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SELFADJOINT OPERATOR CHEBYSHEV–GRÜSS TYPE INEQUALITIES

Abstract. We present very general selfadjoint operator Chebyshev–Grüss type inequalities. We give applications.

1. Motivation.

Here we mention the following inspiring and motivating results.

THEOREM 1 (Chebyshev, 1882, [2]). *Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions. If $f', g' \in L_\infty([a, b])$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \le \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty.$$

THEOREM 2 (Grüss, 1935, [7]). *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions such that $m \le f(x) \le M$ and $\rho \le g(x) \le \sigma$ for all $x \in [a, b]$, where $m, M, \rho, \sigma \in \mathbb{R}$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \le \frac{1}{4}(M-m)(\sigma-\rho).$$

A recent result follows:

THEOREM 3 (Anastassiou, 2011, [1, pp. 312–313]). *Let $n \in \mathbb{N}$ and let $f, g : [a, b] \to \mathbb{R}$ have $f^{(n-1)}, g^{(n-1)}$ absolutely continuous on $[a, b]$. Denote*

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$$F_{n-1}^{f}(x) := \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a}$$

(with $F_0^f(x) = 0$*),*

$$F_{n-1}^{g}(x) := \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{g^{(k-1)}(b)(x-b)^k - g^{(k-1)}(a)(x-a)^k}{b-a}$$

(with $F_0^g(x) = 0$*) and*

$$\Delta_{(f,g)} := \int_a^b f(x)g(x)\,dx - \frac{n}{b-a}\Big(\int_a^b f(x)\,dx\Big)\Big(\int_a^b g(x)\,dx\Big) - \frac{1}{2}\Big[\int_a^b (g(x)F_{n-1}^f(x) + f(x)F_{n-1}^g(x))\,dx\Big].$$

(1) *If* $f^{(n)}, g^{(n)} \in L_\infty([a,b])$*, then*

$$|\Delta_{(f,g)}| \le \frac{(b-a)^{n+1}}{(n+2)!}\big[\|f\|_\infty \|g^{(n)}\|_\infty + \|g\|_\infty \|f^{(n)}\|_\infty\big].$$

(2) *If* $f^{(n)}, g^{(n)} \in L_p([a,b])$*, where* $p, q > 1$ *with* $1/p + 1/q = 1$*, then*

$$|\Delta_{(f,g)}| \le 2^{-1/p}(qn+2)^{-1/q}(B(q(n-1)+1, q+1))^{1/q}\frac{(b-a)^{n-1+2/q}}{(n-1)!} \times \big[\|f\|_p\|g^{(n)}\|_p + \|g\|_p\|f^{(n)}\|_p\big].$$

When $p = q = 2$*,*

$$|\Delta_{(f,g)}| \le \frac{(b-a)^n}{(n-1)!2\sqrt{n(n+1)(4n^2-1)}}\big[\|f\|_2\|g^{(n)}\|_2 + \|g\|_2\|f^{(n)}\|_2\big].$$

(3) *With respect to* $\|\cdot\|_1$*,*

$$|\Delta_{(f,g)}| \le \frac{(b-a)^n}{2(n+1)!}(\|f\|_1\|g^{(n)}\|_\infty + \|g\|_1\|f^{(n)}\|_\infty).$$

2. Background. Let A be a selfadjoint linear operator on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The Gelfand map establishes a $*$-isometric isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all continuous functions defined on the spectrum $\mathrm{Sp}(A)$ of A and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [6, p. 3]):

For any $f, g \in C(\mathrm{Sp}(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (operation composition on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in \mathrm{Sp}(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for all $t \in \mathrm{Sp}(A)$.

With this notation we define

$$f(A) := \Phi(f) \quad \text{for all } f \in C(\mathrm{Sp}(A)),$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$ then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds (with $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H):

(P) $f(t) \geq g(t)$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq g(A)$ in the operator order of $\mathcal{B}(H)$.

We also use the following (see [4, pp. 7–8]):

Let U be a selfadjoint operator on $(H, \langle \cdot, \cdot \rangle)$ with $\mathrm{Sp}(U) \subset [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any continuous function $f : [m, M] \to \mathbb{C}$, we have the following spectral representation in terms of the Riemann–Stieltjes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y \rangle$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on $[m, M]$, and

$$g_{x,y}(m - 0) = 0 \quad \text{and} \quad g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. Furthermore, $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

In this article we will be using a lot the formula

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, x \rangle, \qquad \forall x \in H,$$

or briefly

$$f(U) = \int_{m-0}^{M} f(\lambda) dE_\lambda. \tag{1}$$

Above, $m = \min\{\lambda : \lambda \in \mathrm{Sp}(U)\} = \min \mathrm{Sp}(U)$, $M = \max\{\lambda : \lambda \in \mathrm{Sp}(U)\} = \max \mathrm{Sp}(U)$. The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ satisfy

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;
(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \quad \forall \lambda \in \mathbb{R},$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1 & \text{for } -\infty < s \leq \lambda, \\ 0 & \text{for } \lambda < s < \infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ determines the selfadjoint operator U uniquely and vice versa.

For more on the topic see [8] and [3].

Here are some more basics (we follow [4, pp. 1–5]):

A bounded linear operator A defined on H is selfadjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in H$; and if A is selfadjoint, then

$$\|A\| = \sup_{x \in H:\, \|x\|=1} |\langle Ax, x\rangle|.$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$.

In particular, A is called *positive* if $A \geq 0$.

Denote

$$\mathcal{P} := \Big\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k : n \geq 0,\, \alpha_k \in \mathbb{C},\, 0 \leq k \leq n \Big\}.$$

If $A \in \mathcal{B}(H)$ is selfadjoint, and $\varphi \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)| : \lambda \in \mathrm{Sp}(A)\}.$$

If φ is any function defined on $\mathbb{R}$, we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)| : \lambda \in \mathrm{Sp}(A)\}.$$

If A is selfadjoint and φ is continuous and such that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so is $|\varphi|$; then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [4, p. 4, Theorem 7]).

Hence

$$\begin{aligned} \|\,|\varphi(A)|\,\| &= \|\,|\varphi|\,\|_A = \sup\{|\varphi(\lambda)| : \lambda \in \mathrm{Sp}(A)\} \\ &= \|\varphi\|_A = \|\varphi(A)\|. \end{aligned}$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$. Then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)|\, dE_\lambda$$
$$= \int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2}\, dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

Finally, if $A, B \in \mathcal{B}(H)$, then

$$\|AB\| \le \|A\|\,\|B\|,$$

the Banach algebra property.

3. Main results. Next we present very general Chebyshev–Grüss type operator inequalities based on Fink's [5] identity. Then we specialize them for $n = 1$.

THEOREM 4. *Let $n \in \mathbb{N}$ and $f, g \in C^n([a,b])$ with $[m,M] \subset (a,b)$, $m < M$. Let A be a selfadjoint linear operator on the Hilbert space H with* $\mathrm{Sp}(A) \subseteq [m,M]$. *Let $x \in H$ with $\|x\| = 1$. Then*

$$\begin{aligned}
&(2)\quad \langle(\Delta(f,g))(A)x, x\rangle \\
&:= \bigg|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle \cdot \langle g(A)x,x\rangle - \frac{1}{2(M-m)} \sum_{k=1}^{n-1} \frac{n-k}{k!} \\
&\quad \cdot \big\{g^{(k-1)}(m)\big[\langle f(A)x,x\rangle\langle (A-m1_H)^k x,x\rangle - \langle f(A)(A-m1_H)^k x,x\rangle\big] \\
&\quad + g^{(k-1)}(M)\big[\langle f(A)(A-M1_H)^k x,x\rangle - \langle f(A)x,x\rangle\langle (A-M1_H)^k x,x\rangle\big] \\
&\quad + f^{(k-1)}(m)\big[\langle g(A)x,x\rangle\langle (A-m1_H)^k x,x\rangle - \langle g(A)(A-m1_H)^k x,x\rangle\big] \\
&\quad + f^{(k-1)}(M)\big[\langle g(A)(A-M1_H)^k x,x\rangle - \langle g(A)x,x\rangle\langle (A-M1_H)^k x,x\rangle\big]\big\}\bigg| \\
&\le \frac{1}{(n+1)!(M-m)} \big[\|g^{(n)}\|_{\infty,[m,M]}\|f(A)\| + \|f^{(n)}\|_{\infty,[m,M]}\|g(A)\|\big] \\
&\quad \cdot \big[\|(M1_H - A)^{n+1}\| + \|(A-m1_H)^{n+1}\|\big].
\end{aligned}$$

Proof. Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ and suppose $f, g : [a,b] \to \mathbb{R}$ have $f^{(n)}, g^{(n)}$ continuous on $[a,b]$. Then by Fink [5] we have

$$\begin{aligned}
(3)\qquad f(\lambda) &= \frac{n}{b-a}\int_a^b f(t)\,dt \\
&\quad - \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(a)(\lambda-a)^k - f^{(k-1)}(b)(\lambda-b)^k}{b-a} \\
&\quad + \frac{1}{(n-1)!(b-a)} \int_a^b (\lambda-t)^{n-1} k^*(t,\lambda) f^{(n)}(t)\,dt,
\end{aligned}$$

where

$$k^*(t,\lambda) := \begin{cases} t-a, & a \le t \le \lambda \le b, \\ t-b, & a \le \lambda < t \le b, \end{cases} \qquad \forall \lambda \in [a,b].$$

When $n=1$ the sum $\sum_{k=1}^{n-1}$ in (3) is zero.

Similarly, we get

$$g(\lambda) = \frac{n}{b-a}\int_a^b g(t)\,dt - \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{g^{(k-1)}(a)(\lambda-a)^k - g^{(k-1)}(b)(\lambda-b)^k}{b-a}$$
$$+ \frac{1}{(n-1)!(b-a)}\int_a^b (\lambda-t)^{n-1}k^*(t,\lambda)g^{(n)}(t)\,dt, \qquad \forall \lambda \in [a,b].$$

Therefore

$$(4) \qquad f(\lambda) = \frac{n}{M-m}\int_m^M f(t)\,dt$$
$$- \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(m)(\lambda-m)^k - f^{(k-1)}(M)(\lambda-M)^k}{M-m}$$
$$+ \frac{1}{(n-1)!(M-m)}\int_m^M (\lambda-t)^{n-1}k(t,\lambda)f^{(n)}(t)\,dt,$$

where

$$k(t,\lambda) := \begin{cases} t-m, & m \le t \le \lambda \le M, \\ t-M, & m \le \lambda < t \le M, \end{cases}$$

and

$$(5) \qquad g(\lambda) = \frac{n}{M-m}\int_m^M g(t)\,dt$$
$$- \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{g^{(k-1)}(m)(\lambda-m)^k - g^{(k-1)}(M)(\lambda-M)^k}{M-m}$$
$$+ \frac{1}{(n-1)!(M-m)}\int_m^M (\lambda-t)^{n-1}k(t,\lambda)g^{(n)}(t)\,dt, \ \forall \lambda \in [m,M].$$

By applying the spectral representation theorem to (4) and (5), i.e. integrating against E_λ over $[m,M]$ (see (1)), we obtain

$$f(A) = \left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)1_H$$
$$-\sum_{k=1}^{n-1}\frac{n-k}{k!}\cdot\frac{f^{(k-1)}(m)(A-m1_H)^k - f^{(k-1)}(M)(A-M1_H)^k}{M-m}$$
$$+\frac{1}{(n-1)!(M-m)}\int_{m-0}^{M}\left(\int_m^M(\lambda-t)^{n-1}k(t,\lambda)f^{(n)}(t)\,dt\right)dE_\lambda,$$

and

$$g(A) = \left(\frac{n}{M-m}\int_m^M g(t)\,dt\right)1_H$$
$$-\sum_{k=1}^{n-1}\frac{n-k}{k!}\cdot\frac{g^{(k-1)}(m)(A-m1_H)^k - g^{(k-1)}(M)(A-M1_H)^k}{M-m}$$
$$+\frac{1}{(n-1)!(M-m)}\int_{m-0}^{M}\left(\int_m^M(\lambda-t)^{n-1}k(t,\lambda)g^{(n)}(t)\,dt\right)dE_\lambda.$$

We notice that

$$g(A)f(A) = f(A)g(A),$$

to be used next.

Hence

$$g(A)f(A) = \left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)g(A)$$
$$-\sum_{k=1}^{n-1}\frac{n-k}{k!}\cdot\frac{f^{(k-1)}(m)g(A)(A-m1_H)^k - f^{(k-1)}(M)g(A)(A-M1_H)^k}{M-m}$$
$$+\frac{1}{(n-1)!(M-m)}g(A)\int_{m-0}^{M}\left(\int_m^M(\lambda-t)^{n-1}k(t,\lambda)f^{(n)}(t)\,dt\right)dE_\lambda,$$

and

$$f(A)g(A) = \left(\frac{n}{M-m}\int_m^M g(t)\,dt\right)f(A)$$
$$-\sum_{k=1}^{n-1}\frac{n-k}{k!}\cdot\frac{g^{(k-1)}(m)f(A)(A-m1_H)^k - g^{(k-1)}(M)f(A)(A-M1_H)^k}{M-m}$$
$$+\frac{1}{(n-1)!(M-m)}f(A)\int_{m-0}^{M}\left(\int_m^M(\lambda-t)^{n-1}k(t,\lambda)g^{(n)}(t)\,dt\right)dE_\lambda.$$

As $x \in H$ with $\|x\| = 1$, we immediately get $\int_{m-0}^{M} d\langle E_\lambda x, x\rangle = 1$. Then

$$\begin{aligned}\langle f(A)x, x\rangle &= \frac{n}{M-m}\int_m^M f(s)\,ds \\ &-\sum_{k=1}^{n-1}\frac{n-k}{k!}\cdot\frac{f^{(k-1)}(m)\langle (A-m1_H)^k x, x\rangle - f^{(k-1)}(M)\langle (A-M1_H)^k x, x\rangle}{M-m} \\ &+\frac{1}{(n-1)!(M-m)}\int_{m-0}^{M}\Big(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x, x\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle g(A)x, x\rangle &= \frac{n}{M-m}\int_m^M g(s)\,ds \\ &-\sum_{k=1}^{n-1}\frac{n-k}{k!}\cdot\frac{g^{(k-1)}(m)\langle (A-m1_H)^k x, x\rangle - g^{(k-1)}(M)\langle (A-M1_H)^k x, x\rangle}{M-m} \\ &+\frac{1}{(n-1)!(M-m)}\int_{m-0}^{M}\Big(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x, x\rangle.\end{aligned}$$

Consequently,

$$\begin{aligned}(6)\quad &\langle f(A)x, x\rangle\langle g(A)x, x\rangle \\ &= \Big(\frac{n}{M-m}\int_m^M f(s)\,ds\Big)\langle g(A)x, x\rangle \\ &\quad-\sum_{k=1}^{n-1}\frac{n-k}{k!}\Big(\frac{f^{(k-1)}(m)\langle g(A)x, x\rangle\langle (A-m1_H)^k x, x\rangle}{M-m} \\ &\qquad\qquad - \frac{f^{(k-1)}(M)\langle g(A)x, x\rangle\langle (A-M1_H)^k x, x\rangle}{M-m}\Big) \\ &\quad+\frac{\langle g(A)x, x\rangle}{(n-1)!(M-m)}\int_{m-0}^{M}\Big(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x, x\rangle,\end{aligned}$$

and

$$\begin{aligned}(7)\quad &\langle g(A)x, x\rangle\langle f(A)x, x\rangle = \frac{n}{M-m}\Big(\int_m^M g(s)\,ds\Big)\langle f(A)x, x\rangle \\ &\quad-\sum_{k=1}^{n-1}\frac{n-k}{k!}\Big(\frac{g^{(k-1)}(m)\langle f(A)x, x\rangle\langle (A-m1_H)^k x, x\rangle}{M-m} \\ &\qquad\qquad - \frac{g^{(k-1)}(M)\langle f(A)x, x\rangle\langle (A-M1_H)^k x, x\rangle}{M-m}\Big) \\ &\quad+\frac{\langle f(A)x, x\rangle}{(n-1)!(M-m)}\int_{m-0}^{M}\Big(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x, x\rangle.\end{aligned}$$

Hence

$$\langle f(A)g(A)x,x\rangle = \left(\frac{n}{M-m}\int_m^M f(s)\,ds\right)\langle g(A)x,x\rangle - \sum_{k=1}^{n-1}\frac{n-k}{k!} \cdot \frac{f^{(k-1)}(m)\langle g(A)(A-m1_H)^k x,x\rangle - f^{(k-1)}(M)\langle g(A)(A-M1_H)^k x,x\rangle}{M-m} + \frac{1}{(n-1)!(M-m)}\left\langle\left(g(A)\int_{m-0}^M\left(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\right)dE_\lambda\right)x,x\right\rangle, \tag{8}$$

and

$$\langle f(A)g(A)x,x\rangle = \left(\frac{n}{M-m}\int_m^M g(s)\,ds\right)\langle f(A)x,x\rangle - \sum_{k=1}^{n-1}\frac{n-k}{k!} \cdot \frac{g^{(k-1)}(m)\langle f(A)(A-m1_H)^k x,x\rangle - g^{(k-1)}(M)\langle f(A)(A-M1_H)^k x,x\rangle}{M-m} + \frac{1}{(n-1)!(M-m)}\left\langle\left(f(A)\int_{m-0}^M\left(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\right)dE_\lambda\right)x,x\right\rangle. \tag{9}$$

By (7)–(9) we obtain

$$\begin{aligned} E :=& \langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle = -\sum_{k=1}^{n-1}\frac{n-k}{k!} \\ &\cdot \frac{g^{(k-1)}(m)\langle f(A)(A-m1_H)^k x,x\rangle - g^{(k-1)}(M)\langle f(A)(A-M1_H)^k x,x\rangle}{M-m} \\ &+ \sum_{k=1}^{n-1}\frac{n-k}{k!}\Bigg(\frac{g^{(k-1)}(m)\langle f(A)x,x\rangle\langle (A-m1_H)^k x,x\rangle}{M-m} \\ &\qquad - \frac{g^{(k-1)}(M)\langle f(A)x,x\rangle\langle (A-M1_H)^k x,x\rangle}{M-m}\Bigg) \\ &+ \frac{1}{(n-1)!(M-m)}\Bigg[\left\langle\left(f(A)\int_{m-0}^M\left(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\right)dE_\lambda\right)x,x\right\rangle \\ &\qquad - \langle f(A)x,x\rangle\int_{m-0}^M\left(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\right)d\langle E_\lambda x,x\rangle\Bigg]. \end{aligned}$$

By (6)–(8) we also get

$$E = -\sum_{k=1}^{n-1}\frac{n-k}{k!} \cdot \frac{f^{(k-1)}(m)\langle g(A)(A-m1_H)^k x,x\rangle - f^{(k-1)}(M)\langle g(A)(A-M1_H)^k x,x\rangle}{M-m}$$

$$+\sum_{k=1}^{n-1}\frac{n-k}{k!}\cdot\Bigg(\frac{f^{(k-1)}(m)\langle g(A)x,x\rangle\langle (A-m1_H)^k x,x\rangle}{M-m}$$
$$-\frac{f^{(k-1)}(M)\langle g(A)x,x\rangle\langle (A-M1_H)^k x,x\rangle}{M-m}\Bigg)$$
$$+\frac{1}{(n-1)!(M-m)}\Bigg[\bigg\langle\bigg(g(A)\int_{m-0}^{M}\bigg(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\bigg)\,dE_\lambda\bigg)x,x\bigg\rangle$$
$$-\langle g(A)x,x\rangle\int_{m-0}^{M}\bigg(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\bigg)\,d\langle E_\lambda x,x\rangle\Bigg].$$

Consequently,

$$2E=\frac{1}{M-m}\sum_{k=1}^{n-1}\frac{n-k}{k!}$$
$$\cdot\big\{g^{(k-1)}(m)\big[\langle f(A)x,x\rangle\langle (A-m1_H)^k x,x\rangle-\langle f(A)(A-m1_H)^k x,x\rangle\big]$$
$$+g^{(k-1)}(M)\big[\langle f(A)(A-M1_H)^k x,x\rangle-\langle f(A)x,x\rangle\langle (A-M1_H)^k x,x\rangle\big]$$
$$+f^{(k-1)}(m)\big[\langle g(A)x,x\rangle\langle (A-m1_H)^k x,x\rangle-\langle g(A)(A-m1_H)^k x,x\rangle\big]$$
$$+f^{(k-1)}(M)\big[\langle g(A)(A-M1_H)^k x,x\rangle-\langle g(A)x,x\rangle\langle (A-M1_H)^k x,x\rangle\big]\big\}$$
$$+\frac{1}{(n-1)!(M-m)}\bigg\{\bigg\langle\bigg(f(A)\int_{m-0}^{M}\bigg(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\bigg)\,dE_\lambda\bigg)x,x\bigg\rangle$$
$$-\langle f(A)x,x\rangle\int_{m-0}^{M}\bigg(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\bigg)\,d\langle E_\lambda x,x\rangle$$
$$+\bigg\langle\bigg(g(A)\int_{m-0}^{M}\bigg(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\bigg)\,dE_\lambda\bigg)x,x\bigg\rangle$$
$$-\langle g(A)x,x\rangle\int_{m-0}^{M}\bigg(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\bigg)\,d\langle E_\lambda x,x\rangle\bigg\}.$$

We find

$$\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\langle g(A)x,x\rangle-\frac{1}{2(M-m)}\sum_{k=1}^{n-1}\frac{n-k}{k!}$$
$$\cdot\big\{g^{(k-1)}(m)\big[\langle f(A)x,x\rangle\langle (A-m1_H)^k x,x\rangle-\langle f(A)(A-m1_H)^k x,x\rangle\big]$$
$$+g^{(k-1)}(M)\big[\langle f(A)(A-M1_H)^k x,x\rangle-\langle f(A)x,x\rangle\langle (A-M1_H)^k x,x\rangle\big]$$
$$+f^{(k-1)}(m)\big[\langle g(A)x,x\rangle\langle (A-m1_H)^k x,x\rangle-\langle g(A)(A-m1_H)^k x,x\rangle\big]$$

$$+ f^{(k-1)}(M)\big[\langle g(A)(A-M1_H)^k x,x\rangle - \langle g(A)x,x\rangle\langle (A-M1_H)^k x,x\rangle\big]\Big\}$$

$$= \frac{1}{2(n-1)!(M-m)}\Big\{\Big\langle\Big(f(A)\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,dE_\lambda\Big)x,x\Big\rangle$$

$$-\langle f(A)x,x\rangle\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle$$

$$+\Big\langle\Big(g(A)\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,dE_\lambda\Big)x,x\Big\rangle$$

$$-\langle g(A)x,x\rangle\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle\Big\} =: R.$$

Hence

$$|R| \le \frac{1}{2(n-1)!(M-m)} \tag{10}$$

$$\cdot\Big\{\Big|\Big\langle\Big(f(A)\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,dE_\lambda\Big)x,x\Big\rangle\Big|$$

$$+|\langle f(A)x,x\rangle|\Big|\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle\Big|$$

$$+\Big|\Big\langle\Big(g(A)\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,dE_\lambda\Big)x,x\Big\rangle\Big|$$

$$+|\langle g(A)x,x\rangle|\Big|\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle\Big|\Big\}$$

(here $\big|\int_m^M(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\big| \le \frac{\|g^{(n)}\|_{\infty,[m,M]}}{n(n+1)}[(M-\lambda)^{n+1}+(\lambda-m)^{n+1}]$)

$$\le \frac{1}{2(n-1)!(M-m)}\Big\{\Big\|f(A)\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)dE_\lambda\Big\|$$

$$+\|f(A)\|\frac{\|g^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\big\{\langle (M1_H-A)^{n+1}x,x\rangle+\langle (A-m1_H)^{n+1}x,x\rangle\big\}$$

$$+\Big\|g(A)\int_{m-0}^{M}\Big(\int_m^M(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,dE_\lambda\Big\|$$

$$+\|g(A)\|\frac{\|f^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\big\{\langle (M1_H-A)^{n+1}x,x\rangle+\langle (A-m1_H)^{n+1}x,x\rangle\big\}\Big\}$$

$$\leq \frac{1}{2(n-1)!(M-m)}\Big\{\|f(A)\|\Big[\Big\|\int\limits_{m-0}^{M}\Big(\int\limits_{m}^{M}(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,dE_\lambda\Big\|$$

$$+\frac{\|g^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\big\{\langle(M1_H-A)^{n+1}x,x\rangle+\langle(A-m1_H)^{n+1}x,x\rangle\big\}\Big]$$

$$+\|g(A)\|\Big[\Big\|\int\limits_{m-0}^{M}\Big(\int\limits_{m}^{M}(\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,dE_\lambda\Big\|$$

$$+\frac{\|f^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\big\{\langle(M1_H-A)^{n+1}x,x\rangle+\langle(A-m1_H)^{n+1}x,x\rangle\big\}\Big]\Big\}=:(\xi).$$

Notice here that

$$\Big\|\int\limits_{m-0}^{M}\Big(\int\limits_{m}^{M}(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,dE_\lambda\Big\|$$

$$=\sup_{\|x\|=1}\Big|\int\limits_{m-0}^{M}\Big(\int\limits_{m}^{M}(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle\Big|$$

$$\leq\frac{\|g^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\sup_{\|x\|=1}\big[\langle(M1_H-A)^{n+1}x,x\rangle+\langle(A-m1_H)^{n+1}x,x\rangle\big]$$

$$\leq\frac{\|g^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\Big\{\sup_{\|x\|=1}\langle(M1_H-A)^{n+1}x,x\rangle+\sup_{\|x\|=1}\langle(A-m1_H)^{n+1}x,x\rangle\Big\}$$

$$\leq\frac{\|g^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\big[\|(M1_H-A)^{n+1}\|+\|(A-m1_H)^{n+1}\|\big].$$

We have proved that

$$(11)\qquad \Big\|\int\limits_{m-0}^{M}\Big(\int\limits_{m}^{M}(\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,dE_\lambda\Big\| \leq\frac{\|g^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\big[\|(M1_H-A)^{n+1}\|+\|(A-m1_H)^{n+1}\|\big].$$

A similar estimate holds for $f^{(n)}$.

Hence by (10), (11) we obtain

$$(\xi)\leq\frac{1}{2(n-1)!(M-m)}\Big\{\frac{\|g^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\|f(A)\| \cdot\big\{\|(M1_H-A)^{n+1}\|+\|(A-m1_H)^{n+1}\| +\|(M1_H-A)^{n+1}\|+\|(A-m1_H)^{n+1}\|\big\}$$

$$+ \frac{\|f^{(n)}\|_{\infty,[m,M]}}{n(n+1)}\|g(A)\|\big\{\|(M1_H - A)^{n+1}\| + \|(A - m1_H)^{n+1}\| \\ + \|(M1_H - A)^{n+1}\| + \|(A - m1_H)^{n+1}\|\big\}\Big\}$$

$$= \frac{1}{(n+1)!(M-m)}\big\{\|g^{(n)}\|_{\infty,[m,M]}\|f(A)\| \\ \cdot \big[\|(M1_H - A)^{n+1}\| + \|(A - m1_H)^{n+1}\|\big] \\ + \|f^{(n)}\|_{\infty,[m,M]}\|g(A)\|\big[\|(M1_H - A)^{n+1}\| + \|(A - m1_H)^{n+1}\|\big]\big\}$$

$$= \frac{1}{(n+1)!(M-m)}\big[\|g^{(n)}\|_{\infty,[m,M]}\|f(A)\| + \|f^{(n)}\|_{\infty,[m,M]}\|g(A)\|\big] \\ \cdot \big[\|(M1_H - A)^{n+1}\| + \|(A - m1_H)^{n+1}\|\big].$$

We have proved that

$$|R| \le \frac{1}{(n+1)!(M-m)}\big[\|g^{(n)}\|_{\infty,[m,M]}\|f(A)\| + \|f^{(n)}\|_{\infty,[m,M]}\|g(A)\|\big] \\ \cdot \big[\|(M1_H - A)^{n+1}\| + \|(A - m1_H)^{n+1}\|\big],$$

which yields the claim. ∎

Corollary 5 (case $n = 1$ of Theorem 4). *For every $x \in H$ with $\|x\| = 1$,*

$$|\langle f(A)g(A)x, x\rangle - \langle f(A)x, x\rangle\langle g(A)x, x\rangle| \\ \le \frac{1}{2(M-m)}\big[\|g'\|_{\infty,[m,M]}\|f(A)\| + \|f'\|_{\infty,[m,M]}\|g(A)\|\big] \\ \cdot \big[\|(M1_H - A)^2\| + \|(A - m1_H)^2\|\big].$$

Theorem 6. *Under the assumptions of Theorem 4, let $p, q > 1$ with $1/p + 1/q = 1$. Then*

(12)

$$\langle(\Delta(f,g))(A)x, x\rangle \le \frac{1}{(n-1)!(M-m)}\left(\frac{\Gamma(p+1)\Gamma(p(n-1)+1)}{\Gamma(pn+2)}\right)^{1/p} \\ \cdot \big[\|g^{(n)}\|_{q,[m,M]}\|f(A)\| + \|f^{(n)}\|_{q,[m,M]}\|g(A)\|\big] \\ \cdot \big[\|(M1_H - A)^{n+1/p}\| + \|(A - m1_H)^{n+1/p}\|\big],$$

where Γ is the gamma function.

Proof. We observe that

$$\Big|\int_m^M (\lambda - s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big| \\ \le \Big|\int_m^\lambda (\lambda - s)^{n-1}(s-m)g^{(n)}(s)\,ds\Big| + \Big|\int_\lambda^M (\lambda - s)^{n-1}(s-M)g^{(n)}(s)\,ds\Big|$$

$$
\begin{aligned}
&\leq \int_m^\lambda (\lambda-s)^{n-1}(s-m)|g^{(n)}(s)|\,ds + \int_\lambda^M (M-s)(s-\lambda)^{n-1}|g^{(n)}(s)|\,ds \\
&\leq \Big(\int_m^\lambda ((\lambda-s)^{n-1}(s-m))^p\,ds\Big)^{1/p}\|g^{(n)}\|_{q,[m,M]} \\
&\quad + \Big(\int_\lambda^M ((M-s)(s-\lambda)^{n-1})^p\,ds\Big)^{1/p}\|g^{(n)}\|_{q,[m,M]} \\
&= \|g^{(n)}\|_{q,[m,M]}\Big[\Big(\int_m^\lambda (\lambda-s)^{(p(n-1)+1)-1}(s-m)^{(p+1)-1}\,ds\Big)^{1/p} \\
&\quad + \Big(\int_\lambda^M (M-s)^{(p+1)-1}(s-\lambda)^{(p(n-1)+1)-1}\,ds\Big)^{1/p}\Big] \\
&= \|g^{(n)}\|_{q,[m,M]}\Big(\frac{\Gamma(p+1)\Gamma(p(n-1)+1)}{\Gamma(pn+2)}\Big)^{1/p} \\
&\quad \cdot \big[(M-\lambda)^{n+1/p}+(\lambda-m)^{n+1/p}\big], \qquad \forall\lambda\in[m,M].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\Big|\int_{m-0}^M \Big(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle\Big| \\
&\qquad\qquad \leq \Big(\frac{\Gamma(p+1)\Gamma(p(n-1)+1)}{\Gamma(pn+2)}\Big)^{1/p}\|g^{(n)}\|_{q,[m,M]} \\
&\qquad\qquad\quad \cdot\big[\|(M1_H-A)^{n+1/p}\|+\|(A-m1_H)^{n+1/p}\|\big].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Big|\int_{m-0}^M \Big(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle\Big| \\
&\qquad\qquad \leq \Big(\frac{\Gamma(p+1)\Gamma(p(n-1)+1)}{\Gamma(pn+2)}\Big)^{1/p}\|f^{(n)}\|_{q,[m,M]} \\
&\qquad\qquad\quad \cdot\big[\|(M1_H-A)^{n+1/p}\|+\|(A-m1_H)^{n+1/p}\|\big].
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\Big\|\int_{m-0}^M \Big(\int_m^M (\lambda-s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,dE_\lambda\Big\| \\
&\qquad\qquad \leq \Big(\frac{\Gamma(p+1)\Gamma(p(n-1)+1)}{\Gamma(pn+2)}\Big)^{1/p}\|g^{(n)}\|_{q,[m,M]} \\
&\qquad\qquad\quad \cdot\big[\|(M1_H-A)^{n+1/p}\|+\|(A-m1_H)^{n+1/p}\|\big].
\end{aligned}
$$

A similar estimate can be derived for $f^{(n)}$.

Acting as in the proof of Theorem 4 we find that

$$|R| \le \frac{1}{(n-1)!(M-m)}\left(\frac{\Gamma(p+1)\Gamma(p(n-1)+1)}{\Gamma(pn+2)}\right)^{1/p} \cdot \big[\|g^{(n)}\|_{q,[m,M]}\|f(A)\| + \|f^{(n)}\|_{q,[m,M]}\|g(A)\|\big] \cdot \big[\|(M1_H - A)^{n+1/p}\| + \|(A - m1_H)^{n+1/p}\|\big],$$

proving the claim. ∎

Corollary 7 (to Theorem 6, $n = 1$). *We have*

$$(13)\quad |\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle| \le \frac{1}{(M-m)(p+1)^{1/p}}\big[\|g'\|_{q,[m,M]}\|f(A)\| + \|f'\|_{q,[m,M]}\|g(A)\|\big] \cdot \big[\|(M1_H - A)^{1+1/p}\| + \|(A - m1_H)^{1+1/p}\|\big].$$

Theorem 8. *Under the assumptions of Theorem* 4,

$$(14)\quad \langle(\Delta(f,g))(A)x,x\rangle \le \frac{(M-m)^{n-1}}{(n-1)!}\big[\|g^{(n)}\|_{1,[m,M]}\|f(A)\| + \|f^{(n)}\|_{1,[m,M]}\|g(A)\|\big].$$

Proof. We observe that

$$\Big|\int_m^M (\lambda - s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big| \le \int_m^M |\lambda - s|^{n-1}|k(s,\lambda)|\,|g^{(n)}(s)|\,ds \le (M-m)^n\int_m^M |g^{(n)}(s)|\,ds = (M-m)^n\|g^{(n)}\|_{1,[m,M]}.$$

Hence

$$\Big|\int_{m-0}^M\Big(\int_m^M (\lambda - s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle\Big| \le (M-m)^n\|g^{(n)}\|_{1,[m,M]},$$

and similarly

$$\Big|\int_{m-0}^M\Big(\int_m^M (\lambda - s)^{n-1}k(s,\lambda)f^{(n)}(s)\,ds\Big)\,d\langle E_\lambda x,x\rangle\Big| \le (M-m)^n\|f^{(n)}\|_{1,[m,M]};$$

the last two estimates are valid since $\int_{m-0}^M d\langle E_\lambda x,x\rangle = 1$ for $x \in H$, $\|x\| = 1$.

Similarly, we obtain

$$\Big\|\int_{m-0}^M\Big(\int_m^M (\lambda - s)^{n-1}k(s,\lambda)g^{(n)}(s)\,ds\Big)\,dE_\lambda\Big\| \le (M-m)^n\|g^{(n)}\|_{1,[m,M]},$$

and a similar estimate for $f^{(n)}$.

Acting as in the proof of Theorem 4 we find that

$$|R| \leq \frac{(M-m)^{n-1}}{(n-1)!}\big[\|g^{(n)}\|_{1,[m,M]}\|f(A)\| + \|f^{(n)}\|_{1,[m,M]}\|g(A)\|\big],$$

proving the claim. ∎

Corollary 9 (to Theorem 8, $n = 1$).

$$|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle| \leq \|g'\|_{1,[m,M]}\|f(A)\| + \|f'\|_{1,[m,M]}\|g(A)\|. \tag{15}$$

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