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The Boolean prime ideal theorem does not imply the extension of almost disjoint families to MAD families

by

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Dedicated to Professor Paul E. Howard and
to the memory of Professor Jean E. Rubin

Summary. We establish that the statement "For every infinite set X, every almost disjoint family in X can be extended to a maximal almost disjoint (MAD) family in X" is not provable in ZF + Boolean prime ideal theorem + Axiom of Countable Choice.

This settles an open problem from Tachtsis [*On the existence of almost disjoint and MAD families without* AC, Bull. Polish Acad. Sci. Math. 67 (2019), 101–124].

1. Introduction. In [T19], we initiated the study of almost disjoint and MAD families (for the definitions see Section 2) within mild extensions of ZF (i.e. Zermelo–Fraenkel set theory minus the Axiom of Choice (AC)) and of ZFA (i.e. ZF with the Axiom of Extensionality weakened to allow the existence of atoms), that is, within ZF + Weak Choice and ZFA + Weak Choice.

In particular, the research in [T19] filled several gaps in information via results which shedded light on the open problem of the placement of the following statements (among others) in the hierarchy of weak choice principles: "Every almost disjoint family in an infinite set X can be extended to a MAD family in X"; "No MAD family in an infinite set has cardinality $\aleph_0$"; "Every infinite set has an uncountable (1) almost disjoint family".

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(1) A set X is called *uncountable* if $|X| \not\leq \aleph_0$. That is, X is uncountable if there is no injection $f : X \to \omega$, where (as usual) ω denotes the set of natural numbers.

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In view of the aim (as suggested by the title) of this note, let us mention here what has been proved in [T19] regarding the first of the above statements, and refer the reader to [T19] for the complete results therein on almost disjoint and MAD families. Prior to this, let us note that, in [T19], it has been established that the statement "Every infinite set has an infinite almost disjoint family", which is formally weaker than the third of the above statements, is not provable in $\mathsf{ZF} + \mathsf{BPI}$, where "$\mathsf{BPI}$" denotes the Boolean prime ideal theorem.

So, regarding the set-theoretic strength of the statement

($*$) "Every almost disjoint family in an infinite set X can be extended to a MAD family in X",

the following results have been established in [T19]:

(1) ($*$) is not provable in ZF.
(2) In ZFA, the Axiom of Multiple Choice (MC) implies ($*$). Hence, ($*$) does not imply BPI in ZFA.
(3) ($*$) does not imply MC in ZFA. In particular, ($*$) is true in the Mostowski Linearly Ordered Model of ZFA (Model $\mathcal{N}3$ in Howard–Rubin [HR98]), in which BPI is true but MC is false.

In view of (2) and (3), and especially of the fact that BPI and ($*$) are true in the model $\mathcal{N}3$, as well as of the fact that both BPI and ($*$) are maximality principles, it is natural to inquire whether or not BPI implies ($*$). This open problem (until now) has been posed in [T19] (see [T19, Section 4, Question 1]).

The goal of this note is to settle that problem. In particular, we will provide a strongly *negative answer* by establishing in Theorem 6.1 that

$$(*) \textit{ is not provable in } \mathsf{ZF} + \mathsf{BPI} + \mathsf{AC}^{\omega},$$

where "AC^{ω}" denotes the Axiom of Countable Choice.

Our proof of the above result will comprise two steps: Firstly, we will prove that "Every almost disjoint family in an infinite set X can be extended to a MAD family in X" is false in a certain permutation model of $\mathsf{ZFA} + \mathsf{BPI} + \mathsf{AC}^{\omega}$, and secondly we will apply a theorem of Pincus [P77] in order to transfer the ZFA-independence result to ZF.

Before embarking on the proof, we will provide the following background:

(a) in Section 3, a concise account of the construction of permutation models for the reader's convenience;
(b) in Section 4, the description of the suitable permutation model;
(c) in Section 5, the terminology and the specific theorem of Pincus for the transfer to ZF.

2. Definitions and notation

Definition 2.1. Let X be an infinite set. (That is, $X \neq \emptyset$ and for all $n \in \omega \setminus \{0\}$, there is no bijection $f : n \to X$; otherwise X is called finite.)

(1) X is called *denumerable* if there is a bijection $f : \omega \to X$.
(2) A family $\mathcal{A}$ of infinite subsets of X is called *almost disjoint* in X if for all $A, B \in \mathcal{A}$ with $A \neq B$, the set $A \cap B$ is finite ([2]).
(3) An almost disjoint family $\mathcal{A}$ in X is called *maximal almost disjoint* (MAD) in X if for every almost disjoint family $\mathcal{B}$ in X with $\mathcal{A} \subseteq \mathcal{B}$, we have $\mathcal{A} = \mathcal{B}$.

Next, we provide the statements of BPI and AC^{ω}. For the reader's convenience, we also supply the ones for MC and the Principle of Dependent Choices since the latter two weak choice forms, though not having a key role in this note, are mentioned at specific points.

Definition 2.2.

(1) The *Boolean prime ideal theorem* BPI (Form 14 in [HR98]): Every Boolean algebra has a prime ideal.
(2) The *Axiom of Countable Choice* AC^{ω} (Form 8 in [HR98]): Every denumerable family of non-empty sets has a choice function.
(3) The *Axiom of Multiple Choice* MC (Form 67 in [HR98]): For every family $\mathcal{A}$ of non-empty sets there is a function f with domain $\mathcal{A}$ such that for all $X \in \mathcal{A}$, $f(X)$ is a non-empty finite subset of X. (f is called a *multiple choice function* for $\mathcal{A}$.)
(4) The *Principle of Dependent Choices* DC (Form 43 in [HR98]): If R is a relation on a non-empty set X such that for every $x \in X$ there exists $y \in X$ with xRy, then there is a sequence $(x_n)_{n\in\omega}$ of elements of X such that $x_n R x_{n+1}$ for all $n \in \omega$.

Let us also recall a couple of known facts about BPI and MC.

(a) BPI is equivalent to the statement "Every filter on a set can be extended to an ultrafilter" (see [J73, Theorem 2.2]). It is also a renowned result of Halpern and Levy [HL71] that BPI does not imply AC in ZF. In particular, BPI is true in the Basic Cohen Model (Model $\mathcal{M}1$ in [HR98]) of $\mathsf{ZF} + \neg\mathsf{AC}$.

(b) MC is equivalent to AC in ZF, but it is not equivalent to AC in ZFA (see [J73, Theorems 9.1 and 9.2]).

([2]) Our definition of almost disjoint family (here and in [T19]) differs from the usual one, namely the one which states that given an infinite set X, a family $\mathcal{A} \subseteq [X]^{|X|} = \{Y : Y \subseteq X \text{ and } |Y| = |X|\}$ is almost disjoint in X if for any two distinct members A, B of $\mathcal{A}$, $|A \cap B| < |X|$; that is, there is a one-to-one mapping from $A \cap B$ into X but no one-to-one mapping from X into $A \cap B$.

3. Terminology for permutation models. For the reader's convenience, we provide below a brief account of the construction of permutation models of ZFA; a detailed account can be found in Jech [J73, Chapter 4].

One starts with a model M of $\mathsf{ZFA}+\mathsf{AC}$ which has A as its set of atoms. Let G be a group of permutations of A and also let $\mathcal{F}$ be a filter on the lattice of subgroups of G which satisfies the following:

$$\forall a \in A\ \exists H \in \mathcal{F}\ \forall \phi \in H\ (\phi(a) = a)$$

and

$$\forall \phi \in G\ \forall H \in \mathcal{F}\ (\phi H \phi^{-1} \in \mathcal{F}).$$

Such a filter $\mathcal{F}$ of subgroups of G is called a *normal filter* on G. Every permutation of A extends uniquely to an $\in$-automorphism of M by $\in$-induction, and for any $\phi \in G$, we identify ϕ with its (unique) extension. If H is a subgroup of G and $x \in M$ and for all $\phi \in H$, $\phi(x) = x$, then we say that H *fixes* x. If $E \subseteq A$ and H is a subgroup of G, then $\mathrm{fix}_H(E)$ denotes the (pointwise stabilizer) subgroup $\{\phi \in H : \forall e \in E\ (\phi(e) = e)\}$ of H.

An element x of M is called $\mathcal{F}$-*symmetric* if there exists $H \in \mathcal{F}$ such that H fixes x (equivalently, $\{\phi \in G : \phi(x) = x\} \in \mathcal{F}$), and it is called *hereditarily* $\mathcal{F}$-*symmetric* if x and all elements of its transitive closure, $\mathrm{TC}(x)$, are $\mathcal{F}$-symmetric.

Let $\mathcal{N}$ be the class which consists of all hereditarily $\mathcal{F}$-symmetric elements of M. Then $\mathcal{N}$ is a model of ZFA and $A \in \mathcal{N}$ (see Jech [J73, Theorem 4.1, p. 46]); it is called the *permutation model* determined by M, G and $\mathcal{F}$.

4. The permutation model for the main result. The key ZFA-model for our goal is due to Howard and Rubin [HR96], and it is labeled 'Model $\mathcal{N}38$' in [HR98].

We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a linearly ordered set $(A, \leq)$ of atoms which is order isomorphic to $\mathbb{Q}^\omega$, the set of all sequences of rational numbers, ordered by the lexicographic order, that is,

$$\forall a, b \in \mathbb{Q}^\omega\ (a < b \Longleftrightarrow \exists n \in \omega\ \forall j < n\ (a_j = b_j \wedge a_n < b_n)).$$

We identify the atoms with the elements of $\mathbb{Q}^\omega$ to simplify the description of the permutation model.

Definition 4.1.

(1) Assume $b \in A$ and $n \in \omega$.
   (a) $A_b^n = \{a \in A : a_i = b_i \text{ for } 0 \leq i \leq n\}$ is the *n-level block containing b*. (We note that if $a \in A_b^n$, then $A_a^n = A_b^n$, and if $m, n \in \omega$ with $m \leq n$, then $A_b^n \subseteq A_b^m$. Furthermore, the sets A_b^n will not be in the permutation model defined below.)
   (b) The sequence $(b_{n+1}, b_{n+2}, \ldots)$ is the *position of b in its n-level block.*
   (c) $\mathcal{B}^n = \{A_a^n : a \in A\}$ is the set of n-level blocks.

(d) $\leq_n$ is the relation on $\mathcal{B}^n$ defined by

$$A_c^n \leq_n A_d^n \iff c{\restriction}(n+1) \leq d{\restriction}(n+1).$$

(e) Let f be an order automorphism of $(\mathcal{B}^n, \leq_n)$ (see Facts 4.2 and 4.3 below). We define ϕ_f to be the unique order automorphism of $(A, \leq)$ which satisfies the following two properties:

(i) $\phi_f[A_a^n] = f(A_a^n)$ for all $a \in A$, and

(ii) for all $a \in A$, a and $\phi_f(a)$ have the same position in their n-level blocks. (By item (1b), this means that for every $a \in A$ and every $i > n$, $a_i = (\phi_f(a))_i$.)

(2) For $n \in \omega$, G_n is the group $\{\phi_f : f$ is an order automorphism of $(\mathcal{B}^n, \leq_n)\}$.

(3) G is the group $\bigcup_{n\in\omega} G_n$. (Note that for $n \leq m$, $G_n \subseteq G_m$.)

(4) A set $E \subseteq A$ is called a *support* if it satisfies (a)–(c) below:

(a) E is well-ordered by the ordering $\leq$ on A.

(b) For each $n \in \omega$, $\{A_a^n : a \in E\}$ is finite. (That is, for each $n \in \omega$, the set of nth coordinates of elements of E is finite.)

(c) E is countable.

(5) $\mathcal{F}$ is the filter on the lattice of subgroups of G which is generated by the filter base $\{\mathrm{fix}_G(E) : E$ is a support$\}$.

$\mathcal{F}$ is a normal filter on G. Firstly, note that for every $a \in A$, $\{a\}$ is a support, and thus $\mathrm{fix}_G(\{a\}) \in \mathcal{F}$. Secondly, let $\phi \in G$ and $H \in \mathcal{F}$. Then there exists a support E such that $\mathrm{fix}_G(E) \subseteq H$. It is not hard to verify now that $\phi[E]$ is a support and $\mathrm{fix}_G(\phi[E]) \subseteq \phi H \phi^{-1}$, i.e. $\phi H \phi^{-1} \in \mathcal{F}$.

$\mathcal{N}38$ is the permutation model determined by M, G and $\mathcal{F}$. By the definition of $\mathcal{F}$, it follows that for every $x \in \mathcal{N}38$ there exists a support E such that for all $\phi \in \mathrm{fix}_G(E)$, $\phi(x) = x$. Under these circumstances, we call E a *support of* x.

The following two facts are straightforward; the second of these follows from the observation that $(\mathcal{B}^n, \leq_n)$ is order isomorphic to $\mathbb{Q}^{n+1}$ with the lexicographic order, which is a countable dense linear order without endpoints.

Fact 4.2 ([HR96, Lemma A]). *For each $n \in \omega$ and $a \in A$, A_a^n is an interval in the ordering $\leq$ on A (in the sense that if $c, d \in A_a^n$ and $c \leq b \leq d$, then $b \in A_a^n$).*

Fact 4.3 ([HR96, Lemma B]). *For each $n \in \omega$, the ordering $\leq_n$ defined on $\mathcal{B}^n$ by*

$$A_a^n \leq_n A_b^n \iff a{\restriction}(n+1) \leq b{\restriction}(n+1)$$

is well-defined and the ordered set $(\mathcal{B}^n, \leq_n)$ is order isomorphic to the rational numbers with the usual ordering.

Howard and Rubin [HR96, Sections 5 and 6] established the following result about $\mathcal{N}38$.

THEOREM 4.4. *The permutation model* $\mathcal{N}38$ *satisfies* $\mathsf{BPI} \wedge \mathsf{AC}^{\omega} \wedge \neg\mathsf{DC}$.

5. The suitable transfer theorem of Pincus

DEFINITION 5.1. For any set X, let $\mathcal{P}^{\alpha}(X)$ (where α ranges over ordinal numbers) be defined as follows:

$$\begin{aligned} \mathcal{P}^{0}(X) &= X, \\ \mathcal{P}^{\alpha+1}(X) &= \mathcal{P}^{\alpha}(X) \cup \mathcal{P}(\mathcal{P}^{\alpha}(X)), \\ \mathcal{P}^{\alpha}(X) &= \bigcup_{\beta<\alpha} \mathcal{P}^{\beta}(X) \quad \text{for } \alpha \text{ limit.} \end{aligned}$$

For use in the transfer of our ZFA-independence result to ZF, we provide below some terminology from Jech–Sochor [JS66] and Pincus [P72].

Let us point out that in the forthcoming Definitions 5.2 and 5.3(2), the notation $\mathbf{x}$ stands for a tuple $(x_1, \dots, x_n)$ of variables. In Definition 5.3(2), the variables of $\mathbf{y} = (y_1, \dots, y_n)$ are assumed to be disjoint from those of $\mathbf{x}$. $\exists\mathbf{x}$ ($\forall\mathbf{x}$) stands for $\exists x_1 \cdots \exists x_n$ ($\forall x_1 \cdots \forall x_n$). $\bigcup\mathbf{x}$ stands for $x_1 \cup \dots \cup x_n$.

DEFINITION 5.2. Let C be a class and let $\Phi(\mathbf{x})$ be a formula in the language of set theory with atoms. Then $\Phi^{C}(\mathbf{x})$ is Φ with quantifiers restricted to C. Similarly, if $\sigma(\mathbf{x})$ is a term then $\sigma^{C}(\mathbf{x})$ is defined by the same formula that defines σ but with its quantifiers restricted to C.

$\Phi(\mathbf{x})$ is *boundable* if for some ordinal γ, $\mathsf{ZFA} \vdash \Phi(\mathbf{x}) \leftrightarrow \Phi^{\mathcal{P}^{\gamma}(\bigcup\mathbf{x})}(\mathbf{x})$. Similarly, the term $\sigma(x)$ is boundable if for some ordinal γ, $\mathsf{ZFA} \vdash \sigma(\mathbf{x}) = \sigma^{\mathcal{P}^{\gamma}(\bigcup\mathbf{x})}(\mathbf{x})$.

A *statement* is boundable if it is the existential closure of a boundable formula.

In the following definition, $|y|$ denotes the least ordinal α such that there is a bijection $f : \alpha \to y$; so $|y|$ does not denote the cardinal number of y unless y is well-orderable.

DEFINITION 5.3.

(1) Let x be a set. We define

$$|x|_{-} = \sup\{|y| : \text{there is an injection from } y \text{ to } x\}.$$

$|x|_{-}$ is called the *injective cardinality* of x.

(2) A formula $\Phi(\mathbf{y})$ is *injectively boundable* if it is a conjunction of $\Phi_i(\mathbf{y})$:

$$\Phi_i(\mathbf{y}) = \forall\mathbf{x}\ \Bigg(\bigg(\Big|\bigcup\mathbf{x}\Big|_{-} \le \sigma_i(\mathbf{y}) \wedge \bigcup\mathbf{x} \cap \mathrm{TC}\Big(\bigcup\mathbf{y}\Big) = \emptyset\bigg) \to \Psi_i(\mathbf{x},\mathbf{y})\Bigg),$$

where $\sigma_i(\mathbf{y})$ and $\Psi_i(\mathbf{x},\mathbf{y})$ are boundable.

A statement is injectively boundable if it is the existential closure of an injectively boundable formula.

The following fact was noted in [P72, p. 722].

FACT 5.4. *Boundable formulae and statements are (up to equivalence) injectively boundable.*

THEOREM 5.5 ([P77]). *If a conjunction of injectively boundable statements and* BPI *and* AC^ω *has a permutation model, then it also has a* ZF*-model.*

6. The main result

THEOREM 6.1. *The statement "For every infinite set X, every almost disjoint family in X can be extended to a MAD family in X" is not provable in* $\mathsf{ZF} + \mathsf{BPI} + \mathsf{AC}^\omega$.

Proof. Firstly, we will prove that in the permutation model $\mathcal{N}38$, which (by Theorem 4.4) satisfies $\mathsf{BPI} \wedge \mathsf{AC}^\omega$, there exist an infinite set X and an almost disjoint family in X which cannot be extended to a MAD family in $\mathcal{N}38$.

To this end, we take as our infinite set the set A of atoms of $\mathcal{N}38$. We define

$$e_0 = (0, 0, \ldots),$$

i.e. e_0 is the constant sequence with value 0, and we also define

$$\forall n \in \omega \setminus \{0\}, \qquad (e_n)_i = \begin{cases} i & \text{if } i < n, \\ n & \text{otherwise,} \end{cases}$$

so $e_1 = (0, 1, 1, \ldots)$, $e_2 = (0, 1, 2, 2, \ldots)$, $e_3 = (0, 1, 2, 3, 3, \ldots)$, etc. It is clear that $e_n < e_{n+1}$ for all $n \in \omega$, and that the subset

$$E = \{e_n : n \in \omega\}$$

of A is a support. We let

$$H_0 = (-\infty, e_0] = \{a \in A : a \leq e_0\},$$

and for $n > 0$, we let

$$H_n = (e_{n-1}, e_n] = \{a \in A : e_{n-1} < a \leq e_n\}.$$

We also let

$$H_\infty = \{a \in A : \forall t \in E \ (t < a)\}$$

and

$$\mathcal{H} = \{H_n : n \in \omega\} \cup \{H_\infty\}.$$

Note that E is a support of every member of $\mathcal{H}$, and thus $\mathcal{H} \in \mathcal{N}38$ and $\mathcal{H}$ is denumerable in $\mathcal{N}38$. Furthermore, $\mathcal{H}$ is a partition of A into infinite sets, and hence $\mathcal{H}$ is almost disjoint in A.

$\mathcal{H}$ is not MAD in A. Indeed, let $h \in H_\infty$ and also let $E_0 = E \cup \{h\}$. Then $\mathcal{H}_0 = \mathcal{H} \cup \{E_0\}$ is in $\mathcal{N}38$ since E_0 is a support of $\mathcal{H}_0$, $\mathcal{H}_0$ is almost disjoint in A and $\mathcal{H} \subsetneq \mathcal{H}_0$.

Claim 6.2. *$\mathcal{H}$ cannot be extended to a MAD family in the model $\mathcal{N}38$.*

Proof. Let $\mathcal{G} \in \mathcal{N}38$ be an almost disjoint family in A such that $\mathcal{H} \subsetneq \mathcal{G}$. We will show that $\mathcal{G}$ can be properly extended to an almost disjoint family in A, which is in $\mathcal{N}38$. Let $E' \subset A$ be a support of $\mathcal{G}$, and let

$$E^* = E \cup E'.$$

Clearly, E^* is a support. Without loss of generality, we may assume that

$$\forall a \in A\ [\forall n \in \omega\ (E^* \cap A_a^n \neq \emptyset) \to a \in E^*]. \tag{6.1}$$

This assumption is possible since if $F \subset A$ is a support, then $F \cup \{a \in A : \forall n \in \omega\ (F \cap A_a^n \neq \emptyset)\}$ is a support.

We assert that for every $X \in \mathcal{G} \setminus \mathcal{H}$, $X \subseteq E^*$. Fix $X \in \mathcal{G} \setminus \mathcal{H}$. First of all, we have the following

Subclaim 6.3. *X satisfies condition* (b) *of the definition of support, i.e. for every $n \in \omega$, $\{A_x^n : x \in X\}$ is finite.*

Proof. We will prove the subclaim by induction. Firstly, since $\mathcal{H}$ is contained in the almost disjoint family $\mathcal{G}$ and $X \in \mathcal{G}$, $X \cap H$ is finite for all $H \in \mathcal{H}$. In particular, $X \cap H_0$ and $X \cap H_\infty$ are finite, and thus $\{A_x^0 : x \in X\}$ is finite.

Assume that for some $n > 0$, $\{A_x^{n-1} : x \in X\}$ is finite. If $\{A_x^n : x \in X\}$ is infinite, then, by the pigeonhole principle, there exists an infinite $X' \subseteq X$ such that $x_i = y_i$ for $x, y \in X'$ and $i < n$, and $x_n \neq y_n$ for any distinct $x, y \in X'$. But then it is reasonably clear that for some $H \in \mathcal{H}$, $X' \cap H$ is infinite, which is impossible. Thus, $\{A_x^n : x \in X\}$ is finite, concluding the inductive step and the proof. ∎

Suppose that $X \not\subseteq E^*$. Since $\mathcal{H}$ is a partition of A, $(X \setminus E^*) \cap H \neq \emptyset$ for some $H \in \mathcal{H}$, and since $\mathcal{G}$ is almost disjoint, $(X \setminus E^*) \cap H$ is finite. Assume that

$$(X \setminus E^*) \cap H = \{x^{(1)}, \ldots, x^{(r)}\},$$

where $x^{(1)} < \cdots < x^{(r)}$. There exists $b \in H \setminus E^*$ such that $x^{(r)} < b$ and

$$[x^{(r)}, b] \cap (E^* \cup X) = \{x^{(r)}\}. \tag{6.2}$$

For such a b, $[x^{(r)}, b] \subseteq H$ since $x^{(r)}, b \in H$ and H is an interval in the ordering $\leq$ on A. Let $L = \{e \in E^* \cap H : x^{(r)} < e\}$. If $L = \emptyset$ (which yields $H = H_\infty$), then for any $b \in H$ with $x^{(r)} < b$, (6.2) holds. If $L \neq \emptyset$, then since

E^* is well-ordered by the ordering $\leq$ on A, we let $e^* = \min(L)$ and we also let $b \in H$ be such that $x^{(r)} < b < e^*$. Then, for this b, (6.2) holds.

Fixing a b as above, we let, for $i = 1, \ldots, r-1$, $n_i \in \omega$ be such that $x^{(i)}_{n_i} < x^{(i+1)}_{n_i}$, and we let $n_r \in \omega$ be such that $x^{(r)}_{n_r} < b_{n_r}$. Then $A^{n_i}_{x^{(i)}} <_{n_i} A^{n_i}_{x^{(i+1)}}$ and $A^{n_r}_{x^{(r)}} <_{n_r} A^{n_r}_b$. Since $x^{(r)}, b \notin E^*$, by (6.1) there exist $k, \ell \in \omega$ such that $A^k_{x^{(r)}} \cap E^* = \emptyset$ and $A^\ell_b \cap E^* = \emptyset$. Let $m = \max\{n_1, \ldots, n_r, k, \ell\}$. Then

$$A^m_{x^{(1)}} <_m \cdots <_m A^m_{x^{(r)}} <_m A^m_b \tag{6.3}$$

and

$$A^m_{x^{(r)}} \cap E^* = A^m_b \cap E^* = \emptyset. \tag{6.4}$$

Observe that for every $x \in X \setminus \{x^{(r)}\}$, $A^m_x \neq A^m_{x^{(r)}}$ and $A^m_x \neq A^m_b$. Indeed, if for some $x \in X \setminus \{x^{(r)}\}$, $A^m_x = A^m_{x^{(r)}}$ or $A^m_x = A^m_b$, then $x \in H$ since $A^m_{x^{(r)}}$ and A^m_b are contained in H. By (6.4), we deduce that $x \in (X \setminus E^*) \cap H$, which yields a contradiction to (6.3).

Let $K = \{A^m_e : e \in E^*\} \cup \{A^m_x : x \in X \setminus \{x^{(r)}\}\}$. By the previous observation, (6.2), (6.4) and the definition of $\leq_m$, we conclude that

$$[A^m_{x^{(r)}}, A^m_b] \cap K = \emptyset.$$

Furthermore, by Subclaim 6.3 and the fact that E^* is a support, it follows that K is finite.

Hence, as $(\mathcal{B}^m, \leq_m)$ is isomorphic to the rational numbers with the usual ordering (see Fact 4.3), there exists an order automorphism f of $(\mathcal{B}^m, \leq_m)$ such that $f(A^m_{x^{(r)}}) = A^m_b$ and f fixes all elements of K. Let ϕ_f be the corresponding order automorphism of $(A, \leq)$. Then $\phi_f \in \mathrm{fix}_{G_m}(E^*)$, and thus $\phi_f(\mathcal{G}) = \mathcal{G}$, since E^* is a support of $\mathcal{G}$. It follows that $\phi_f(X) \in \mathcal{G}$. However, since ϕ_f fixes all elements of $X \setminus \{x^{(r)}\}$, and since $\phi_f(x^{(r)}) \in A^m_b$ and $A^m_b \cap A^m_{x^{(r)}} = \emptyset$, we have $\phi_f(X) \cap X = X \setminus \{x^{(r)}\}$, i.e. $\phi_f(X) \cap X$ is infinite, contradicting $\mathcal{G}$'s being almost disjoint. Thus, $X \subseteq E^*$.

Let $\mathcal{U} = \{H \setminus E^* : H \in \mathcal{H}\}$. Since $\mathcal{H}$ is disjoint, so is $\mathcal{U}$, and since E^* is a support of every member of $\mathcal{U}$ and $\mathcal{H}$ is denumerable in $\mathcal{N}38$, $\mathcal{U} \in \mathcal{N}38$ and $\mathcal{U}$ is denumerable in $\mathcal{N}38$. Moreover, all members of $\mathcal{U}$ are infinite.

As AC^ω is true in $\mathcal{N}38$, there exists a choice function for $\mathcal{U}$ in $\mathcal{N}38$, g_0 say. Since $\mathrm{ran}(g_0) \notin \mathcal{H}$ and $\mathrm{ran}(g_0) \cap E^* = \emptyset$, we conclude (by the first part of this proof) that $\mathrm{ran}(g_0) \notin \mathcal{G}$. Thus, letting $\mathcal{G}_0 = \mathcal{G} \cup \{\mathrm{ran}(g_0)\}$, we find that $\mathcal{G}_0$ is almost disjoint in A and $\mathcal{G} \subsetneq \mathcal{G}_0$, and note that $\mathcal{G}_0 \in \mathcal{N}38$ since $E^* \cup \mathrm{ran}(g_0)$ is a support of $\mathcal{G}_0$. This completes the proof of the claim. ∎

We are now ready to transfer the above ZFA-independence result to ZF. Consider the following formula: $\Phi(x) =$ “x is infinite and there exists an almost disjoint family $\mathcal{A}$ in x which cannot be extended to a MAD family in x". Letting "AD" stand for "almost disjoint", we may write $\Phi(x)$ as

$$\Phi(x) = (x \text{ is infinite}) \wedge \exists \mathcal{A} \big(\mathcal{A} \text{ is AD in } x \wedge \forall \mathcal{B} ((\mathcal{B} \text{ is AD in } x \wedge \mathcal{A} \subseteq \mathcal{B}) \\ \rightarrow \exists \mathcal{C} (\mathcal{C} \text{ is AD in } x \wedge \mathcal{B} \subsetneq \mathcal{C}))\big),$$

where "$\mathcal{U}$ is AD in x" is the formula

$$\forall u \big((u \in \mathcal{U}) \rightarrow (u \subseteq x \wedge u \text{ is infinite})\big) \\ \wedge \forall u \, \forall v \big((u \in \mathcal{U} \wedge v \in \mathcal{U} \wedge u \neq v) \rightarrow (u \cap v \text{ is finite})\big).$$

Since for any x, every $n \in \omega$ is a member of $\mathcal{P}^{n+1}(x)$, and thus of $\mathcal{P}^{\omega+\omega}(x)$ (see Definition 5.1), and for every $y \subseteq x$ and every function $f : n \to y$, f is a member of $\mathcal{P}^{n+3}(x)$, and thus of $\mathcal{P}^{\omega+\omega}(x)$, it follows that "$y$ is infinite" and "y is finite" in $\Phi(x)$ can be respectively expressed by

$$\forall n \in \mathcal{P}^{\omega+\omega}(x) \; \forall f \in \mathcal{P}^{\omega+\omega}(x) \; ((n \in \omega \wedge f : n \to y) \rightarrow (f \text{ is not a bijection}))$$

and

$$\exists n \in \mathcal{P}^{\omega+\omega}(x) \; \exists f \in \mathcal{P}^{\omega+\omega}(x) \; (n \in \omega \wedge f : n \to y \text{ is a bijection}).$$

Furthermore, every almost disjoint family in x is a member of $\mathcal{P}^2(x)$, and thus of $\mathcal{P}^{\omega+\omega}(x)$, and $\mathcal{P}^{\omega+\omega}(x)$ is transitive. Hence, all quantifiers in $\Phi(x)$ can be restricted to $\mathcal{P}^{\omega+\omega}(x)$, and thus $\Phi(x)$ is equivalent to $\Phi^{\mathcal{P}^{\omega+\omega}(x)}(x)$, i.e. $\Phi(x)$ is a boundable formula.

It follows that the existential closure of $\Phi(x)$,

$$\Psi = \exists x \; (\Phi(x)),$$

is a boundable statement, and hence (by Fact 5.4) an injectively boundable statement.

Now, since the statement $\Omega = \Psi \wedge \mathsf{BPI} \wedge \mathsf{AC}^\omega$ is a conjunction of the injectively boundable statement Ψ, BPI and AC^ω, and has a permutation model, namely $\mathcal{N}38$, it follows (by Theorem 5.5) that Ω has a ZF-model.

The above arguments complete the proof of the theorem. ∎

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