms on the l.h.s. of (3.46) by $\Phi_1(t,\widehat{\Omega})$.

In view of (3.41) and (3.46) we have

$$
(3.47)\qquad \frac{d}{dt}\varphi_2(t,\widehat{\Omega})+\Phi_1(t,\widehat{\Omega})+\int\limits_{\mathbb{R}^2}d\tau\int\limits_{\widehat{\Omega}}dz\,\frac{|\Delta(\tau)\nabla_3^2\widetilde{u}_3|^2}{|\tau|^{2+2\alpha}}\le cX.
$$

Denote the sum of the second and third terms on the l.h.s. of (3.47) by $\Phi_2(t,\widehat{\Omega})$.

From (3.47) and (3.37) we obtain

$$
(3.48)\qquad \frac{d}{dt}\varphi_2(t,\widehat{\Omega})+\Phi_3(t,\widehat{\Omega})\le cX,
$$

where

$$
(3.49)\qquad \Phi_3(t,\widehat{\Omega})=\Phi_2(t,\widehat{\Omega})+\int\limits_{\mathbb{R}^2}\bigg(\frac{\|\Delta(\tau)\widetilde{u}\|_{2,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}+\frac{\|\Delta(\tau)\widetilde{q}_\sigma\|_{1,\widehat{\Omega}}^2}{|\tau|^{2+2\alpha}}\bigg)d\tau.
$$

Using (3.42) in (3.43) and next applying (3.48) we obtain

$$
(3.50)\qquad \frac{d}{dt}\varphi_3(t,\widehat{\Omega})+\Phi_4(t,\widehat{\Omega})\le cX,
$$

where

$$\begin{aligned}\varphi_3(t,\widehat{\Omega}) &= \varphi_2(t,\widehat{\Omega}) + \int\limits_{\mathbb{R}^1_+} dn \int\limits_{\widehat{\Omega}} dz\, \frac{\mu+\nu}{p_1} \frac{|\Delta_3(n)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|n|^{1+2\alpha}},\\ \Phi_4(t,\widehat{\Omega}) &= \Phi_3(t,\widehat{\Omega}) + \int\limits_{\mathbb{R}^1_+} dn \int\limits_{\widehat{\Omega}} dz\, \frac{|\Delta_3(n)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|n|^{1+2\alpha}}.\end{aligned} \tag{3.51}$$

In view of (3.42), (3.44) and (3.50) we get

$$\frac{d}{dt}\varphi_3(t,\widehat{\Omega}) + \Phi_5(t,\widehat{\Omega}) \le cX, \tag{3.52}$$

where

$$\Phi_5(t,\widehat{\Omega}) = \Phi_4(t,\widehat{\Omega}) + \int\limits_{\mathbb{R}^1_+} dn \int\limits_{\widehat{\Omega}} dz\, \frac{|\Delta_3(n)\widetilde{u}_{z_3z_3}|^2}{|n|^{1+2\alpha}}. \tag{3.53}$$

Summarizing the above considerations we have come to

$$\begin{aligned}\varphi_3(t,\widehat{\Omega}) = {} & \int\limits_{\widehat{\Omega}} \left(\widehat{\eta}\,\widetilde{u}^2 + \frac{1}{p_1}\widetilde{q}_\sigma^2\right) d\xi\\ & + \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}} dz \left(\widehat{\eta}\frac{|\Delta^2(\tau)\widetilde{u}|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{1}{p_1}\frac{|\Delta^2(\tau)\widetilde{q}_\sigma|^2}{|\tau|^{2+2(1+\alpha)}}\right)\\ & + \int\limits_{\mathbb{R}^1_+} dn \int\limits_{\widehat{\Omega}} dz\, \frac{\mu+\nu}{p_1}\frac{|\Delta_3(n)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|\tau|^{1+2\alpha}},\end{aligned} \tag{3.54}$$

where for $\widetilde{q}_\sigma$ we have all directional dervatives $\partial_\tau^{1+\alpha}\widetilde{q}_\sigma$, $\partial_n^{1+\alpha}\widetilde{q}_\sigma$, so the other derivatives can be calculated by interpolation. However for $\widetilde{u}$ we only have the tangential derivatives $\partial_\tau^{1+\alpha}\widetilde{u}$.

Next, we have

$$\begin{aligned}\Phi_5(t,\widehat{\Omega}) = {} & \|\widetilde{u}\|^2_{1,\widehat{\Omega}} + \int\limits_{\mathbb{R}^2} d\tau\, \frac{\|\Delta^2(\tau)\widetilde{u}\|^2_{1,\widehat{\Omega}}}{|\tau|^{2+2(1+\alpha)}} + \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}} dz\, \frac{|\Delta(\tau)\nabla_3^2\widetilde{u}_3|^2}{|\tau|^{2+2\alpha}}\\ & + \int\limits_{\mathbb{R}^2} d\tau\, \frac{\|\Delta(\tau)\widetilde{u}\|^2_{2,\widehat{\Omega}}}{|\tau|^{2+2\alpha}} + \int\limits_{\mathbb{R}^1_+} dn \int\limits_{\widehat{\Omega}} dz\, \frac{|\Delta_3(n)\widetilde{u}_{z_3z_3}|^2}{|n|^{1+2\alpha}}\\ & + \int\limits_{\mathbb{R}^2} d\tau \int\limits_{\widehat{\Omega}} dz\, \frac{|\Delta(\tau)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|\tau|^{2+2\alpha}} + \int\limits_{\mathbb{R}^2} d\tau\, \frac{\|\Delta(\tau)\widetilde{q}_\sigma\|^2_{1,\widehat{\Omega}}}{|\tau|^{2+2\alpha}}\\ & + \int\limits_{\mathbb{R}^1_+} dn \int\limits_{\widehat{\Omega}} dz\, \frac{|\Delta_3(n)\widehat{\nabla}_3\widetilde{q}_\sigma|^2}{|\tau|^{1+2\alpha}},\end{aligned} \tag{3.55}$$

where for $\widetilde{u}$ we have the derivatives $\partial_\tau^\alpha\widetilde{u}_{zz}$, $\partial_n^\alpha\widetilde{u}_{z_3z_3}$, so to complete all highest derivatives we need $\partial_n^\alpha\widetilde{u}_{zz}$, which can be calculated from the interpolation inequality (3.42). Concerning $\widetilde{q}_\sigma$, we have $\partial_\tau^\alpha \widetilde{q}_{\sigma z}$, $\partial_n^\alpha \widetilde{q}_{\sigma z_3}$, so the missing derivative $\partial_n^\alpha \widetilde{q}_{\sigma z}$ is calculated from (3.42) where $\widetilde{u}_z$ is replaced by $\widetilde{q}_\sigma$.

Therefore we have shown

$$
\begin{aligned}
(3.56)\qquad c_1(\|\widetilde{u}\|^2_{2+\alpha,\widehat{\Omega}} + \langle\langle \widehat{q}_\sigma\rangle\rangle^2_{1+\alpha,\widehat{\Omega}}) &\le \Phi_5(t,\widehat{\Omega}) \\
&\le c_2(\|\widetilde{u}\|^2_{2+\alpha,\widehat{\Omega}} + \langle\langle \widehat{q}_\sigma\rangle\rangle^2_{1+\alpha,\widehat{\Omega}}),
\end{aligned}
$$

where c_1, c_2 depend on the bound for the local solution and $\langle\langle\omega\rangle\rangle_{\beta,\widehat{\Omega}}$ denotes that only the highest derivatives β of ω appear.

To obtain an estimate for $\|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}}$ we use

$$
(3.57)\qquad \|\widetilde{q}_\sigma\|_{0,\widehat{\Omega}} \le \|\widehat{q}_\sigma\|_{0,\widehat{\Omega}}.
$$

To estimate the expression $\|\widetilde{u}_t\|_{0,\widehat{\Omega}}$ which appears in X we use the inequality (see [12])

$$
(3.58)\qquad \int_0^t \|\widetilde{u}_t\|^2_{\alpha,\widehat{\Omega}}\,dt' \le \varepsilon\int_0^t \|\widetilde{u}_t\|^2_{2+\alpha,\widehat{\Omega}}\,dt' + c(\varepsilon)\int_{\widehat{\Omega}} \|\widetilde{u}_t\|^2_{\alpha/2,(0,t)}\,d\xi,
$$

where to estimate the last integral we repeat the considerations leading to (3.28). Therefore we get

$$
\begin{aligned}
(3.59)\qquad &\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_0^{T-h_0}\psi(t)\,dt\int_{\widehat{\Omega}}\widehat{\eta}|\Delta_t\widetilde{u}_t|^2\,d\xi \\
&\quad + \int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_{\widehat{\Omega}}\left(\frac{\mu}{4}|S(\Delta_t\widetilde{u})|^2 + \frac{\nu-\mu}{2}|\operatorname{div}\Delta_t\widetilde{u}|^2\right)d\xi\bigg|_{t=T-h_0} \\
&\le \varepsilon(\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^T} + \|\widehat{q}_\sigma\|^2_{1+\alpha,\widehat{\Omega}^T}) + c(\varepsilon)(\|\widehat{u}\|^2_{0,\widehat{\Omega}^T} + \|\widehat{q}_\sigma\|^2_{0,\widehat{\Omega}^T}) \\
&\quad + c(1+\sup_t\|\widehat{q}_\sigma\|^2_{1+\alpha,\widehat{\Omega}})\|\widehat{g}\|^2_{\alpha,\widehat{\Omega}^T} + c(\sup_t\|\widehat{q}_\sigma\|^2_{1+\alpha,\widehat{\Omega}} + \|\widehat{u}_t\|^2_{2,\widehat{\Omega}^T})\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^T} \\
&\quad + c\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^T}(1+\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^T})(\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^T} + \|\widehat{q}_\sigma\|^2_{1+\alpha,\widehat{\Omega}^T}).
\end{aligned}
$$

From (1.14) we have

$$
(3.60)\qquad \widetilde{q}_{\sigma t} + \Psi(\widehat{\eta})\widehat{\nabla}\cdot\widehat{u}\widehat{\zeta} = 0,
$$

so

$$
\begin{aligned}
(3.61)\qquad \int_{\widehat{\Omega}}\|\widetilde{q}_{\sigma t}\|^2_{\alpha/2,(0,t)}\,dz &\le c\int_{\widehat{\Omega}}\|\nabla\cdot\widehat{u}\|^2_{\alpha/2,(0,t)}\,dz \\
&\quad + c(\|\widehat{q}_\sigma\|^2_{1+\alpha,\widehat{\Omega}^t} + \|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t})\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t},
\end{aligned}
$$

where the first term on the r.h.s. is bounded by

$$
\varepsilon\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t} + c(\varepsilon)\|\widehat{u}\|^2_{0,\widehat{\Omega}^t}.
$$

Let us introduce a partition of unity such that $\Omega^{(k)}$, $k\in\mathfrak{M}$, is an interior subdomain and $\Omega^{(k)}$, $k\in\mathfrak{N}$, is a boundary subdomain. We denote by χ_k,

$k \in \mathfrak{N}$, the transformation χ in $\Omega^{(k)}$, $k \in \mathfrak{N}$. Then we introduce the quantity

$$\varphi(t,\Omega) = \sum_{k\in\mathfrak{M}} \varphi_1(t,\Omega^{(k)}) + \sum_{k\in\mathfrak{N}} \varphi_3(t,\widehat{\Omega}^{(k)})\Big|_{z=\chi_k(\xi)}, \tag{3.62}$$

where φ_1 is defined by (3.29) and φ_3 by (3.51).

Integrating (3.52) with respect to time, passing to the old variables ξ and adding to (3.31) we obtain

$$\begin{aligned}\varphi(t,\Omega) + \|u\|^2_{2+\alpha,\Omega^t} + \|q_\sigma\|^2_{1+\alpha,\Omega^t} &\le c(\|u\|^2_{0,\Omega^t} + \|q_\sigma\|^2_{0,\Omega^t})\\ &\quad + c\|g\|^2_{\alpha,\Omega^t} + cX_1^2 + \varphi(0,\Omega),\end{aligned} \tag{3.63}$$

where $t \le T$, T is the time of local existence,

$$X_1 = \|u\|^2_{2+\alpha,\Omega^t} + \|q_\sigma\|^2_{1+\alpha,\Omega^t}, \tag{3.64}$$

and the constants c depend on the bound for the local solution. This concludes the proof.

4. Global existence. From $(1.14)_1$ we have

$$\eta u_t - \operatorname{div}_u \mathbb{D}_u(u) + \nabla_u q_\sigma = \eta g. \tag{4.1}$$

Let us introduce a function φ which is a solution to the problem

$$\begin{aligned}\operatorname{div}\varphi &= q_\sigma \quad &&\text{in } \Omega,\\ \varphi|_S &= 0 &&\text{on } S.\end{aligned} \tag{4.2}$$

Let $q_\sigma \in L_2(\Omega)$. Then $\varphi \in H^1(\Omega)$ and

$$\|\varphi\|_{1,\Omega} \le c\|q_\sigma\|_{0,\Omega}. \tag{4.3}$$

Multiplying (4.1) by φ, integrating over Ω^t and using the fact that φ is a solution to (4.2) we obtain

$$\begin{aligned}\|q_\sigma\|^2_{0,\Omega^t} &\le c(\|u_t\|^2_{0,\Omega^t} + \|u_x\|^2_{0,\Omega^t} + \|g\|^2_{0,\Omega^t})\\ &\le \varepsilon_1\|u\|^2_{2+\alpha,\Omega^t} + c(\varepsilon_1)\|u\|^2_{0,\Omega^t} + c\|g\|^2_{0,\Omega^t},\end{aligned} \tag{4.4}$$

where the constants on the r.h.s. depend on the estimate for the local solution.

Assuming that the data for the local solution are such that

$$\|v_0\|_{1+\alpha,\Omega} + \|\varrho_{\sigma 0}\|_{1+\alpha,\Omega} + \|g\|_{0,\Omega^t} \le \varepsilon, \quad t \le T, \tag{4.5}$$

T is the time of local existence, we deduce from (1.13) that

$$\|u\|_{2+\alpha,\Omega^t} + \|q_\sigma\|_{1+\alpha,\Omega^t} \le c\varepsilon, \quad t \le T. \tag{4.6}$$

Assuming that ε is sufficiently small we obtain from (3.36) the inequality

$$\begin{aligned}\varphi(t,\Omega) + \tfrac{3}{4}(\|u\|^2_{2+\alpha,\Omega^t} &+ \|q_\sigma\|^2_{1+\alpha,\Omega^t})\\ &\le c(\|u\|^2_{0,\Omega^t} + \|q_\sigma\|^2_{0,\Omega^t})\\ &\quad + c\|g\|^2_{\alpha,\Omega^t} + \varphi(0,\Omega), \quad t \le T.\end{aligned} \tag{4.7}$$

Let us introduce the quantities

$$\varphi_4(t,\Omega) = \int_\Omega \left(\frac{1}{2}\varrho v^2 + \frac{p_\sigma^2}{2p_\varrho \varrho}\right) dx \tag{4.8}$$

and

$$\psi(t,\Omega)=\varphi(t,\Omega)+\varphi_4(t,\Omega). \tag{4.9}$$

Integrating (3.1) with respect to time, adding to (4.7), using (4.8), (4.9) and the fact that ε is sufficiently small we obtain

$$\begin{aligned}\psi(t,\Omega)+\tfrac{1}{2}(\|u\|^2_{2+\alpha,\Omega^t}+\|q_\sigma\|^2_{1+\alpha,\Omega^t}) &\le c_1(\|f\|^2_{0,\Omega^t}+\|g\|^2_{\alpha,\Omega^t})\\ &\quad+\psi(0,\Omega),\qquad t\le T,\end{aligned} \tag{4.10}$$

where the constant c_1 depends on T.

Assume that

$$c_1(\|f\|^2_{0,\Omega^\infty}+\|g\|^2_{\alpha,\Omega^\infty})+\psi(0,\Omega)\le\varepsilon_0. \tag{4.11}$$

Then for $t\le T$ we have

$$c_1(\|f\|^2_{0,\Omega^t}+\|g\|^2_{\alpha,\Omega^t})+\psi(0,\Omega)\le\varepsilon_0. \tag{4.12}$$

Hence by (4.10) we see for $t\le T$ that

$$\|u(t)\|^2_{1+\alpha,\Omega}+\|q_\sigma(t)\|^2_{1+\alpha,\Omega}\le c_2\varepsilon_0. \tag{4.13}$$

Assuming that ε_0 is so small that (4.5) holds for $v(T)$, $\varrho_\sigma(T)$ and $\|g\|_{\Omega\times(T,2T)}$ we can prove the local existence in the time interval $(T,2T)$ and also the inequality (4.10) in the form

$$\begin{aligned}&\varphi(t,\Omega)+\tfrac{1}{2}(\|u\|^2_{2+\alpha,\Omega\times(T,t)}+\|q_\sigma\|^2_{1+\alpha,\Omega\times(T,t)})\\ &\quad\le c_1(\|f\|^2_{0,\Omega\times(T,t)}+\|g\|^2_{\alpha,\Omega\times(T,t)})+\psi(T,\Omega)\\ &\quad\le c_1(\|f\|^2_{0,\Omega\times(T,t)}+\|g\|^2_{\alpha,\Omega\times(T,t)})+c_1(\|f\|^2_{0,\Omega\times(0,T)}+\|g\|^2_{\alpha,\Omega\times(0,T)})\\ &\qquad+\psi(0,\Omega)\\ &\quad=c_1(\|f\|^2_{0,\Omega^t}+\|g\|^2_{\alpha,\Omega^t})+\psi(0,\Omega),\qquad t\in(T,2T).\end{aligned} \tag{4.14}$$

Continuing in this way, we prove the global-in-time existence of solutions to (1.1). Thus Theorem 1.3 has been proved.

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