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REMARKS ON SOME MONOTONICITY CONDITIONS FOR THE PERIOD FUNCTION

Abstract. We are interested in the optimality of monotonicity criteria for the period function of some planar Hamiltonian systems. This study is illustrated by examples.

1. Introduction. In this paper we study the period function of some ODE and its dependence on the energy. We extend results of the work [2] by the first author and A. Kelfa. We consider several monotonicity conditions on the period function $T(c)$ for the periodic solutions with c the energy constant. This period depends on the energy for some planar Hamiltonian systems with the Hamiltonian of the form

$$\mathcal{H}(u,v) = \tfrac{1}{2}v^2 + G(u), \tag{E}$$

where $G(u)$ is an integral of $g(u)$, with a nondegenerate relative minimum at the origin. The function $g(u)$ is smooth and defined on $(-\infty,\infty)$. We also show some relations between these different criteria.

The conditions denoted by $(\mathcal{C}_4)$ and $(\mathcal{C}_6)$ (see below) seem the best of the known conditions except $(\mathcal{C}_0)$ (with an additional convexity assumption). They are relatively easy to check. In a sense, they are more general than the others. A natural question is to ask if $(\mathcal{C}_4)$ is more general than $(\mathcal{C}_6)$, or conversely.

Notice that monotonicity criteria on the periods for planar Hamiltonian systems given by various authors should be logically related, as suggested by F. Rothe (see Section 2 of [3]).

Nevertheless, we only remark in this paper that condition $(\mathcal{C}_4)$ may be more restrictive than $(\mathcal{C}_6)$. We show that by giving examples of functions satisfying only one of the two conditions.

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Moreover, we also prove that none of the conditions $(\mathcal{C}_4)$, $(\mathcal{C}_6)$ and $(\mathcal{C}_0)$ is optimal. Indeed, we give examples so that the period function associated with the equation (E) is monotonic, but does not satisfy any of the above conditions.

2. Preliminaries. Consider a real smooth function g satisfying

$$(A) \qquad \begin{cases} xg(x) > 0, \quad g''(x) > 0 \quad \text{if } x \in (\alpha, \beta), \\ \alpha < 0 < \beta, \quad g(0) = 0 \quad \text{and} \quad g'(0) > 0, \end{cases}$$

with $g'(x)$, $g''(x)$, $g'''(x)$ denoting the successive derivatives of $g(x)$.

Let G be the primitive of g satisfying

$$G(0) = 0 \quad \text{and} \quad G(a) = G(b) = c,$$

with $\alpha < a < 0$ and $0 < b < \beta$.

Let γ be a constant such that $0 < c < \gamma$.

We are interested in the solution of the Newtonian Hamiltonian system (E), starting at the origin. The origin is surrounded by a continuous family of periodic orbits. For a local parametrization of the periodic trajectories, we define the corresponding energy-period function $T : (0, \gamma) \to \mathbb{R}$, which assigns to each periodic trajectory its minimum period. Thus, each orbit lies on an energy level, $\mathcal{H}(x, y) = c$, and is uniquely determined by c. The energy-period function is then given by

$$T(c) = \sqrt{2} \int_a^b \frac{dx}{\sqrt{c - G(x)}}.$$

The qualitative behavior of this function determines its critical points. One of the open problems is to find the number of these points. The monotonicity case occurs when the period function has no critical value in $]0, \gamma[$.

Several authors have studied the energy-period function, in order to find some monotonicity conditions easy to verify for many functions g.

For convenience, we list below some known sufficient conditions, each of them implying that $T(c)$ is nondecreasing under the above hypothesis (A) on $g(x)$.

$$(\mathcal{C}_0) \qquad H_0 = (x) = g(x)^2 + \frac{g''(0)}{3g'(0)^2} g(x)^3 - 2G(x)g'(x) \geq 0 \qquad \text{for } x \in (a, b).$$

$$(\mathcal{C}_1) \qquad \begin{cases} \text{(i) } g''(x) > 0 & \text{for } x \in (a, b), \\ \text{(ii) } H_1(x) = x(g''(0)g'(x) - g'(0)g''(x)) \geq 0 & \text{for } x \in (a, b). \end{cases}$$

$(\mathcal{C}_2)$ $\quad \Psi(x) = G(x)/g(x)^2$ is a convex function for $x \in (a, b)$.

$$(\mathcal{C}_3) \qquad \begin{cases} \text{(i) } g''(x) > 0 & \text{for } x \in (a, b), \\ \text{(ii) } H_3(x) = 5g''(x)^2 - 3g'(x)g^{(3)}(x) \geq 0 & \text{for } x \in ({g'}^{-1}(0), b). \end{cases}$$

$$(\mathcal{C}_4) \qquad H_4(x) = x\left[3g'(x)^2 - g(x)g''(x) - \left(3\frac{g'(0)^2}{g''(0)}\right)g''(x)\right] \geq 0 \text{ for } x \in (a,b).$$

We recall that conditions $(\mathcal{C}_0)$ and $(\mathcal{C}_1)$ are given by Chow and Wang (see Corollary (2-5) and Proposition (3-1) of [6]). $(\mathcal{C}_2)$ appears in Chicone [1]. $(\mathcal{C}_3)$ is due to R. Schaaf (see [4], [5]) but (i) is replaced by a weaker condition:

$$\text{if } g'(x) = 0, \text{ then } g(x)g''(x) < 0.$$

Note that $(\mathcal{C}_3)$(ii) is equivalent to $(G'')^{-2/3}$ being convex. Finally, $(\mathcal{C}_4)$ was proposed by F. Rothe, and was denoted f_4 (see [3]). Observe that the assumption

$$xg''(x) < 0 \qquad \text{for all } x \neq 0$$

implies $(\mathcal{C}_4)$. But then necessarily $g''(0) = 0$, which is a very strong condition actually.

Our new criterion $(\mathcal{C}_5)$ (see [2]) is more restrictive than $(\mathcal{C}_0)$ but relatively easy to check. This condition is more general than conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_3)$, and, with an additional assumption, more general than $(\mathcal{C}_2)$ and $(\mathcal{C}_4)$.

$$(\mathcal{C}_5) \quad \begin{cases} \text{(i) } g''(x) > 0 & \text{for } x \in (a,b), \\ \text{(ii) } 3g'(x)^2 - g(x)g''(x) - \dfrac{3g'(0)^2}{g''(0)}g''(x) \leq 0 & \text{for } x \in ({g'}^{-1}(0), 0), \\ \text{(iii) } \dfrac{g'(x)g''(0)}{g''(x)g'(0)^2} \geq \dfrac{2G(x)}{g(x)^2} & \text{for } x \in (0,b). \end{cases}$$

As we have seen in our preceding work, each of these five conditions implies $(\mathcal{C}_0)$.

In particular, as a corollary we may deduce

PROPOSITION 1. *The condition*

$$(\mathcal{C}_6) \qquad H_6(x) = x\left(\frac{g'(x)g''(0)}{g''(x)g'(0)^2} - \frac{2G(x)}{g(x)^2}\right) > 0 \quad (\textit{resp.} \; < 0) \qquad \textit{for } x \in (a,b), \; x \neq 0,$$

implies $T'(c) > 0$ (*resp.* $T'(c) < 0$) *for* $0 < c < \gamma$.

Proof. Indeed, it suffices to prove that $(\mathcal{C}_6)$ implies $(\mathcal{C}_0)$. Notice that the derivative

$$H_0'(x) = \frac{g''(0)}{3g'(0)^2}g(x)^2g'(x) - 2G(x)g''(x) \tag{2.1}$$

is connected with $H_6(x)$ by $g(x)^2g''(x)H_6(x) = H_0'(x)$. Thus, condition $(\mathcal{C}_6)$ implies $H_0(x) > 0$ (resp. $H_0(x) < 0$). ■

Furthermore, as we have seen before, an interesting question is to compare these sufficient conditions. Indeed it is quite easy to give examples satisfying $(\mathcal{C}_3)$ and not $(\mathcal{C}_1)$, or satisfying $(\mathcal{C}_2)$ and not $(\mathcal{C}_3)$. On the other hand, the existence of a function satisfying condition $(\mathcal{C}_5)$ and none of the others is harder to show. A relevant example was given in [2].

The starting point of our investigation was motivated by the example

$$g(x) = e^x - 1$$

considered first by Chow–Wang [6], and later by Chicone [1]. They remark that for this g the previous methods cannot be applied. These authors have tested their monotonicity conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$, respectively.

The function we produce below is a modification of $e^x - 1$. Let

$$g_s(x) = \left(\frac{x+s}{2}\right)\sinh(2x) - \frac{\cosh(2x)}{4} + \frac{1}{4}, \qquad s > 0.$$

This function plays an important role in comparing these different conditions. Indeed, there exist different values of s such that the corresponding function

$$H_{0,s}(x) = g_s(x)^2 + \frac{g_s''(0)}{3g_s'(0)^2} g_s(x)^3 - 2G_s(x)g_s'(x)$$

may vanish for an $x = z_0 \neq 0$.

We recall that for the parameter value $s = 0.636$, $(\mathcal{C}_5)$ is satisfied but neither $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ nor $(\mathcal{C}_3)$ is (see [2]). In fact, as one could easily see, the conditions $(\mathcal{C}_4)$ and $(\mathcal{C}_6)$ are also applicable.

Unfortunately, we are not able to produce an example of a function g satisfying condition $(\mathcal{C}_5)$, but neither $(\mathcal{C}_3)$, $(\mathcal{C}_2)$ nor $(\mathcal{C}_1)$.

3. Nonoptimality of the criteria. We recall below the result of Chow and Wang [6], which plays a fundamental role in the monotonicity questions of the energy-period function. Here, their condition is denoted by $(\mathcal{C}_0)$.

PROPOSITION 2 (Chow–Wang). *Under the above conditions* (A) *on the function* $g(x)$ *and*

$$\frac{R(x)}{g(x)^3} - \frac{R(A(x))}{g(A(x))^3} < 0 \quad (\text{or} \ > 0) \qquad \text{for } x \in \,]\alpha, 0[, \tag{3.1}$$

where $R(x) = g(x)^2 - 2G(x)g'(x)$ *and* $A(x)$ *is defined by*

$$G(A(x)) = G(x), \qquad x \in \,]\alpha, 0[, \;\; A(x) \in \,]0, \beta[,$$

we have $T'(c) > 0$ $(\text{or} < 0)$ *for* $0 < c < \gamma$.

To prove that, they use the following expression for the derivative:

$$cT'(c) = \int_a^0 \frac{g(x)}{\sqrt{c - G(x)}}\left(\frac{R(x)}{g(x)^3} - \frac{R(A(x))}{g(A(x))^3}\right)dx.$$

Remark 1. The criterion (3.1) appears to be more general than criteria $(\mathcal{C}_0)$ to $(\mathcal{C}_6)$. But the function A is only implicitly known:

$$A'(x) = \frac{g(x)}{g(A(x))}.$$

For testing the monotonicity of the period function, (3.1) seems to be more difficult to apply.

In the following, we show that for some value of the parameter s, the above function g_s satisfies $(\mathcal{C}_6)$ but not $(\mathcal{C}_4)$.

As we have remarked above, $(\mathcal{C}_6)$ and $(\mathcal{C}_4)$ are the more general conditions. Then we deduce that these examples do not satisfy any other conditions (except $(\mathcal{C}_0)$).

Thus, if we assume condition (A) holds, then the less general monotonicity criteria $(\mathcal{C}_4)$ and $(\mathcal{C}_6)$ for the energy-period function of the planar Hamiltonian system

$$\mathcal{H}(u,v) = \tfrac{1}{2}v^2 + G(u)$$

are not optimal.

In fact we have more:

Theorem 1. *Under the hypothesis* (A), *the most general condition* $(\mathcal{C}_0)$ *is not an optimal criterion.*

Proof. To prove that, it suffices to produce a function g for which $(\mathcal{C}_0)$ is not satisfied. But the period function $T(c)$ depending on the energy of the (periodic) solutions for the differential equation

$$\frac{d^2u}{dt^2} + g(u) = 0$$

has a derivative $T'(c) \neq 0$ for all $c \in (0,\gamma)$.

We need the following result, which improves the above Proposition 2.

Proposition 3. *Suppose that the function* $H_0(x)$ *vanishes for* $z_0 \neq 0$. *Using the notation of Proposition* 2,

(i) *if* $H_0(x) > 0$ *for* $y_0 = A^{-1}(z_0) < x < z_0$, *then*

$$\frac{R(x)}{g(x)^3} - \frac{R(A(x))}{g(A(x))^3} < 0 \quad \textit{for } y_0 \le x < 0, \tag{3.3}$$

(ii) *if* $H_0(x) < 0$ *for* $z_0 < x < y_0 = A(z_0)$, *then*

$$\frac{R(x)}{g(x)^3} - \frac{R(A(x))}{g(A(x))^3} > 0 \quad \textit{for } z_0 \le x < 0. \tag{3.4}$$

REMARK 2. In case (i), this proposition implies that $T(c)$ is increasing in $[0, G(z_0)]$. In the other case, $T(c)$ is decreasing in $[0, G(z_0)]$.

Proof (of Proposition 3). Indeed, we deduce from $H_0(z_0) = 0$ that

$$\frac{R(z_0)}{g(z_0)^3} = -\frac{g''(0)}{3g'(0)^2}.$$

Moreover, to prove (i) we use (2.1) and $y_0 < g'(0)^{-1}$, we deduce $H_0'(y_0) < 0$, and then $z_0 > 0$. Under the hypothesis on H, we have in particular $H_0(y_0) > 0$, implying that

$$\frac{R(x)}{g(x)^3} - \frac{R(A(x))}{g(A(x))^3} < 0 \quad \text{for } y_0 < x < 0,$$

and

$$\frac{R(y_0)}{g(y_0)^3} < -\frac{g''(0)}{3g'(0)^2} = \frac{R(A(y_0))}{g(A(y_0))^3}.$$

Thus, we obtain (3.3).

Similarly, we prove (ii). ∎

This proposition will be illustrated below by some examples.

4. Examples. Consider again the function

$$g_s(x) = \left(\frac{x+s}{2}\right)\sinh(2x) - \frac{\cosh(2x)}{4} + \frac{1}{4}.$$

As we shall see in Figures 3 and 6, we have

$$H_{0,s}(x) = g_s(x)^2 + \frac{g_s''(0)}{3g_s'(0)^2} g_s(x)^3 - 2G_s(x)g_s'(x) \ge 0 \quad (\text{resp. } \le 0),$$

for a suitable choice of α, β, γ such that $G_s(\alpha) = G_s(\beta) = \gamma$. z_0 is defined by Proposition 4. The condition $H_{0,s}(x) \ge 0$ occurs only if we assume that s satisfies $s < s_+$ (resp. $s > s_-$) where $s_+ = 0.63848\ldots$ (resp. $s_- = 0.6458\ldots$).

In the following, we consider g_s for different values of the parameter s. We use *Maple*. The details of the calculation are available upon request.

4.1. *The value* $s = 0.63845$. With $\alpha = -0.073736\ldots$ and $\beta = 0.071$ the function g_s satisfies condition $(\mathcal{C}_6)$ but not $(\mathcal{C}_4)$. See Figures 1 and 2. We take β between the zero of $H_{4,s}$ and the zero of $H_{6,s}$, denoted by $z_{4,s}$ and $z_{6,s}$ respectively.

For $\alpha = -\infty$ and $\beta = \infty$ the function g_s satisfies condition $(\mathcal{C}_0)$ but neither $(\mathcal{C}_4)$ nor $(\mathcal{C}_6)$. See Figures 1, 2 and 3.

This choice of s is motivated by the fact that in $]-\infty, \infty[$, $H_{0,s}(x) > 0$ for $x \in \mathbb{R} - \{0\}$ and $z_{4,s} < z_{6,s}$. Thus, in the interval $[-0.073736, 0.071]$, condition $(\mathcal{C}_6)$ is satisfied but $(\mathcal{C}_4)$ is not.

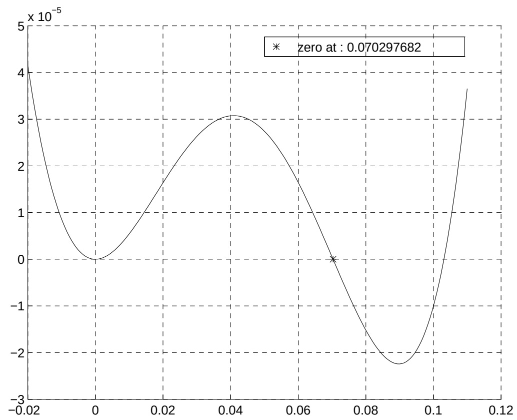

Fig. 1. $H_{4,s}(x)$ with $s = 0.63845$, $z_{4,s} = 0.070297\ldots$

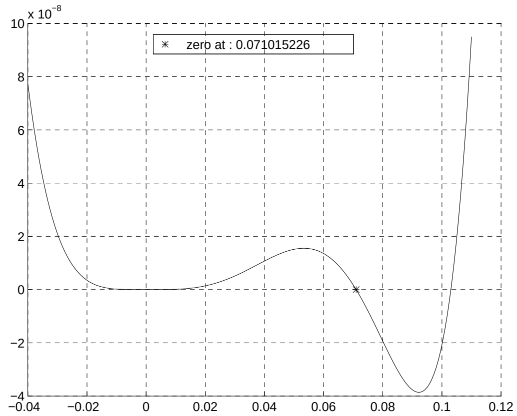

Fig. 2. $H_{6,s}(x)$ with $s = 0.63845$, $z_{6,s} = 0.071015\ldots$

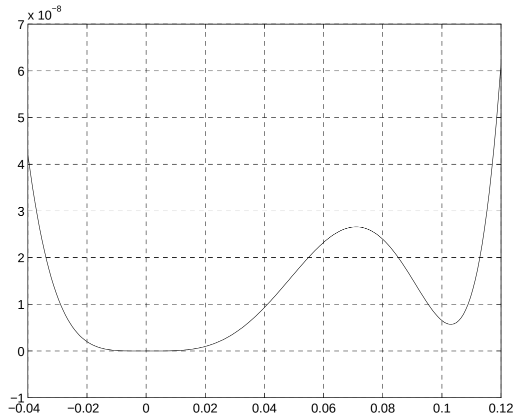

Fig. 3. $H_{0,s}(x)$ with $s = 0.63845$; here $H_{0,s}(x) > 0$ for $x \in \,]-\infty, \infty[$ and $x \neq 0$

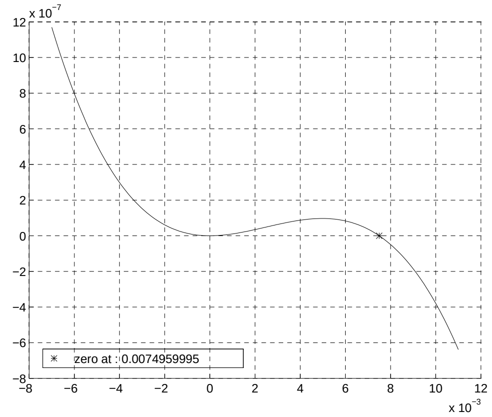

Fig. 4. $H_{4,s}(x)$ with $s = 0.6443$, $z_{4,s} = 0.007495\ldots$

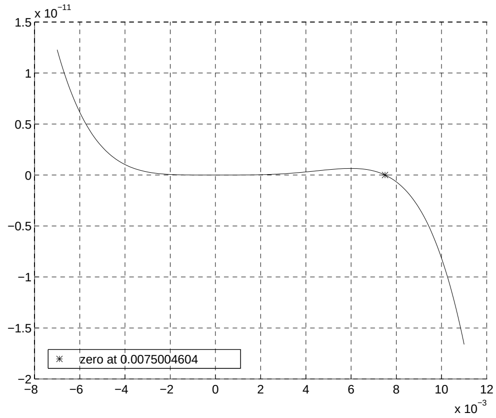

Fig. 5. $H_{6,s}(x)$ with $s = 0.6443$, $z_{6,s} = 0.0075004\ldots$

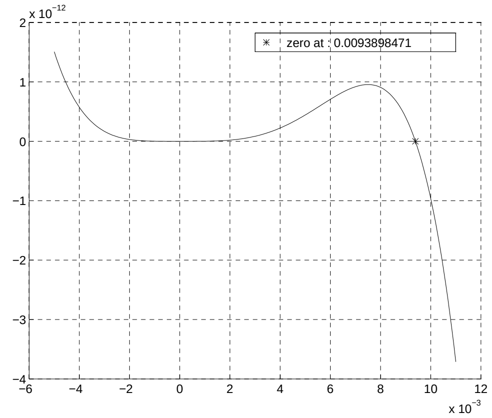

Fig. 6. $H_{0,s}(x)$ with $s = 0.6443$, $z_{0,s} = 0.009389\ldots$;

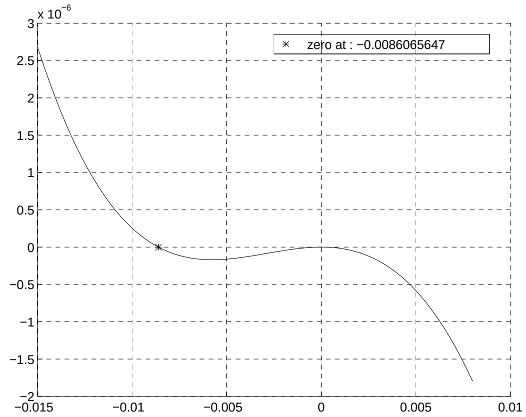

Fig. 7. $H_{4,s}(x)$ with $s = 0.647$, $z_{4,s} = -0.008606\ldots$

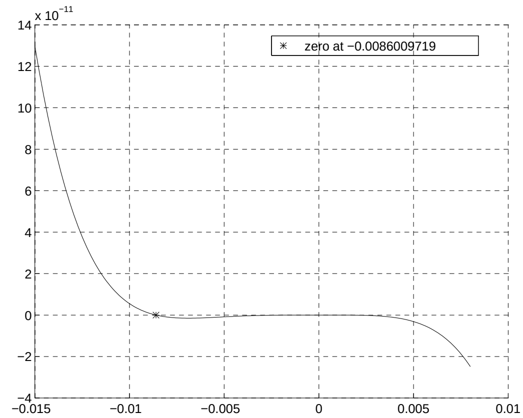

Fig. 8. $H_{6,s}(x)$ with $s = 0.647$, $z_{6,s} = -0.0086009\ldots$

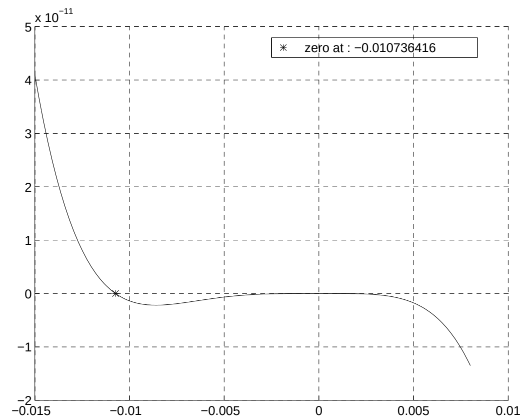

Fig. 9. $H_{0,s}(x)$ with $s = 0.647$, $z_{0,s} = -0.010736\ldots$; $H_{0,s}(x) < 0$ for $x \in]\alpha, \beta]$, $x > z_{0,s}$ and $x \neq 0$

4.2. *The value* $s = 0.6443$. For $\alpha = -1.38184\ldots$ and $\beta = 0.009$ the function g_s satisfies condition $(\mathcal{C}_0)$ but neither $(\mathcal{C}_4)$ nor $(\mathcal{C}_6)$. See Figures 4 and 5.

For $\alpha = -1.38184\ldots$ and $\beta = z_{0,s} = 0.0093898\ldots$ the function g_s does not satisfy condition $(\mathcal{C}_0)$, we have $T'(c) > 0$ for $0 < c < \gamma = G_s(z_{0,s}) = 2.85424\ldots \cdot 10^{-5}$. See Figures 4–6.

4.3. *The value* $s = 0.647$. For $\alpha = -0.01$ and $\beta = 9.94874\ldots \cdot 10^{-3}$ the function g_s satisfies condition $(\mathcal{C}_0)$ but neither $(\mathcal{C}_4)$ nor $(\mathcal{C}_6)$. See Figures 7 and 8.

For $\alpha = z_{0,s} = -0.010736\ldots$ and $\beta = 0.0106773\ldots$ the function g_s does not satisfy condition $(\mathcal{C}_0)$, we have $T'(c) < 0$ for $0 < c < \gamma = G_s(z_{0,s}) = 3.7085\ldots \cdot 10^{-5}$. See Figures 7, 8 and 9.

References

[1] C. Chicone, *The monotonicity of the period for planar hamiltonian vector fields*, J. Differential Equations 69 (1987), 310–321.

[2] A. Kelfa and R. Chouikha, *Monotonicity conditions for the period function of some planar hamiltonian systems*, Comm. Appl. Nonlinear Anal. 4 (1996), no. 3, 99–120.

[3] F. Rothe, *Remarks on periods of planar hamiltonian systems*, SIAM J. Math. Anal. 24 (1993), 129–154.

[4] R. Schaaf, *A class of hamiltonian systems with increasing periods*, J. Reine Angew. Math. 363 (1985), 96–109.

[5] —, *Global Solution Branches of Two Point Boundary Value Problems*, Lecture Notes in Math. 1458, Springer, 1990.

[6] D. Wang and S. N. Chow, *On the monotonicity of the period function of some second order equations*, Časopis Pěst. Mat. 111 (1986), 14–25.

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