BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 68, No. 2, 2020

*MANIFOLDS AND CELL COMPLEXES*

# Symplectic structure on colorings, Lagrangian tangles and Tits buildings

by

Jan DYMARA, Tadeusz JANUSZKIEWICZ and Józef H. PRZYTYCKI

*Presented by the Editors*

**Summary.** We define a symplectic form $\widehat{\varphi}$ on a free $R$-module $R^{2n-2}$ associated to $2n$ points on a circle. Using this form, we establish a relation between submodules of $R^{2n-2}$ induced by Fox $R$-colorings of an $n$-tangle and Lagrangians or virtual Lagrangians in the symplectic structure $(R^{2n-2}, \widehat{\varphi})$ depending on whether $R$ is a field or a PID. We prove that when $R = \mathbb{Z}_p$, $p > 2$, all Lagrangians are induced by Fox $R$-colorings of some $n$-tangles and note that for $p = 2$ and $n > 3$ this is no longer true. For any ring, every $2\pi/n$-rotation of an $n$-tangle yields an isometry of the symplectic space $R^{2n-2}$. We analyze invariant Lagrangian subspaces of this rotation and we partially answer the question whether an operation of rotation (generalized mutation) defined in [A-P-R] preserves the first homology group of the double branched cover of $S^3$ along a given link.

**1. Alternating form on the space of boundary points colorings.** Importance of symplectic structures in knot theory was probably first realized by R. H. Fox in his review of A. Plans's 1953 paper [Pla]. In the present paper we construct a symplectic structure on the space of $2n$-boundary points colorings of an $n$-tangle. We note that our symplectic structure is naturally related to the symplectic structure on the first homology group of a surface ([1]). Using our symplectic structure, we are able to draw several far-reaching conclusions. In particular, we partially answer the question

2020 *Mathematics Subject Classification*: Primary 57K10; Secondary 57R17, 51E24, 20E42.
*Key words and phrases*: knot, link, rotor, tangle, branched cover, symplectic form, Lagrangian, Fox coloring, Tits building.
Received 30 December 2020; revised 28 February 2021.
Published online 19 March 2021.

([1]) This relation might be viewed as a parallel of the well-known result that 3-manifolds yield Lagrangians in $H_1(\partial M, \mathbb{Q})$.

DOI: 10.4064/ba201230-2-3  

whether a rotation operation (generalized mutation) as defined in [A-P-R] preserves the first homology group of the double branched cover of $S^3$ along a given link; and we address the problem of "imprisoned colorings".

Let $R$ be a commutative ring with identity and $p_1, \ldots, p_{2n}$ be points on a circle (or a square) as in Figure 1.

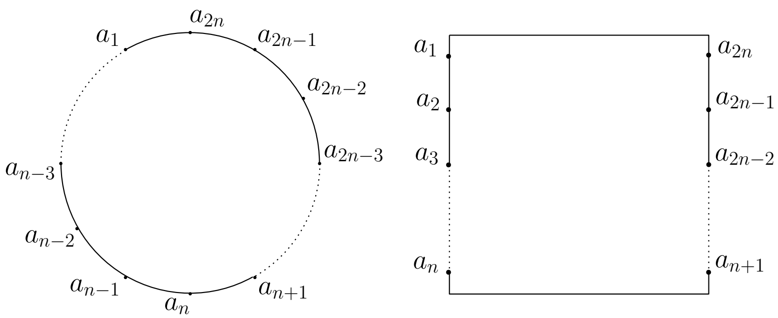



Fig. 1. $R$-coloring of $2n$ points $p_1, \ldots, p_{2n}$ by $a_1, \ldots, a_{2n} \in R$. Different shapes of a tangle are useful for different applications.

Regarding $R$ as a set of colors allows us to identify the set of all $R$-colorings of $2n$ points with the elements of the free $R$-module $R^{2n}$ with basis $e_1, \ldots, e_{2n}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with $1 \in R$ in the $i$th position.

Let

$$W_{2n-1} = \Big\{ \sum_{i=1}^{2n} a_i e_i \in R^{2n} : \sum_{i=1}^{2n} (-1)^i a_i = 0 \Big\} \cong R^{2n-1},$$

$f_i = e_i + e_{i+1}$, $i = 1, \ldots, 2n-1$, and $f_{2n} = e_{2n} + e_1$. Clearly $f_1, \ldots, f_{2n-1}$ are in $W_{2n-1}$ and form a basis (2). Notice that for any $w \in R^{2n}$ we have

$$\begin{aligned} w &= \sum_{i=1}^{2n} a_i e_i \\ &= a_1 f_1 + (a_2 - a_1) f_2 \\ &\quad + \cdots + (a_{2n} - a_{2n-1} + \cdots - a_1) f_{2n-1} + \Big( \sum_{i=1}^{2n} (-1)^i a_i \Big) e_{2n}. \end{aligned}$$

Let $W$ denote an $R$-module. Recall that an $R$-bilinear form $\varphi : W \times W \to R$ is called *alternating* (or *skew-symmetric*) if for all $u, v \in W$,

$$\varphi(u, v) = -\varphi(v, u).$$

Note that if 2 is invertible in $R$, then $\varphi$ is skew-symmetric if and only if $\varphi(u, u) = 0$ for all $u \in W$.

(2) We note that $f_{2n} = e_{2n} + e_1 = \sum_{i=1}^{n} f_{2i-1} - \sum_{i=1}^{n-1} f_{2i}$.

If a skew-symmetric form $\varphi$ is non-degenerate we call it a *symplectic form* (3).

Consider an alternating form $\varphi$ on $W_{2n-1}$ of nullity 1 given by

$$\varphi(f_i, f_j) = \begin{cases} j - i & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| \neq 1, \end{cases}$$

and let

$$v_0 = \sum_{i=1}^{2n} e_i = \sum_{i=1}^{n} f_{2i-1} = \sum_{i=1}^{n} f_{2i}.$$

Then $\varphi(v_0, v) = 0$ for all $v \in W_{2n-1}$, i.e. $v_0$ is an isotropic vector. We will write $v_0 \perp W_{2n-1}$ as $v_0$ is $\varphi$-orthogonal to $W_{2n-1}$. It is useful to introduce the free $R$-module $W_{2n}$ with the basis $f_1, \ldots, f_{2n-1}, f_{2n}$. We can extend $\varphi$ to a symplectic form $\varphi'$ on $W_{2n}$ by setting $\varphi'(f_{2n}, f_j) = 0$, $j = 2, \ldots, 2n-2$, $\varphi'(f_{2n}, f_{2n-1}) = 1$, and $\varphi'(f_{2n}, f_1) = -1$.

Let $w_0 = \sum_{i=1}^{n}(f_{2i-1} - f_{2i})$ and let

$$Rw_0 = \{rw_0 : r \in R\}$$

be the $R$-submodule of $W_{2n}$ generated by $w_0$. We have

$$W_{2n-1} \cong W_{2n}/Rw_0,$$

and if $Rv_0$ denotes the $R$-submodule of $W_{2n-1}$ consisting of all monochromatic $R$-colorings of $2n$ points on the circle, then

$$W_{2n-2} = W_{2n-1}/Rv_0.$$

As one verifies, $\varphi$ descends to a symplectic form $\widehat{\varphi}$ on the free $R$-module $W_{2n-2}$.

Let $r : W_{2n} \to W_{2n}$ be defined by $r(f_i) = f_{i+1}$ for $i = 1, \ldots, 2n-1$ and $r(f_{2n}) = f_1$. It is easy to see that $r$ is induced by a counterclockwise $\pi/n$-rotation of the circle with $2n$ points and clearly $r$ is a $\varphi'$-isometry of $W_{2n}$, i.e.

$$\varphi'(r(f_i), r(f_j)) = \varphi'(f_i, f_j)$$

for all $i, j \in \{1, \ldots, 2n\}$.

LEMMA 1.1. *Let $r : W_{2n} \to W_{2n}$ be the $\varphi'$-isometry of $W_{2n}$ defined above.*

(i) *$r$ descends to a $\varphi$-isometry of $W_{2n-1}$ and a $\widehat{\varphi}$-isometry of $W_{2n-2}$.*
(ii) *The map $m_{yz} : W_{2n} \to W_{2n}$, $m_{yz}(f_i) = f_{2n-i}$, induced by the reflection of the circle with $2n$ points in the $yz$ plane in $\mathbb{R}^3$ is an anti-isometry, i.e. $\varphi'(m_{yz}(u), m_{yz}(v)) = -\varphi'(u, v)$ for all $u, v \in W_{2n}$.*

(3) If a skew-symmetric form $\varphi$ has nullity $k$, we often call it a *symplectic form of nullity* $k$ [Tur].

*Proof.* Since for $w_0 = \sum_{i=1}^{n}(f_{2i-1} - f_{2i})$,

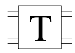
$$r(w_0) = -w_0,$$

we see that $-1$ and $w_0$ is an eigenpair for $\varphi'$. Also $Rv_0 \subset W_{2n-1}$ is an invariant $R$-submodule of $r$. Thus (i) follows (recall that $Rv_0$ is the null $R$-submodule for $\varphi$).

Since $\varphi'(m_{yz}(f_i), m_{yz}(f_j)) = -\varphi'(f_i, f_j)$ for all $i, j = 1, \ldots, 2n$, the statement (ii) also follows. ∎

Let $V$ be a free $R$- module and $\varphi$ be a symplectic form on $V$. We say that $W$ is an *isotropic R-submodule* of $V$ if $\varphi(w_1, w_2) = 0$ for all $w_1, w_2 \in W$. A maximal isotropic $R$-submodule $W$ of $V$ is called a *Lagrangian submodule.* If $R$ is a field, as we will see it later, the isotropic subspaces $W$ of $W_{2n-2}$ form a flag structure in $(W_{2n-2}, \widehat{\varphi})$, hence a Tits building. An $(n-1)$-dimensional isotropic subspace of $(W_{2n-2}, \widehat{\varphi})$ is called a *Lagrangian subspace* and when $R = \mathbb{Z}_p$ ($p$ prime) there are $\prod_{i=1}^{n-1}(p^i + 1)$ of them (see [O]).

The above lemma will be used in Section 4 to analyze Fox $R$-colorings of rotors (see Section 2 and [Pr-1] for introduction to Fox 3-colorings).

**2. Tangles as Lagrangians.** For every $n$-tangle $T =$  define the $R$-module $\mathrm{Col}_R(T)$ of its Fox $R$-colorings ([4]) as follows. Let $x_1, \ldots, x_m$ be arcs of $T$ (parts of the tangle diagram from tunnel to tunnel) and $c_1, \ldots, c_t$ be its crossings. Then $\mathrm{Col}_R(T)$ is a quotient of the free $R$-module $R^m$ with basis $\{x_1, \ldots, x_m\}$ modulo its submodule generated by $2x_i - x_j - x_k$ for each crossing $c_s$, $1 \le s \le t$ (see Figure 2) An element $c \in \mathrm{Col}_R(T)$ is called

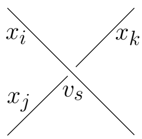



Fig. 2. Fox $R$-coloring relation $2x_i = x_j + x_k$

a *Fox R-coloring* of $T$ and it can be regarded as a mapping from the arcs of $T$ to $R$. Every $R$-coloring of $T$ induces a coloring of its $2n$ boundary points $p_1, \ldots, p_{2n}$. Therefore, there is a homomorphism of $R$-modules $\psi$ : $\mathrm{Col}_R(T) \to R^{2n}$ with $\psi(\mathrm{Col}_R(T)) \subseteq W_{2n-1}$ (see Proposition 2.2), i.e. $\psi$ : $\mathrm{Col}_R(T) \to W_{2n-1}$. Moreover, since $Rv_0 \subseteq \mathrm{Col}_R(T)$, the homomorphism $\psi$ descends to a homomorphism of the quotient $R$-modules

$$\hat{\psi} : \mathrm{Col}_R(T)/Rv_0 \to W_{2n-1}/Rv_0.$$

It is natural to ask which subspaces of $W_{2n-2} \simeq W_{2n-1}/Rv_0$ are induced by $n$-tangles. We answer this question partially when $R$ is a field.

([4]) If $R = \mathbb{Z}_p$ we write $\mathrm{Col}_p(T)$ instead $\mathrm{Col}_{\mathbb{Z}_p}(T)$.

THEOREM 2.1. *Let $\hat{\psi} : \mathrm{Col}_R(T)/Rv_0 \to W_{2n-2}$ be the homomorphism of $R$-modules defined above.*

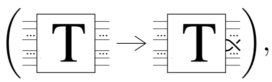
(i) *If $R$ is a field then $\hat{\psi}(\mathrm{Col}_R(T)/Rv_0)$ is a Lagrangian subspace of $(W_{2n-2}, \hat{\varphi})$.*
(ii) *If $R$ is a commutative ring with identity and $T$ is a rational $n$-tangle then $\hat{\psi}(\mathrm{Col}_R(T)/Rv_0)$ is a Lagrangian submodule of $(W_{2n-2}, \hat{\varphi})$.*

Before we give the proof of Theorem 2.1, let us recall some standard terminology concerning tangles and sketch a rather general inductive procedure which will be used later.

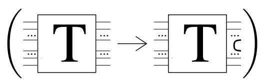
Define an $(n,k)$-*tangle* as a tangle with $n$ inputs (on the left) and $k$ outputs (on the right). Tangles can be composed if the number of outputs of the first one equals the number of inputs of the second ([5]). Let $\sigma_i$ be a crossing between the $i$th and $(i+1)$th strands, $U_i^m$ be a "local minimum" and $U_i^M$ be a "local maximum" (while moving in the direction of the $x$-axis) that involve $i$ and $i+1$ strands ([6]) respectively.

We say that an $n$-tangle is *rational* if it is obtained from an $n$-tangle with no crossings and trivial components ([7]) by adding a finite number of crossings $\sigma_i^{\pm 1}$ $(i = 1, \ldots, 2n)$. We see that the generators $\sigma_i$, $U_i^m$ and $U_i^M$ (for $n < i < 2n$) yield maps

$$\sigma_i : \mathcal{T}_{2n} \to \mathcal{T}_{2n}$$

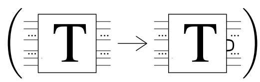


$$U_i^m : \mathcal{T}_{2n} \to \mathcal{T}_{2n+2}$$



and

$$U_i^M : \mathcal{T}_{2n} \to \mathcal{T}_{2n-2}$$



where $\mathcal{T}_{2k}$ denotes the set of all tangles with $k$ inputs and $k$ outputs. We

([5]) Note that there is a general category of tangles with boundary points as objects and tangles as morphisms.

([6]) If inputs and outputs are labeled as in Figure 1 (right), then the standard $i$th generator of the braid group $B_n$ will be denoted by $\sigma_{2n-i}$. However, it is convenient to be able to add crossings, minima and maxima at any place in a given tangle $T$ as we can then take advantage of rotational symmetry that is important for this paper.

([7]) Any $n$-tangle with no crossings or trivial components can be chosen as a starting tangle.

observe that each $n$-tangle $T$ is a composition ([8]) of crossings $\sigma_i$, maxima $U_i^m$, and minima $U_i^M$ (for $n < i < 2n$). Thus, to prove a property $P$ for all tangles, it is sufficient to argue that $P$ holds for "basic" tangles and then show by induction if a tangle $T$ has property $P$, then so do the tangles $T\sigma_i^{\pm 1}, TU_i^m$, and $TU_i^M$. This idea will be used several times (including the proof of our main theorem) and, in particular, we use it below to show that an $R$-Fox coloring of a tangle $T$ induces an $R$-coloring of its $2n$ boundary points that satisfies the "alternating" condition ([9]).

PROPOSITION 2.2. *Let $T$ be an $n$-tangle and $\psi : \mathrm{Col}_R(T) \to R^{2n}$ be the homomorphism of $R$-modules which assigns to an $R$-Fox coloring of arcs of $T$ the corresponding $R$-coloring of their $2n$ boundary points. If $c = \sum_{i=1}^{2n} a_i e_i$ and $c \in \psi(\mathrm{Col}_R(T))$ then $c$ satisfies the* alternating condition*, i.e.* $\sum_{i=1}^{2n} (-1)^i a_i = 0$.

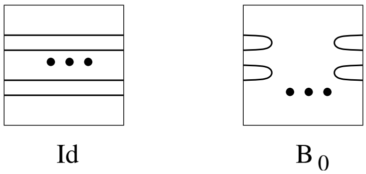



Fig. 3. Tangles Id and $B_0$

*Proof.* Consider an $n$-tangle ([10]) $B_0$ shown in Figure 3. Since each Fox $R$-coloring $a_1, \ldots, a_n$ of the $n$ arcs of $B_0$ that join $x_1$ and $x_2$, $x_3$ and $x_4$, $\ldots$, $x_{2n-1}$ and $x_{2n}$ gives the corresponding $R$-coloring

$$c = a_1 e_1 + a_1 e_2 + \cdots + a_n e_{2n-1} + a_n e_{2n} \in \psi(\mathrm{Col}_R(T))$$

of the boundary points of $B_0$, clearly the alternating condition holds for $B_0$. Assume that the alternating condition holds for a tangle $T$, and consider tangles $T\sigma_i, T\sigma_i^{-1}, TU_i^m$ and $TU_i^M$. The local modification of $T$ shown in Figure 4 yields $a - b = d - c$ for the first two tangles and $a - b = 0$ for the last two. Hence the alternating condition holds for $T\sigma_i, T\sigma_i^{-1}, TU_i^m$ and $TU_i^M$. Therefore, it must also be true for an arbitrary tangle $T'$, which completes the proof. ∎

([8]) Since an $n$-tangle $T$ is a 1-dimensional manifold properly embedded in a cube, $T$ is a Morse function along the $x$-axis with $\sigma_i$'s between extremal points.

([9]) In [Pr-1], we gave a proof of this result in the case when 2 is not a zero divisor in $R$.

([10]) We could take the identity $n$-tangle Id shown in Figure 3 as an $n$-tangle with no crossings or trivial components as a "basic" state, but due to our later considerations it is better for us to choose $B_0$.

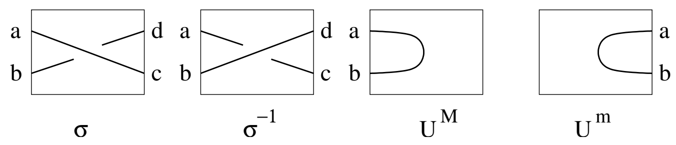



Fig. 4. The tangles $\sigma^{\pm 1}$, $U^M$, and $U^m$

*Proof.* To prove Theorem 2.1, we use an inductive argument similar to that above.

(0) We easily verify that the $R$-submodule induced by the Fox $R$-colorings of $B_0$ is a Lagrangian. In fact $\hat{\psi}(B_0) \subseteq W_{2n-2}$ is a free $R$-module with basis $f_1, f_3, \ldots, f_{2n-3}$.

(1) If a tangle $T$ yields a Lagrangian then $T'$ obtained from $T$ by adding one crossing is also a Lagrangian. In fact, we can view adding a crossing as a transvection in $(W_{2n-2}, \hat{\varphi})$. More precisely, a crossing between the $i$th and $(i+1)$th strands ($1 \leq i \leq 2n$ and indices are mod $2n$) yields a transvection on $W_{2n-2}$ defined by

$$t_i(v) = v \pm \hat{\varphi}(v, f_i) f_i,$$

where the sign $\pm$ depends on the type of the crossing. It is well-known (see [O] for example) that transvections are isometries of a symplectic space, thus in particular they map Lagrangians in $W_{2n-2}$ into Lagrangians in $W_{2n-2}$.

(2) Minima as "expansions". The symplectic space $R^{2n}$ is obtained from the symplectic space $R^{2n-2}$ by adding a 2-dimensional symplectic space (with the symplectic form given by a standard $2 \times 2$ "hyperbolic" matrix). Lagrangians in $R^{2n}$ are built from Lagrangians in $R^{2n-2}$ by adding a vector from the 2-dimensional subspace that was added.

(3) Maxima as "contractions". Adding a "maximum" to $T$ results in connecting the $i$th and $(i+1)$th strands. As a result an $n$-tangle $T$ reduces to an $(n-1)$-tangle $T_i^M$. Consider the projection $p_i$ from $R^{2n}$ to $R^{2n-2}$ defined by

$$p_i\Big(\sum_{j=1}^{2n} x_i e_i\Big) = \sum_{j=1,\, j\neq i,\, (i+1)}^{2n} x_i e_i.$$

One can verify that $p_i$ restricted to the subspace spanned by

$$\{f_1, \ldots, f_{i-2}, f_{i-1} - f_i + f_{i+1}, f_{i+2}, \ldots, f_{2n-1}\}$$

is a symplectic isometry. If $R$ is a field then by a simple dimension counting argument ([11]), the $(n-1)$-tangle $T^M$ also induces a Lagrangian in $R^{2n-2}$.

([11]) This is a special case of Turaev's "contraction lemma" (see [Tur, p. 180]).

This completes the proof of Theorem 2.1(i). Theorem 2.1(ii) also follows, because we note that step (1) also holds for an arbitrary ring. ∎

It is also important to note that Theorem 2.1(i) is no longer true if $R$ is a ring rather than a field (not even for $R = \mathbb{Z}$).

EXAMPLE 2.3. Let $R = \mathbb{Z}$ and $T$ be the 2-tangle $U_2^m \sigma_1^2 \sigma_3^{-2} U_2^M$ shown in Figure 5. Then $\hat{\psi}(T) \subset W_2$ is generated by $2f_1$.

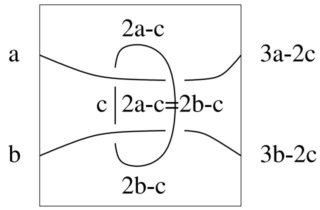



Fig. 5. Tangle $T = U_2^m \sigma_1^2 \sigma_3^{-2} U_2^M$

Using our previous considerations we obtain the following:

THEOREM 2.4. *If $R$ is a PID then $\hat{\psi}(\mathrm{Col}_R(T)/Rv_0)$ is a virtual Lagrangian of $(W_{2n-2}, \hat{\varphi})$, i.e. a finite index $R$-submodule of a maximal isotropic $R$-submodule of $W_{2n-2}$.*

An important question that arises at this point is whether every Lagrangian subspace can be realized by a tangle. The answer is affirmative when $R = \mathbb{Z}_p$, $p > 2$.

THEOREM 2.5. *For $p > 2$, every Lagrangian subspace of $\mathbb{Z}_p^{2n-2}$ is induced by an $n$-tangle.*

*Proof.* This follows from the work of J. Assion [A] (for $p = 3$) and B. Wajnryb [Wa-2, Wa-3] (for general $p > 2$). ∎

When $R = \mathbb{Z}_2$ or $\mathbb{Z}$, Theorem 2.5 is true for 2- and 3-tangles only and it is related via the double branched cover to the fact that the mapping class group of a torus and a genus two surface is generated by "horizontal" Dehn twists. An algebraic proof is based on the observation that the symplectic group $\mathrm{Sp}(2n-2, R)$, $n < 4$, is generated by $2n-1$ transvections $t_i(v) = v + \hat{\varphi}(v, f_i) f_i$. However, Theorem 2.5 is not true in general when $p = 2$.

PROPOSITION 2.6. *For $p = 2$ the number of Lagrangians of $(\mathbb{Z}_2^{2n-2}, \hat{\varphi})$ is $\prod_{i=1}^{n-1}(2^i + 1)$ while the number of Lagrangians induced by $n$-tangles equals $\prod_{i=1}^{n-1}(2i+1)$. In particular, for $n \geq 4$ not all Lagrangians of $(\mathbb{Z}_2^{2n-2}, \hat{\varphi})$ are realizable by $n$-tangles.*

*Proof.* When $p = 2$, the space $\mathrm{Col}_2(T)$ is an invariant of homotopy of tangles (fixed on the boundary). Equivalently $\mathrm{Col}_2(T)$ is preserved by a crossing change. Thus, $\psi(\mathrm{Col}_2(T))$ depends only on connections between $2n$ boundary points. Since there are $(2n-1)(2n-3)\dots 3\cdot 1$ such connections and each of them yields a different $n$-dimensional subspace of $\mathbb{Z}_2^{2n-1}$ (and $\mathbb{Z}_2^{2n-2}$), the statement follows. ∎

COROLLARY 2.7. *Let $\psi : \mathrm{Col}_R(T) \to R^{2n}$ be the homomorphism of $R$-modules defined as before.*

(a) $\dim \psi(\mathrm{Col}_p(T)) = n$ *for any $n$-tangle $T$.*
(b) *For every $p$-coloring $c$ of $2n$ boundary points (an element of $\mathbb{Z}_p^{2n-1}$) there is an $n$-tangle $T$ and a Fox $p$-coloring of $T$ that induces $c$, i.e.*

$$\mathbb{Z}_p^{2n-1} = \bigcup_T \psi(\mathrm{Col}_p(T)).$$

*Proof.* (a) Follows from Theorem 2.1.

(b) Follows from Theorem 2.5 for $p \geq 3$ as every vector in a symplectic space is in some Lagrangian.

For $p = 2$, let $(\epsilon_1, \dots, \epsilon_{2n}) \in \mathbb{Z}_2^{2n}$ and assume that $\sum_{i=1}^{2n} \epsilon_i = 0$. Then the number of $i \in \{1, \dots, 2n\}$ such that $\epsilon_i = 1$ is even. Let $T$ be an $n$-tangle with no arc that connects boundary points labeled by 1 to those labeled by 0. If we assign 1 to each arc of $T$ with endpoints labeled by 1, and 0 to the remaining ones, we see that such a Fox 2-coloring of $T$ induces $(\epsilon_1, \dots, \epsilon_{2n})$ (see Figure 6). ∎

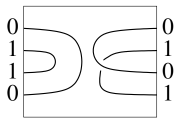



Fig. 6. A tangle $T$ inducing $(\epsilon_1, \dots, \epsilon_{2n})$

REMARK 2.8 (Surfaces analogy). The best known example of a symplectic $\mathbb{Z}$-module in topology is $H_1(F_g, \mathbb{Z})$ (i.e. the first homology of a closed surface $F_g$ of genus g), with the symplectic form defined as the algebraic crossing number of closed curves representing elements of $H_1(F_g, \mathbb{Z})$. In fact, we can make the analogy more precise by considering the double branched cover of $(D^3, \partial D^3)$ along a given $n$-tangle. In particular, $\partial D^3$ branched over $2n$ points lifts to a surface $F_{n-1}$. The symplectic structure on the space $\mathbb{Z}_p^{2n-2}$ of boundary points colorings lifts to the symplectic structure on $H_1(F_g, \mathbb{Z}_p)$. As is well-known, any 3-manifold with boundary $F_g$ yields a Lagrangian subspace in $H_1(F_g, \mathbb{Z}_p)$. In particular, the Lagrangian induced by an $n$-tangle corresponds to the Lagrangian subspace induced by a 3-manifold of the double branched cover. As noticed before, (for $p = 2$) not every Lagrangian is induced by an $n$-tangle. This in turn has its consequences for branched coverings and mapping class groups (see Remark 2.9).

REMARK 2.9. The mapping class group of a surface is generated by a finite number of Dehn twists. Furthermore, one shows that it is enough to take a "horizontal" Dehn twist (for genus $g \le 2$) together with one additional non-horizontal Dehn twist (for $g > 2$) [Lic, Hu, Wa-1]. In fact horizontal Dehn twists correspond to crossing changes, and the argument we used for $\mathbb{Z}_2$-colorings shows the known fact (see [B-H]) that for $g > 2$ horizontal twists are insufficient to generate the mapping class group of $F_g$. On the other hand the above reasoning shows that for 3-tangles we produce all Lagrangians.

As is easy to see, the set $\mathcal{T}_{2n}$ of all $n$-tangles with the operation of composition is an $R$-algebra. We will prove the following lemma.

LEMMA 2.10. *Let $m : \mathcal{T}_{2n} \to \mathcal{T}_{2n}$ be an isomorphism of $R$-algebras that sends an $n$-tangle $T$ to its mirror image $\overline{T}$ in the $xy$ plane. Then $m$ induces an anti-isometry $m_* : W_{2n-2} \to W_{2n-2}$ of the symplectic module $(W_{2n-2}, \hat{\varphi})$ given by*

$$m_*(f_i) = (-1)^i f_i$$

*that maps Lagrangian submodules of $W_{2n-2}$ determined by $T$ to Lagrangian submodules of $W_{2n-2}$ determined by $m(T) = \overline{T}$.*

*Proof.* We see that

$$m_*(f_{2n-1}) = m_*(-f_1 - f_3 - \cdots - f_{2n-3}) = f_1 + f_3 + \cdots + f_{2n-3} = -f_{2n-1}.$$

and analogously

$$m_*(f_{2n}) = m_*(-f_2 - f_4 - \cdots - f_{2n-2}) = -f_2 - f_4 - \cdots - f_{2n-2} = f_{2n}.$$

Using the method from our proof of Theorem 2.1, we first show that Lemma 2.10 holds for a basic tangle $B_0$. Then we show that if the lemma holds for an $n$-tangle $T$, it also holds for any $n$-tangle $T'$ obtained from $T$ by adding a crossing, a minimum or a maximum.

(i) (Crossing) As noted earlier, adding a crossing $\sigma_i^\epsilon$ induces a transvection on $W_{2n-2}$:

$$t_i(v) = v - \epsilon\tilde{\varphi}(v, f_i) f_i.$$

Since the operation of taking the mirror image changes only $\epsilon$ to $-\epsilon$ in the transvection $t_i$, one easily verifies that the conclusion of Lemma 2.10 also holds for $T\sigma_i^\epsilon$.

(ii) (Minimum or maximum) Clearly a minimum and a maximum are fixed when taking the mirror image. Since, as we saw in the proof of Theorem 2.1, adding a maximum or a minimum changes by 2 the dimension of the symplectic $R$-module associated to the boundary points $R$-coloring, the conclusion holds for $TU_i^m$ and $TU_i^M$. ∎

We note that an extension of the anti-isomorphism $m_*: W_{2n-2} \to W_{2n-2}$, $m_*(f_i) = (-1)^i f_i$, to $W_{2n-1}$ is defined by

$$\begin{aligned} m_*(f_{2n}) &= m_*(f_1 + f_3 + \cdots + f_{2n-1} - f_2 - f_4 - \cdots - f_{2n-2}) \\ &= -f_1 - f_3 - \cdots - f_{2n-1} - f_2 - f_4 - \cdots - f_{2n-2} \\ &= f_{2n} - (f_1 + f_2 + \cdots + f_{2n}) = f_{2n} - 2v_0. \end{aligned}$$

Using Lemmas 1.1(ii) and 2.10 we obtain as a corollary the following result, important for our later analysis of rotors.

COROLLARY 2.11. *Let $m_y : \mathcal{T}_{2n} \to \mathcal{T}_{2n}$ be an anti-isomorphism of the $R$-algebra of tangles induced by the $\pi$-rotation about the $y$-axis in $\mathbb{R}^3$. Then the corresponding isometry of $R$-modules $m_{*y} : W_{2n-2} \to W_{2n-2}$ defined by*

$$m_{*y}(f_i) = (-1)^i f_{2n-i}$$

*establishes a bijection between the sets of Lagrangians induced by $T$ and by $m_y(T)$.*

**3. Buildings.** Let $C$ be the Coxeter group defined by the presentation

$$G = \langle s_1, \ldots, s_{d+1} \mid (s_i s_j)^{m_{ij}} \rangle,$$

where $m_{ii} = 1$, $m_{ij} = m_{ji}$ and $m_{ij} \geq 2$ is an integer for $i \neq j$ (or $\infty$ in which case there is no $ij$th relation). We also let $l(w)$ be the length of the shortest word representing $w$ in the generators $s_i$. A *Tits building* is a set $X$ with a $C$-valued distance function $d : X \times X \to C$ that has the following properties:

1. $d(x, y) = 1$ iff $x = y$.
2. If $d(x, y) = w$ and $d(y, z) = s_i$ then $d(x, z) = w$ or $d(x, z) = ws_i$.
3. For each $x$, $y \in X$ and $s_i$ there is $z \in X$ such that

$$d(y, z) = s_i, \quad d(x, z) = d(x, y)s_i.$$

Moreover, $z$ is unique if $l(d(x, y)s_i) < l(d(x, y))$.

We note that $X$ can be realized geometrically as a simplicial complex $X_t$ as follows:

- Label the $(d-1)$-dimensional faces of the simplex $\Delta^d$ with $s_1, \ldots, s_{d+1}$.
- Form $X \times \Delta^d$ and, for any $w, w' \in X$ with $d(w, w') = s_i$, glue $w \times \Delta^d$ and $w' \times \Delta^d$ along the face labeled ([12]) by $s_i$.

The same construction for $C$ equipped with $d(w, w') = w^{-1}w'$ in place of $X$ yields the *Coxeter complex* $C_t$ of $C$. The geometric realization of $X_t$

([12]) Note that faces that are glued together have induced labeling of their faces by the remaining generators of $C$, and the gluing has to preserve them.

contains plenty of embedded copies of $C_t$ (called *apartments*). Any pair of *chambers* ($d$-dimensional simplices of $X_t$) is contained in an apartment ([13]).

REMARK 3.1. Let $c \in X$ be a chamber. Then the folding map $f_c : X \to C$ defined by

$$f_c(c') = d(c, c')$$

induces a simplicial morphism $f_c : X_t \to C_t$ which is an isomorphism when restricted to an apartment that contains $c$.

Assume that $R$ is a field and consider the symplectic space $(W_{2n-2}, \widehat{\varphi})$. We show how to construct a Tits building using $(W_{2n-2}, \widehat{\varphi})$ by describing its geometrical realization as a simplicial complex. Indeed, as shown in [Brown, Ga], the simplicial complex $X_{n-1}$ whose vertices are isotropic subspaces of $W_{2n-2}$ and which has flags ([14]) of isotropic subspaces as simplices forms a building. More precisely, each vertex of $X_{n-1}$ is labeled by the dimension of the corresponding isotropic subspace of $W_{2n-2}$, and codimension one faces of chambers are labeled using the corresponding labels of the opposite vertices. Furthermore, the Lagrangian subspaces of $W_{2n-2}$ are in one-to-one correspondence with vertices of $X_{n-1}$ labeled by $n-1$. We observe that chambers of $X_{n-1} = X_t$ (elements of $X$) correspond to maximal isotropic flags in $W_{2n-2}$. In our case, the Coxeter group $C$ is a signed permutation group determined by

$$m_{i,j} = \begin{cases} 4 & \text{if } i = n-1, j = n-1, \\ 3 & \text{if } 1 \le i < n-2, \\ 2 & \text{if } |i-j| > 1, \end{cases}$$

$$m_{n-2,n-1} = 4, \qquad m_{i,i+1} = 3$$

and the corresponding Coxeter complex can be identified with the first barycentric subdivision of the boundary of an $(n-1)$-dimensional cube $I^{n-1}$.

We now describe how to define apartments for our Coxeter complex. Choose a basis $B = \{v_1, w_1, \dots, v_{n-1}, w_{n-1}\}$ of $W_{2n-2}$ with the property

$$\widehat{\varphi}(v_i, w_i) = -\widehat{\varphi}(w_i, v_i) = 1,$$
$$\widehat{\varphi}(v_i, v_j) = -\widehat{\varphi}(w_i, w_j) = \widehat{\varphi}(v_i, w_j) = 0$$

for $i \neq j$ and consider all isotropic subspaces ([15]) spanned by subsets of $B$. Vertices corresponding to these subspaces span an apartment $A(B)$ in $X_t$. Moreover, the $2^{n-1}$ Lagrangian subspaces in $A(B)$ consisting of all subspaces

([13]) One may use the family of apartments to give another definition of buildings (see [Brown, Ron]).

([14]) A *flag* in a vector space is a strictly increasing sequence of subspaces.

([15]) Such a subspace can be obtained by choosing at most one vector from each pair $\{v_i, w_i\}$.

of $W_{2n-2}$ with exactly one vector from each pair $\{v_i, w_i\}$ correspond to vertices of the cube $I^{n-1}$.

LEMMA 3.2. *Let $\mathcal{L}$ be the family of Lagrangian subspaces corresponding to $(n-1)$-labeled vertices of an apartment $A \subset X_{n-1}$. Then for each Lagrangian $M \subset W_{2n-2}$ there exists an $L \in \mathcal{L}$ with $M \cap L = \{0\}$.*

*Proof.* Suppose this is not the case. Then for each $L \in \mathcal{L}$ there is a length $\leq 2$ path $P_L$ in the 1-skeleton of $X_{n-1}$ connecting $M$ and $L$. Choose a chamber $c \in A$ and consider the folding map $f_c$. The family $\{f_c(P_L) \mid L \in \mathcal{L}\}$ is a family of length $\leq 2$ paths in the 1-skeleton of $C_t$ connecting $f_c(M)$ to every vertex of the cube $I^{n-1}$. However, $f_c(M)$ cannot be connected by such a path to the vertex opposite to it. In other words, if $L$ corresponds to the vertex antipodal in $A$ to $f_c(M)$, then $L$ intersects trivially with $M$. ■

To finish this section, we give an example of a family $\mathcal{L}$ as in the lemma describing its elements in terms of Catalan tangles (i.e. $n$-tangles with no crossings). Let $B = \{v_i, w_i : i = 1, \ldots, n-1\}$, where

$$\begin{aligned} v_1 &= f_{n-1}, & w_1 &= f_n, \\ v_2 &= f_{n-2} + f_n, & w_2 &= f_{n-1} + f_{n+1}, \\ v_3 &= f_{n-3} + f_{n-1} + f_{n+1}, & w_3 &= f_{n-2} + f_n + f_{n+2}, \end{aligned}$$

etc.

PROPOSITION 3.3. *Each Lagrangian subspace of $W_{2n-2}$ corresponding to a vertex of $A(B)$ is represented by exactly one $n$-tangle from the following family $\mathcal{L}$. A tangle $T$ is in $\mathcal{L}$ iff*

(a) *$T$ is a Catalan $n$-tangle;*
(b) *each boundary point $p_i$ is connected to one of $p_{i-1}, p_{i+1}, p_{2n-i}, p_{2n-i-2}$.*

*Proof.* We construct tangles from the family $\mathcal{L}$ inductively, starting from the left of the square (see Figure 1), adding the $i$th arc while choosing a vector from the pair $\{v_i, w_i\}$ at the same time. In the first step the choice of $v_1$ results in joining $p_n$ to $p_{n-1}$ by an arc and the choice of $w_1$ corresponds to connecting $p_n$ and $p_{n+1}$ by an arc. If in the $k$th step $v_k$ was chosen, then in the next step one can choose $v_{k+1}$ (and, at the same time, add an arc connecting the lowest available boundary points) or $w_{k+1}$ (and, at the same time, connecting the two lowest available boundary points on the right hand side of the square).

If in the $k$th step $w_k$ was chosen, then in the next step one can choose $w_{k+1}$ (and add a strand across the square connecting the lowest available boundary points) or $v_{k+1}$ (and connect the two lowest available boundary points on the left hand side of the square).

Finally, one connects the remaining two boundary points. ■

**4. Rotors.** In this section we given criteria when a dihedral flype (see Definition 4.2) preserves the space of Fox $p$-colorings $\mathrm{Col}_p(L)$ ($p > 2$) modulo trivial colorings (the first homology with $\mathbb{Z}_p$ coefficients of the double branched cover of $S^3$ branched along a link [Pr-1]).

Definition 4.1 ([A-P-R]). Consider an $n$-tangle which is a part of the link diagram placed in the regular $n$-gon with $2n$ boundary points ($n$ inputs and $n$ outputs). We say that such an $n$-tangle is an *$n$-rotor* if it has rotational symmetry, that is, the tangle is invariant with respect to rotation about the $z$-axis by $2\pi/n$ (see Figure 7).

Definition 4.2 ([A-P-R]; rotation operation on a link diagram). Consider a link $L_1$ whose diagram is divided into two $n$-tangles: the rotor part, as described in Definition 4.1, and its complement called a *stator*. A *rotant* of a link $L_1$ is the link $L_2$ (Figure 7) obtained from $L_1$ by a dihedral flype of the rotor part. Note that $L_2$ does not depend on the choice of the dihedral flype. We say that $L_2$ is obtained from $L_1$ by a *rotation*.

As noted before, there is a $\mathbb{Z}_{2n}$-action on the set $\mathcal{T}_{2n}$ induced by the rotation $r$ of $n$-tangles counterclockwise about the $z$-axis by the angle $\pi/n$. Moreover, on the space $\mathbb{Z}_p^{2n}$ of $p$-colorings of $2n$ boundary points, $r$ yields an isomorphism defined by $r(e_i) = e_{i+1}$ (where indices are modulo $2n$) and, when restricted to $W_{2n-1}$, the map $r$ is a $\varphi$-isometry which maps basis vectors as follows:

$$\begin{aligned} r(f_i) &= f_{i+1}, \\ r(f_{2n-1}) &= f_{2n} = \sum_{i=1}^{n-1}(f_{2i-1} - f_{2i}) + f_{2n-1}, \\ r(f_{2n}) &= f_1. \end{aligned}$$

We analyze invariant subspaces of $r^2$ and, in particular, we look for invariant

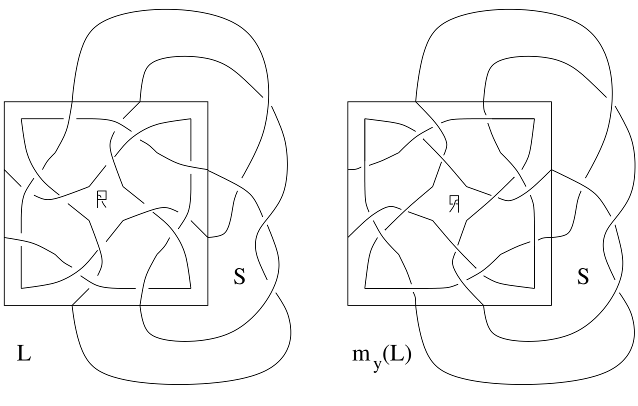



Fig. 7. A link $L$ and its rotant $m_y(L)$

Lagrangians of the $2\pi/n$-rotation about the $z$-axis. We use our analysis to partially answer the question whether an operation of rotation (in the sense of Definition 4.2) preserves the first homology of the double branched cover of $S^3$ along a link.

In particular, we prove the following result.

THEOREM 4.3. *Let $L$ be a link diagram with an $n$-rotor part $R$. Let the rotant $m_y(L)$ be obtained from $L$ by rotating $R$ around the $y$-axis by the angle $\pi$ and keeping the stator, $L-R$, unchanged. Assume that either $n = p$, where $p$ is a prime number, or $n$ is co-prime to $p$ and such that there exists $s$ with $p^s \equiv -1 \bmod n$. Then the space $\mathrm{Col}_p(L)$ is preserved by any $n$-rotation.*

In the proof of Theorem 4.3 we analyze eigenspaces of the symplectic space $\mathbb{Z}_p^{2n-2}$ with respect to rotation. This will allow us to obtain conditions under which a rotation invariant Lagrangian subspace of $\mathbb{Z}_p^{2n-2}$ is also invariant under the dihedral flype $s$. Figure 7 illustrates a pair of 4-rotants which have different spaces of Fox 5-colorings, $\mathbb{Z}_5^2$ and $\mathbb{Z}_5^3$, respectively. One can also check for the tangle $R$ in Figure 7 that $\ker(\psi) = \mathbb{Z}_5$. Therefore, there exists a nontrivial 5-coloring of $R$ which is 0 on the boundary of $R$ ([16]). Analogous examples can be constructed whenever $\gcd(p-1, n) > 2$.

Most of this section is devoted to the proof of Theorem 4.3. Let $\mu_n$ be the group of $n$th roots of 1, and let $k$ be a finite algebraic extension of $\mathbb{Z}_p$ containing $\mu_n$. To understand $r^2$-invariant Lagrangians of $W = W_{2n-2}$ we first consider the action of $r^2$ on two invariant subspaces

$$W_+ = \langle f_0, f_2, \ldots, f_{2n-2}\rangle \quad \text{and} \quad W_- = \langle f_1, f_3, \ldots, f_{2n-1}\rangle.$$

After extending scalars to $k$ we can diagonalize this action. For each $\alpha \in \mu_n - \{1\}$ the vector

$$D_\alpha = \sum_{j=0}^{n-1} \alpha^j f_{2j}$$

(respectively $C_\alpha = \sum_{j=0}^{n-1} \alpha^j f_{2j+1}$) is an eigenvector of $r^2$ with eigenvalue $\alpha$. The vectors $D_\alpha$ (resp. $C_\alpha$) form a basis of $W_+$ (resp. $W_-$). One shows

FORMULA 4.4.

$$\widehat{\varphi}(C_\alpha, D_\beta) = \begin{cases} n(\alpha-1) & \text{if } \alpha\beta = 1,\\ 0 & \text{otherwise,}\end{cases} \tag{1}$$

$$\widehat{\varphi}(C_\alpha, C_\beta) = \widehat{\varphi}(D_\alpha, D_\beta) = 0 \ (W_+ \text{ and } W_- \text{ are Lagrangians})$$

*where $\widehat{\varphi}$ is also used for the extension $k$ of $\mathbb{Z}_p$.*

---

([16]) This follows from the fact that any rotation preserves the determinant of $L$ which is equal to the order of the first homology, $H_1(M_L^{(2)}, \mathbb{Z})$, where $M_L^{(2)}$ is the 2-fold branched cover of $S^3$ along $L$ [Pr-2, Tra, DIPY]. Thus, if a rotation changes the space of 5-colorings then always $\ker(\psi) \neq \{0\}$.

The action of $m_y$ on our eigenvectors is given by

$$m_y(C_\alpha) = -\alpha^{-1} C_{\alpha^{-1}}, \quad m_y(D_\alpha) = D_{\alpha^{-1}}. \tag{2}$$

Let

$$W_{\alpha^{\pm 1}} = \langle C_\alpha, C_{\alpha^{-1}}, D_\alpha, D_{\alpha^{-1}} \rangle.$$

It follows from (1) that the subspaces $W_{\alpha^{\pm 1}}$ are symplectic and $\widehat{\varphi}$-orthogonal to each other. Moreover, they are all 4-dimensional except the 2-dimensional $W_{-1} = \langle C_{-1}, D_{-1} \rangle$ which appears when $n$ is even.

Using the above description we see that $r^2$-invariant Lagrangian subspaces of $W \otimes k$ are direct sums $L_{-1} \oplus \bigoplus L_{\alpha^{\pm 1}}$, where

$$L_{-1} \subset W_{-1},\ L_{\alpha^{\pm 1}} \subset W_{\alpha^{\pm 1}} \quad \text{and} \quad \dim L_{-1} = 1,\ \dim L_{\alpha^{\pm 1}} = 2.$$

Moreover, while $L_{-1}$ can be an arbitrary 1-dimensional subspace of $W_{-1}$, $L_{\alpha^{\pm 1}}$ has to be $r^2$-invariant and isotropic in $W_{\alpha^{\pm 1}}$. There are three possibilities for the latter to hold:

(a) $L_{\alpha^{\pm 1}} = \langle C_\alpha, D_\alpha \rangle$,
(b) $L_{\alpha^{\pm 1}} = \langle C_{\alpha^{-1}}, D_{\alpha^{-1}} \rangle$,
(c) $L_{\alpha^{\pm 1}} = \langle v, w \rangle$, where $v$ is a vector in $\langle C_\alpha, D_\alpha \rangle$ and $w$ is the unique (up to scaling) vector in $\langle C_{\alpha^{-1}}, D_{\alpha^{-1}} \rangle$ such that $\widehat{\varphi}(v, w) = 0$.

Lemma 4.5. *The vectors $w$ in* (c) *and $m_y(v)$ are proportional.*

*Proof.* One checks using (1) and (2) that $\widehat{\varphi}(v, m_y(v)) = 0$. ■

Proposition 4.6. *An $r^2$-invariant subspace of $W \otimes k$ is also $m_y$-invariant unless it contains a subspace of the form $\langle C_\alpha, D_\alpha \rangle$.*

The next step is to analyze which $r^2$-invariant Lagrangian subspaces of $W \otimes k$ arise as $r^2$-invariant Lagrangian subspaces of $W$ by extension of scalars.

As $r^2$ has no multiple eigenvalues in $W_+$ and $W_-$, both of these split into direct sums of $r^2$-irreducible subspaces:

$$W_\pm = \bigoplus_\phi W_\pm^\phi,$$

where $\phi$ runs through the factors of $\frac{x^n - 1}{x - 1}$ over $\mathbb{Z}_p$. Put $W^\phi = W_+^\phi \oplus W_-^\phi$. Every $r^2$-invariant Lagrangian subspace $L$ of $W$ decomposes as

$$L = \bigoplus_\phi (L \cap W^\phi).$$

Furthermore, each $L \cap W^\phi$ is either $\{0\}$, $W^\phi$, or the graph of an $r^2$-equivariant linear map $W_-^\phi \to W_+^\phi$ (including the degenerate case of $W_+^\phi$). Clearly, only when for some $\phi$, $L \cap W^\phi = W^\phi$, does the space $L \otimes k$ contain a $\langle C_\alpha, D_\alpha \rangle$. Now $W^\phi$ is isotropic if and only if for each pair $\alpha, \alpha^{-1} \in \mu_n$ at most one of $\alpha, \alpha^{-1}$ is a root of $\phi$.

LEMMA 4.7. *Consider the following condition:*

(3) *There exists $s$ such that* $p^s \equiv -1 \bmod n$.

(i) *If* (3) *holds, then for each $\alpha$ and some $\phi$,*
$$\phi(\alpha) = 0 \quad \textit{if and only if} \quad \phi(\alpha^{-1}) = 0$$

(ii) *If* (3) *does not hold, then there exist $\alpha$ and $\phi$ such that $\phi(\alpha) = 0$ while $\phi(\alpha^{-1}) \neq 0$.*

*Proof.* The Galois group $\mathrm{Gal}(k/\mathbb{Z}_p)$ is generated by the Frobenius automorphism $x \mapsto x^p$. If $\phi(\alpha) = 0$, then the roots of $\phi$ are exactly the conjugates of $\alpha$, i.e.
$$\alpha, \alpha^p, \alpha^{p^2}, \alpha^{p^3}, \ldots.$$

If (3) holds, this sequence contains $\alpha^{-1}$. If $\alpha$ is a primitive $n$th root of 1, (3) is also necessary for this sequence to contain $\alpha^{-1}$. This proves the lemma. ■

THEOREM 4.8.

(i) *If* (3) *holds, then all $r^2$-invariant Lagrangians in $W$ are $m_y$-invariant.*

(ii) *If* (3) *does not hold, then there are $r^2$-invariant Lagrangians in $W$ which are not $m_y$-invariant.*

*Proof.* For (i) we observe that (3) implies that no $W^\phi$ is isotropic. That is, $L \cap W^\phi$ has to be the graph of a map $W^\phi_- \to W^\phi_+$ (or $W^\phi_+$). Then $(L \cap W^\phi) \otimes k$ is the graph of $W^\phi_- \otimes k \to W^\phi_+ \otimes k$ (or $W^\phi_+ \otimes k$), and does not contain the subspace $\langle C_\alpha, D_\alpha \rangle$.

For (ii), we pick $\phi$ as in Lemma 4.7(ii). Among all the other irreducible factors of $\frac{x^n - 1}{x - 1}$ exactly one, call it $\phi'$, has as its roots the inverses of the roots of $\phi$. Put
$$L = W^\phi \oplus \bigoplus_{\psi \neq \phi, \phi'} W^\psi_+.$$
We see that $L$ is isotropic and
$$m_y(L) = W^{\phi'} \oplus \bigoplus_{\psi \neq \phi, \phi'} W^\psi_+. \ ■$$

For $n \leq 17$ we list of all pairs $(n, p \bmod n)$ that satisfy (3):

- $(3,2)$, $(4,3)$, $(5,2)$, $(5,3)$, $(5,4)$, $(6,5)$, $(7,3)$, $(7,5)$, $(7,6)$, $(8,7)$, $(9,2)$, $(9,5)$, $(9,8)$, $(10,3)$, $(10,7)$, $(10,9)$, $(11,2)$, $(11,6)$, $(11,7)$, $(11,8)$, $(11,10)$, $(12,11)$, $(13,2)$, $(13,4)$, $(13,5)$, $(13,6)$, $(13,7)$, $(13,8)$, $(13,10)$, $(13,11)$, $(13,12)$, $(14,3)$, $(14,5)$, $(14,13)$, $(15,14)$, $(16,15)$, $(17,2)$, $(17,3)$, $(17,4)$, $(17,5)$, $(17,6)$, $(17,7)$, $(17,8)$, $(17,9)$, $(17,10)$, $(17,11)$, $(17,12)$, $(17,13)$, $(17,14)$, $(17,15)$, $(17,16)$.

COROLLARY 4.9. *If $(n, p)$ satisfies* (3) (*e.g. for the pairs above*), *then the number of $p$-colorings is the same for a link and its rotant.*

For instance, when $\gcd(p-1, n) > 2$ condition (3) does not hold, i.e. $x^n - 1$ has a root $\alpha$ in $\mathbb{Z}_p$ different from 1 and $-1$. We will show that in such a case there exist rotors whose Lagrangians are not $m_y$-invariant. In particular, the number of $p$-colorings can be different for a link and its rotant. Consider a rotor $R$ with fundamental domain shown in Figure 8.

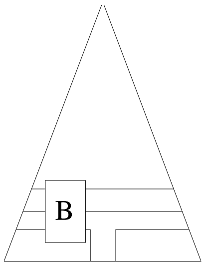



Fig. 8. Rotor $R$ with 3-tangle $B$

We find a 3-tangle $B$ such that

$$\widehat{\psi}(\mathrm{Col}_p(R)) \otimes k \supseteq \langle C_\alpha, D_\alpha \rangle,$$

i.e. each $k$-coloring $v$ of $2n$ boundary points of $R$ satisfying $r^2(v) = \alpha v$ is induced by the Fox $p$-coloring of $R$. The following condition on $B$ ensures that $R$ has the above property (as one can see using $r^2$-equivariant colorings): *For all $a, b \in \mathbb{Z}_p$, there are $c, d \in k$ such that the $k$-coloring of the six boundary points shown in Figure* 9 *is induced by a Fox $p$-coloring of $B$.*

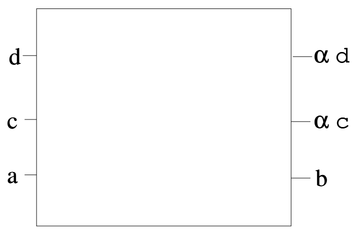



Fig. 9. Condition for boundary coloring

We observe that it is sufficient to verify the above condition for two $k$-linearly independent pairs $(a, b)$. Thus, for instance, we take $(1, 1)$ and $(\alpha, 1)$. For each such pair, we find $(c, d)$ and then check that the resulting boundary $k$-colorings are $\widehat{\varphi}$-orthogonal (in $W_4$). For $(1, 1)$ we take $c = d = \frac{1}{\alpha+1}$ and for $(\alpha, 1)$ let $c = 0$ and $d = 1$. The resulting colorings are represented in $W_4$

by the following vectors:

$$v_1 = \frac{1}{\alpha+1} f_1 + f_3 + \frac{\alpha}{\alpha+1} f_5,$$
$$v_2 = \alpha f_3 + (1-\alpha) f_4 + (\alpha - 1) f_5 + f_6.$$

It is straightforward to check that these are $\widehat{\varphi}$-orthogonal. It follows from Theorem 2.5 that the subspace of $W_4$ spanned by $v_1, v_2$ is induced by some 3-tangle $B$.

REMARK 4.10. All rotors constructed above have non-trivial $p$-colorings (corresponding to $v_1$) equal to 1 on the boundary and non-constant in the interior. After subtracting from this coloring the constant coloring of $B$ (with value 1), one gets a non-trivial Fox $p$-coloring of $B$ which can be extended $r^2$-equivariantly to a $p$-coloring of $R$ with zero values on the boundary.

Two such cases are discussed in the next example.

EXAMPLE 4.11. As one can verify, for $n = 4$, $p = 5$, we can take the braid $B = \sigma_2^{-2}\sigma_1\sigma_2^{-1}$ and $\alpha = 2$ ($2^4 - 1 = 15 \equiv 0 \bmod 5$). For $n = 3$, $p = 7$ we take the braid $B = \sigma_2^{-1}\sigma_1^3\sigma_2^2\sigma_1^{-1}$ and $\alpha = 2$ ($2^3 - 1 = 7 \equiv 0 \bmod 7$).

REMARK 4.12. In Example 4.11, $\widehat{\psi}(\mathrm{Col}_5(R))$ and $\widehat{\psi}(\mathrm{Col}_5(m_y(R)))$ intersect along their $L_{-1}$-part, so that if we use $R$ as both the rotor and the stator then

$$\log_5 |\mathrm{Col}_5(R \cup R)| = 2 + \log_5 |\mathrm{Col}_5(R \cup m_y(R))|.$$

Let us now consider the case $n = p > 2$. As before, we split $W = W_{2n-2}$ into a direct sum $W = W_+ \oplus W_-$, where

$$W_+ = \langle f_0, f_2, \ldots, f_{2n-2} \rangle \quad \text{and} \quad W_- = \langle f_1, f_3, \ldots, f_{2n-1} \rangle.$$

For $j = 1, \ldots, p-1$ put

$$w_{p-j} = \sum_{k=0}^{j-1} (-1)^{j+k-1} \binom{j-1}{k} f_{2k}, \qquad v_{p-j} = \sum_{k=0}^{j-1} (-1)^{j+k-1} \binom{j-1}{k} f_{2k+1}. \tag{4}$$

The vectors $w_{p-j}$ $[v_{p-j}]$ form a basis of $W_+$ $[W_-]$. In these bases, the action of $r^2$ is

$$\begin{aligned} r^2 w_i &= w_i + w_{i-1}, \qquad & r^2 v_i &= v_i + v_{i-1}, \\ r^2 w_1 &= 0, & r^2 v_1 &= 0. \end{aligned} \tag{5}$$

One checks directly that

$$\widehat{\varphi}(w_{p-i}, v_{p-j}) = (-1)^{i+j} \binom{i+j-1}{j}, \tag{6}$$

i.e.

$$\widehat{\varphi}(w_i, v_j) = (-1)^{i+j}\binom{2p-i-j-1}{p-j}, \tag{7}$$

where the binomial coefficient should be read modulo $p$. In particular $\widehat{\varphi}(w_i, v_j) = 0$ when $2p-i-j-1 \geq p$, i.e. when $i+j \leq p-1$.

COROLLARY 4.13. *Let $\Lambda_s = \langle w_1, v_1, \ldots, w_s, v_s\rangle$. Then $\Lambda_s$ is an isotropic subspace of $W$ if and only if $s \leq (p-1)/2$.*

We want to describe the $r^2$-invariant Lagrangian subspaces $L$ of $W$. Let $s$ be the largest number such that $\Lambda_s \subset L$ (it may well be 0) and put $t = p-s-1$.

LEMMA 4.14. $L \subset \Lambda_t$.

*Proof.* Suppose $L \not\subseteq \Lambda_t$. Then we can find in $L$ a vector

$$u = a_k w_k + a_{k-1}w_{k-1} + \cdots + a_1 w_1 + b_{p-1}v_{p-1} + b_{p-2}v_{p-2} + \cdots + b_1 v_1$$

with $k > t$, $a_k \neq 0$ (or a vector of this form with $w$ and $v$ switched) that is not in $\Lambda_t$. Then $\varphi((r^2-1)^{k-t-1}u, v_s) \neq 0$. ∎

LEMMA 4.15. *$L$ contains a unique vector of the form*

$$u_t = w_t + \sum_{i=s+1}^{t} a_i v_i$$

(*or one with $w$ and $v$ switched*), *and then it has the following basis*: $w_1, \ldots, w_s$, $v_1, \ldots, v_s, u_{s+1}, \ldots, u_t$ *where*

$$u_k = w_k + \sum_{i=s+1}^{k} a_{i+t-k}v_i. \tag{8}$$

*Proof.* If $u \in L \cap \Lambda_k$ is non-zero in $\Lambda_k/\Lambda_{k-1}$, then so is $(r^2-1)^l u$ in $\Lambda_{k-l}/\Lambda_{k-l-1}$. Therefore, $(L\cap\Lambda_k)/\Lambda_{k-1}$ is 1-dimensional for $s+1 \leq k \leq t$ (it is 0-dimensional for $k > t$, 2-dimensional for $k \leq s$, while $L$ has dimension $t+s$ and does not contain $\Lambda_{s+1}$).

Pick a vector $u \in L$ that spans $L/\Lambda_{t-1}$. We may assume that $w_t$ (or $v_t$) appears in $u$ with coefficient 1. If $c_{t-l}w_{t-l}$ appears in $u$, it can be canceled by subtracting $c_{t-l}(r^2-1)^l u$. We can also remove $w_i$ and $v_i$ with $i \leq s$, as these are in $\Lambda_s \subset L$. After these operations, we get the vector $u_t$ as in (8). Now $u_k$ equals $(r^2-1)^{t-k}u_t$ (modulo $\Lambda_s$), thus it belongs to $L$. Thus all the vectors $w_1, \ldots, w_s, v_1, \ldots, v_s, u_{s+1}, \ldots, u_t$ are in $L$, and are clearly linearly independent. Since $\dim L = t+s$, these vectors form a basis of $L$. ∎

LEMMA 4.16. $a_t = 0$.

*Proof.* We see that

$$\begin{aligned} 0 = \varphi(u_t, u_{s+1}) &= \varphi(w_t, a_t v_{s+1}) + \varphi(a_t v_t, w_{s+1}) \\ &= -a_t\left(\binom{p-1}{t} - \binom{p-1}{p-t}\right) = 2(-1)^{t+1}a_t. \ \blacksquare \end{aligned}$$

Every $r^2$-invariant Lagrangian subspace $L$ of $W$ is therefore determined by choosing $s, a_{s+1}, \ldots, a_{t-1}$, where $t = p - s - 1$. However, not every choice of these parameters gives rise to an isotropic subspace of $W$—there are extra conditions $\varphi(u_u, u_k) = 0$, which can be written as

$$(9)_{u,k} \qquad \sum_{i=p+t-u-k}^{t-1} a_i(-1)^{u+k+i-t} \times \left(\binom{2p-u-k+t-i-1}{p-k-i+t} - \binom{2p-u-k+t-i-1}{p-k-1}\right) = 0.$$

The above has to hold for each pair $u$, $k$ such that $t \geq u > k \geq s+2$, $u + k > p$.

LEMMA 4.17. $(-1)^k\binom{n}{k} = \binom{p-1+k-n}{k} \bmod p$.

*Proof.* We have

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!},$$

$$\binom{p-1+k-n}{k} = \frac{(p-n+k-1)\ldots(p-n+1)(p-n)}{k!}.$$

Now $p - n \equiv -n \bmod p$, $p - n + 1 \equiv -(n-1) \bmod p$, etc. $\blacksquare$

Let us now turn to the invariance of $L$ under $m_y$. First, using (4), (3) and Lemma 4.17 one gets

$$m_y(w_j) = (-1)^j \sum_{i=0}^{j} \binom{j+1}{i+1} w_i, \qquad m_y(v_j) = (-1)^{j+1} \sum_{i=0}^{j} \binom{j}{i} v_i.$$

Therefore (calculating modulo $m_y$-invariant $\Lambda_s$)

$$\begin{aligned} m_y(u_u) &= m_y\Big(w_u + \sum_{i=s+1}^{u-1} a_{t+i-u} v_i\Big) \\ &= (-1)^u \sum_{j=0}^{u} \binom{u+1}{j+1} w_j + \sum_{i=s+1}^{u-1} a_{t+i-u}(-1)^{i+1} \sum_{l=s+1}^{i} \binom{i}{l} v_l \\ &= (-1)^u \sum_{j=0}^{u} \binom{u+1}{j+1} w_j + \sum_{l=s+1}^{u-1} \Big(\sum_{i=l}^{u-1} a_{t+i-u}(-1)^{i+1}\binom{i}{l}\Big) v_l. \end{aligned}$$

Note that $m_y(u_u) \in L$ if $m_y(u_u)$ is a linear combination of $u_j$ with coefficients matching those of $w_j$ in the above sum. Comparing the $v_l$ terms (in the above sum) gives

$$(10)_{u,l} \qquad \sum_{i=t+l-u}^{t-1} a_i(-1)^u\left(\binom{u+1}{t+l+1-i} + (-1)^{i-t}\binom{i+u-t}{l}\right) = 0.$$

To ensure $m_y$-invariance of $L$, this formula should hold for all pairs $u$, $l$ such that $t \geq u > l \geq s+1$.

THEOREM 4.18. *If $n = p > 2$ then each $r^2$-invariant Lagrangian subspace of $W$ is also $m_y$-invariant.*

*Proof.* We will prove that equations (9) imply (10).

Applying Lemma 4.17 to (10) we find that $(10)_{u,l}$ is equivalent to $(11)_{u+1,l}$, where

$$(11)_{u,l}$$
$$\sum_{i=t+l-u+1}^{t-1} a_i(-1)^{t+l-u+i}\left(\binom{p+t+l-u-i}{l+t+1-i} - \binom{p+t+l-u-i}{l}\right) = 0.$$

So, we want to prove $(11)_{u+1,l}$ for all $u$, $l$ such that $t \geq u > l \geq s+1$. In other words, we want $(11)_{v,l}$ for all $v$, $l$ such that $t+1 \geq v \geq s+3$, $v-2 \geq l \geq s+1$.

On the other hand, $(9)_{u,k}$ is equivalent to $(11)_{u,l}$ with $l = p-k-1$, so $(11)_{x,p-y-1}$ for $x$, $y$ such that $t \geq x > y \geq s+2$, $x+y > p$, in other words, $(11)_{x,z}$ for $x$, $z$ such that $t \geq x \geq s+3$, $x-2 \geq z \geq p-x-2$. This directly covers part of the equations that we want to have. There are two cases left:

(a) $p - v - 2 > l \geq s+1$,
(b) $v = t+1$.

If (a) holds, we notice that $(11)_{v,l} \Leftrightarrow (11)_{p-l-1,p-v-1}$, and the latter is on our list of assumptions.

If (b) is true, we first notice that $(11)_{t+1,s+1}$ is a linear combination of $(11)_{t,s+1}$ and $(11)_{t,s}$. But $(11)_{t,s}$ has zero coefficients since $\binom{n}{k} = \binom{n}{n-k}$. If $l > s+1$, then $(11)_{t+1,l}$ is a linear combination of $(11)_{t,l-1}$ and $(11)_{t,l}$. ∎

**5. Symplectic form on $t$-colorings of tangles.** In this section, we consider a generalization of the symplectic structure on Fox $p$-colorings that we studied in previous sections to *Alexander–Burau–Fox $t$-colorings* (briefly *ABF-colorings*) and a related $t$-invariant symplectic structure. Here we only sketch an idea which we plan to develop in the future. Consider a 2-dimensional disk with $2n$ boundary points $p_1, \ldots, p_{2n}$ ($n$ inputs and $n$ outputs, see Figure 1). For a fixed commutative ring $R$ with identity we consider the free $R[t^{\pm 1}]$-module $(R[t^{\pm 1}])^{2n}$ with a basis $e_1, \ldots, e_{2n}$ and we view elements of this module as colorings of boundary points by elements of $R[t^{\pm 1}]$. Consider the submodule $W_{2n-1}$ of $R[t^{\pm 1}]^{2n}$ generated by all elements of the form $e_i - ue_{i+1}$, where $u$ equals $-1, t$, or $t^{-1}$ depending on whether $p_i$, $p_{i+1}$ are inputs or outputs. On the $R[t^{\pm 1}]$-module $W_{2n-1}$ regarded as an $R$-module, we define a $t$-invariant alternating form. The quotient of $W_{2n-1}$ by trivial (constant) colorings is a symplectic $R$-module. We show that $R[t^{\pm 1}]$-colorings of an oriented $n$-tangle $T$ which satisfy the Alexander–Burau relation at each crossing yield a Lagrangian in our symplectic $R$-module. We discuss a relation between $t$-colorings and homologies of the universal cyclic coverings and even-fold cyclic branched coverings. We note that our considerations in the previous sections correspond to the case when $t = -1$ and the double branched covering.

Let $R$ be a commutative ring with identity and assume that the ring $\mathbb{Z}[t^{\pm 1}]$ acts on $R$. Thus $R$ is also a $\mathbb{Z}[t^{\pm 1}]$-module. An *ABF-coloring* of an oriented tangle $T$ is an assignment of elements of $R$ to arcs of $T$, such that at each crossing the *ABF relation*:

$$(1-t)a + tb - c = 0$$

(or equivalently, $(1-t^{-1})a + t^{-1}c - b = 0$); see Figure 10. We are mostly interested in the cases when $R$ is $\mathbb{Z}[t^{\pm 1}]$, $\mathbb{Z}[t^{\pm 1}]/\left(\frac{t^{2k}-1}{t-1}\right)$ or $\mathbb{Z}_p[t^{\pm 1}]/(q(t))$, where $p$ is a prime number and $q(t)$ an irreducible polynomial. For a boundary point we put $\varepsilon_i = 1$ if the point is an output and $\varepsilon_i = -1$ for an input. Define the *$t$-alternating condition* for an element $\sum_{i=1}^{2n} a_i e_i \in R^{2n}$ to be $\sum_{i=1}^{2n} c_i a_i = 0$, where $c_{i+1} = \varepsilon_i \varepsilon_{i+1} t^{(\varepsilon_i + \varepsilon_{i+1})/2} c_i$.

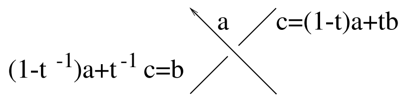



Fig. 10. *ABF*-coloring relation

Let $\mathrm{Col}_{R,t}(T)$ be the $R$-module of $ABF$-colorings of a tangle $T$. There is a map

$$\psi : \mathrm{Col}_{R,t}(T) \to R^{2n}$$

which assigns to an $ABF$-coloring of $T$ a coloring of its $2n$ boundary points.

LEMMA 5.1. *Elements of $\psi(\mathrm{Col}_{R,t}(T))$ satisfy the $t$-alternating condition.*

*Proof.* The proof proceeds in the same manner as in the case $t = -1$. The most important observation is that the $t$-alternating condition holds locally. In fact for a crossing we can write the $ABF$-relation in the form $a + tb - ta - c = 0$. ∎

As in the case $t = -1$ we can find a basis for the subspace satisfying the $t$-alternating condition. Let $W_{2n-1} = W_{2n-1}(\varepsilon_1, \ldots, \varepsilon_{2n})$ denote this $R$-module.

LEMMA 5.2. *$W_{2n-1}$ is a free $R$-module of rank $2n-1$ with a basis $\{f_1,\dots,f_{2n-1}\}$, where $f_i = e_i + u_i e_{i+1}$ and $u_i$ satisfies $c_{i+1} = -u_i^{-1} c_i$, i.e.*

$$u_i = -\varepsilon_i \varepsilon_{i+1} t^{-(\varepsilon_i+\varepsilon_{i+1})/2}.$$

Our plan is to define a $t$-invariant alternating form of nullity one on $W_{2n-1}$ that, as before, satisfies the condition

$$\varphi(f_i, f_j) = \begin{cases} 0 & \text{if } |j-i| \neq 1, \\ j-i & \text{otherwise,} \end{cases}$$

and additionally

$$\varphi(-tf_i, f_j) = \varphi(f_i, f_j).$$

Of course, we did this before for $t=-1$, but to achieve it in a more general setting ([17]), we need to regard $R$ as a ring with a subring $R_0$. Let $j : R_0 \to R$ be a homomorphism of rings and $\varphi$ be defined over $R_0$. We show how it works in one important case:

EXAMPLE 5.3. Let $I$ be an ideal of $R_0[t^{\pm1}]$ such that $(t+1) \in I$ and $R = R_0[t^{\pm1}]/I$. Then the $t$-invariant alternating form $\varphi$ with nullity one described above is well defined.

THEOREM 5.4. *If $T$ is a rational tangle then $\psi(\mathrm{Col}_{R,t}(T))$ is a Lagrangian of nullity one in $W_{2n-1}$.*

*Proof.* We proceed as in the proof of Theorem 2.1. We observe that the homomorphism corresponding to a crossing is $(t+1)$ approximation to a transvection, so $\varphi$ is an isometry. ∎

**6. History of the paper.** In May of 2000 while J. H. Przytycki visited Banach Center in Warsaw he attended T. Januszkiewicz's talk on Coxeter groups and buildings. After the talk Przytycki asked a question concerning his student M. Dąbkowski's calculations of the number of Fox 3-colorings induced on the boundary by 2-, 3- and 4-tangles. As Przytycki told Januszkiewicz, his student got the numbers 4, 40 and 1120. Januszkiewicz immediately noticed that these numbers suggested the number of Lagrangians in symplectic spaces over $\mathbb{Z}_3$. Soon after, J. Dymara and Przytycki found the appropriate symplectic form on the space of $p$-colorings of $2n$ boundary points on a circle. It was the beginning of our collaboration. The first version of this paper was ready in 2001, but its publication was delayed ([18]). Przytycki gave several talks related to this paper, starting from [Pr-6]. Meanwhile

([17]) The general setting is a future project.

([18]) There was also a mathematical reason for this. In early winter of 2002 Dąbkowski and Przytycki solved the Montesinos–Nakanishi conjecture using the nonabelian version of Fox $n$-colorings called Burnside groups of links [DP-1, DP-2].

some parts of this work were cited in [Pr-3, Pr-4, Pr-5] and the approach described in Section 5 was developed by Cimasoni and Turaev [C-T-1, C-T-2].

**Acknowledgments.** The first two authors were partially supported by Polish grant 5 P03A 035 20. The third author was partially supported by the National Science Foundation; Grant No. DMS-9808955, and by the Simons grant (award number: 637794). He also acknowledges discussions with M. Dąbkowski, A. Sikora, and D. Silver.

## References


[A-P-R] R. P. Anstee, J. H. Przytycki and D. Rolfsen, *Knot polynomials and generalized mutation*, Topology Appl. 32 (1989), 237–249.

[A] J. Assion, *Einige endliche Faktorgruppen der Zopfgruppen*, Math. Z. 163 (1978), 291–302.

[B-H] J. S. Birman and H. M. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, in: Advances in the Theory of Riemann Surfaces, Princeton Univ. Press, Ann. of Math. Stud. 66, Princeton, NJ, 1972, 81–115.

[Brown] K. S. Brown, *Buildings*, Springer, New York, 1989.

[C-T-1] D. Cimasoni and V. Turaev, *A Lagrangian representation of tangles*, Topology 44 (2005), 747–767.

[C-T-2] D. Cimasoni and V. Turaev, *A Lagrangian representation of tangles. II*, Fund. Math. 190 (2006), 11–27.

[DIPY] M. Dąbkowski, M. Ishiwata, J. H. Przytycki and A. Yasuhara, *Signature of rotors*, Fund. Math. 184 (2004), 79–97.

[DP-1] M. K. Dąbkowski and J. H. Przytycki, *Burnside obstructions to the Montesinos–Nakanishi 3-move conjecture*, Geom. Topol. 6 (2002), 355–360.

[DP-2] M. K. Dąbkowski and J. H. Przytycki, *Unexpected connection between Burnside groups and knot theory*, Proc. Nat. Acad. Sci. USA 101 (2004), 17357–17360.

[Ga] P. Garrett, *Buildings and Classical Groups*, Chapman & Hall, London, 1997.

[Hu] S. P. Humphries, *Generators for the mapping class group*, in: Topology of Low-Dimensional Manifolds (Chetwood Gate, 1977), R. Fenn (ed.), Lecture Notes in Math. 722, Springer, 1979, 44–47.

[Lic] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) 76 (1962), 531–540.

[O] O. T. O'Meara, *Symplectic Groups*, Math. Surveys 16, Amer. Math. Soc., Providence, RI, 1978.

[Pla] A. Plans, *Contribution to the study of the homology groups of the cyclic ramified coverings corresponding to a knot*, Rev. Acad. Ciencias Madrid 47 (1953), 161–193 (in Spanish).

[Pr-1] J. H. Przytycki, *3-coloring and other elementary invariants of knots*, in: Knot Theory, Banach Center Publ. 42, Inst. Math., Polish Acad. Sci., Warszawa, 1998, 275–295.

[Pr-2] J. H. Przytycki, *Search for different links with the same Jones' type polynomials: Ideas from graph theory and statistical mechanics*, in: Panoramas of Mathematics, Banach Center Publ. 34, Inst. Math., Polish Acad. Sci., Warszawa, 1995, 121–148.

[Pr-3] J. H. Przytycki, *Skein module deformations of elementary moves on links*, in: Invariants of Knots and 3-Manifolds (Kyoto, 2001), Geom. Topol. Monogr. 4 (2002), 313–335.

[Pr-4] J. H. Przytycki, *From 3-moves to Lagrangian tangles and cubic skein modules*, in: Advances in Topological Quantum Field Theory (Kananaskis Village, 2001), Kluwer, 2004, 71–125.

[Pr-5] J. H. Przytycki, *The Trieste look at knot theory*, in: Introductory Lectures on Knot Theory, Ser. Knots and Everything 46, World Sci., 2012, 407–441.

[Pr-6] J. H. Przytycki, *Symplectic structure on colorings and Lagrangian tangles*, Abstracts Amer. Math. Soc. 21 (2000), 545.

[Ron] M. Ronan, *Lectures on Buildings*, Perspectives in Math. 7, Academic Press, Boston, MA, 1989.

[Tra] P. Traczyk, *Conway polynomial and oriented rotant links*, Geom. Dedicata 110 (2005), 49–61.

[Tur] V. G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, de Gruyter Stud. Math. 18, de Gruyter, Berlin, 1994.

[Wa-1] B. Wajnryb, *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. 45 (1983), 157–174.

[Wa-2] B. Wajnryb, *Markov classes in certain finite symplectic representations of braid groups*, in: Braids (Santa Cruz, CA, 1986), Contemp. Math. 78, Amer. Math. Soc., Providence, RI, 1988, 687–695.

[Wa-3] B. Wajnryb, *A braidlike presentation of* $\mathrm{Sp}(n,p)$, Israel J. Math. 76 (1991), 265–288.

Jan Dymara
Instytut Matematyczny
Uniwersytet Wrocławski
Wrocław, Poland
ORCID: 0000-0002-7127-3298
E-mail: Jan.Dymara@math.uni.wroc.pl

Józef H. Przytycki
Department of Mathematics
The George Washington University
Washington, DC, U.S.A.
and
Institute of Mathematics
University of Gdańsk
Gdańsk, Poland
ORCID: 0000-0002-1600-8889
E-mail: przytyck@gwu.edu

Tadeusz Januszkiewicz
Instytut Matematyczny PAN
Warszawa, Poland
and
Instytut Matematyczny
Uniwersytet Wrocławski
Wrocław, Poland
ORCID: 0000-0002-6570-0281
E-mail: T.Januszkiewicz@impan.pl