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Minimal geodesics of pencils of pairs of projections

by

Miaomiao Cui (Xi'an)

Abstract. The set $\mathcal{P}_{T,\lambda}$ of pairs (P,Q) of orthogonal projections with pencil $T = \lambda P + Q$ at any given $\lambda \in \mathbb{R} \setminus \{-1,0\}$ is shown to be an analytic Banach homogeneous space. In generic position, $\mathcal{P}_{T,\lambda}$ with a natural connection and the quotient Finsler metric of the operator norm becomes a classical Riemannian space in which any two pairs are joined by a minimal geodesic. Moreover, given a pair $(P,Q) \in \mathcal{P}_{T,\lambda}$, pairs in an open dense subset of $\mathcal{P}_{T,\lambda}$ can be joined to (P,Q) by a unique minimal geodesic. In general, two pairs $(P_0,Q_0),(P,Q)$ in $\mathcal{P}_{T,\lambda}$ can be joined by a minimal geodesic in $\mathcal{P}_{T,\lambda}$ of length $\le \pi/2$ if and only if

$$\begin{cases} \dim[R(P|_{N(T-I)}) \cap N(P_0|_{N(T-I)})] \\ \quad = \dim[N(P|_{N(T-I)}) \cap R(P_0|_{N(T-I)})], \quad \lambda = 1, \\ \dim N(T-\lambda I) = \dim N(T-I), \quad \lambda \in \mathbb{R} \setminus \{0,-1,1\}. \end{cases}$$

1. Introduction. Self-adjoint operator pencils arise in quantum mechanics and in dynamical problems. Pencils of orthogonal projections have attracted much attention [3, 1, 2, 8, 14, 15, 20].

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators in $\mathcal{H}$. Denote by $\mathcal{B}(\mathcal{H})_{\mathrm{s}}$, $\mathcal{U}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ the set of self-adjoint operators, the set of unitary operators, and the set of orthogonal projections in $\mathcal{B}(\mathcal{H})$ respectively. Throughout the article, an orthogonal projection is referred to as a *projection* for short. A self-adjoint unitary is called a *symmetry*. The nullspace and range of $T \in \mathcal{B}(\mathcal{H})$ are denoted by $N(T)$ and $R(T)$ respectively.

In order to find a complete set of unitary invariants for a pair of closed subspaces M and N, Dixmier [10] obtained a characterization of the set of differences of pairs of projections, $\mathcal{D} = \{P-Q : P,Q \in \mathcal{P}(\mathcal{H})\}$. Subsequently, Raeburn and Sinclair [19] showed that there exists a unitary operator U such

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that $UPU^* = P_1$ and $UQU^* = Q_1$ if and only if the operator $\lambda P + Q$ is unitarily equivalent to $\lambda P_1 + Q_1$ for all $\lambda \in (0, +\infty) \setminus \{1\}$. Later, Cui and Ji [8] extended that result to $\lambda \in \mathbb{R} \setminus \{-1, 0\}$, and discussed some relevant algebraic structures.

On the other hand, more and more scholars began to focus on the algebraic and geometric characteristics of sets associated with pencils of pairs of projections. Koliha and Rakočević [14] considered the Fredholm properties of pairs of projections. Andruchow [1] discussed operators which are the difference of two projections. Afterwards, Shi, Ji and Du [20] provided a complete description, and studied the algebraic structure of $\mathcal{D}$. Later, Andruchow, Corach and Recht [3] further investigated the geometric features and geodesics of $\mathcal{D}$. In addition, Cui and Ji [8] examined the class $\mathcal{P}_\lambda$ in $\mathcal{B}(\mathcal{H})$ which consists of the pencils of pairs of projections at $\lambda \in \mathbb{R}$:

$$\mathcal{P}_\lambda = \{\lambda P + Q : P, Q \in \mathcal{P}(\mathcal{H})\},$$

and explored the algebraic and geometric structures of the set

$$\mathcal{P}_{T,\lambda} = \{(P, Q) \in \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) : T = \lambda P + Q\}$$

for $T \in \mathcal{P}_\lambda$. In this paper, we study the reductive homogeneous structure and minimal geodesics of $\mathcal{P}_{T,\lambda}$.

Pencils of self-adjoint operators are defined in [15]. The operator $\lambda P + Q$ is called the *pencil* of the pair (P, Q) of projections at $\lambda \in \mathbb{R}$ [8]. Assume that T is a self-adjoint operator and $\lambda \in \mathbb{R}$. Then $N(T)$, $N(T - I)$, $N(T - \lambda I)$, $N(T - (1 + \lambda)I)$ and the complement $\mathcal{H}_0$ of the sum of these all reduce T. According to the space decomposition

$$\mathcal{H} = N(T - (1 + \lambda)I) \oplus N(T) \oplus N(T - \lambda I) \oplus N(T - I) \oplus \mathcal{H}_0, \tag{1.1}$$

we have

$$T = (1 + \lambda)I \oplus 0 \oplus \lambda I \oplus I \oplus T_0, \tag{1.2}$$

where $T_0 = T|_{\mathcal{H}_0}$. We call T_0 the *generic part* of T with respect to λ. If

$$N(T) = N(T - I) = N(T - \lambda I) = N(T - (1 + \lambda)I) = \{0\},$$

we say that T is *in generic position with respect to* λ.

If T is the pencil of a pair (P, Q) of projections at some real number λ, then

$$\begin{cases} N(T) = N(P) \cap N(Q), \quad N(T - I) = N(P) \cap R(Q), \\ N(T - \lambda I) = R(P) \cap N(Q), \quad N(T - (1 + \lambda)I) = R(P) \cap R(Q). \end{cases} \tag{1.3}$$

In this case, it is known that T is in generic position with respect to λ if and only if the pair (P, Q) of projections is in generic position in the sense of [12]. Moreover, there exists an isometric isomorphism between the generic part $\mathcal{H}_0$ and a product space $\mathcal{K} \times \mathcal{K}$ which carries $P|_{\mathcal{H}_0}$ and $Q|_{\mathcal{H}_0}$ to the operator matrices (called Halmos decomposition)

$$P|_{\mathcal{H}_0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q|_{\mathcal{H}_0} = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

respectively. Without loss of generality, we can assume

$$\mathcal{H}_0 = \mathcal{K} \oplus \mathcal{K}. \tag{1.4}$$

The content of the paper is as follows. In Section 2, we establish a bijection between $\mathcal{P}_{T,\lambda}$ and $E = \{W \in \mathcal{B}(\mathcal{H}) : W$ is a symmetry and $WT_\lambda = -T_\lambda W\}$ in generic position. Based on this, we show that the action of the Banach–Lie group $\mathcal{U}_{\mathcal{A}}$ on $\mathcal{P}_{T,\lambda}$ is locally transitive, which makes $\mathcal{P}_{T,\lambda}$ an analytic Banach homogeneous space of $\mathcal{A}$. In Section 3, we mainly consider the reductive structure of $\mathcal{P}_{T,\lambda}$ in generic position, and establish a bijection

$$\exp_{(P,Q)} : \{X \in H_1^{(P,Q)} : \|X\| < \pi/2\} \to \exp_{(P,Q)}(\{X \in H_1^{(P,Q)} : \|X\| < \pi/2\})$$

by use of symmetries anti-commuting with T_λ in order to compute geodesics. In Section 4, we endow $\mathcal{P}_{T,\lambda}$ with the quotient Finsler metric of the operator norm in generic position, and prove that any two pairs in $\mathcal{P}_{T,\lambda}$ can be joined by a minimal geodesic. Moreover, pairs in an open dense subset in $\mathcal{P}_{T,\lambda}$ of a given pair $(P,Q) \in \mathcal{P}_{T,\lambda}$ can be joined to (P,Q) by a unique minimal geodesic. Finally, we also provide a characterization of when two pairs in $\mathcal{P}_{T,\lambda}$ can be joined by a minimal geodesic in $\mathcal{P}_{T,\lambda}$ of length $\leq \pi/2$ in the general case.

2. Geometric structure of pencils of pairs of projections. Let $T \in \mathcal{B}(\mathcal{H})_{\mathrm{s}}$ and $\lambda \in \mathbb{R}$. We write T_λ for $T - \frac{1+\lambda}{2}I$. Define a mapping

$$f : \mathcal{P}_{T,\lambda} \to \mathcal{B}(\mathcal{H})_{\mathrm{s}}$$

by

$$f(P,Q) = \begin{cases} -\lambda P + Q - \frac{1-\lambda}{2}\big[1 + \frac{1+\lambda}{2}T_\lambda^{+}\big], & \lambda \in \Lambda_1 = \mathbb{R} \setminus \{0,-1,1\}, \\ I - P + Q, \quad \lambda = 1, \end{cases} \tag{2.1}$$

and a mapping

$$S : \mathcal{P}_{T,\lambda} \to E := \{W \in \mathcal{B}(\mathcal{H}) : W \text{ is a self-adjoint partial isometry and } WT_\lambda = -T_\lambda W\}$$

by

$$S(P,Q) = \operatorname{sgn}(f(P,Q)), \tag{2.2}$$

where T_λ^+ stands for the Moore–Penrose inverse of T_λ and $\operatorname{sgn}(\cdot)$ denotes the sign function. Clearly,

$$
\tag{2.3}
\begin{cases}
T = P + Q, & T_\lambda = P + Q - 1, & f(P,Q) = Q - P, & \lambda = 1,\\
T = Q, & T_\lambda = Q - \frac{1}{2}, & f(P,Q) = \frac{3}{4}Q - \frac{1}{2}, & \lambda = 0,\\
T = Q - P, & T_\lambda = Q - P, & f(P,Q) = P + Q - 1, & \lambda = -1.
\end{cases}
$$

$f(P,Q)$ is called *Davis' characteristic* [9] of T when $\lambda = -1$. Since the cases of $\lambda = 0$ [17] and $\lambda = -1$ [3, 1] have already been discussed, we shall study the cases of $\mathbb{R} \setminus \{0, -1\}$ below.

REMARK 2.1. Let $T \in \mathcal{P}_\lambda$. For any pair $(P,Q) \in \mathcal{P}_{T,\lambda}$, we know from [8, Theorem 2.1] that there exists a positive operator B satisfying $\frac{|1-|\lambda||}{2} I < B < \frac{1+|\lambda|}{2} I$ such that

$$
\mathcal{P}_{T,\lambda} = \big\{ (P_{\lambda,U}, Q_{\lambda,U}) : T = \lambda P_{\lambda,U} + Q_{\lambda,U},\ U \in \{B\}' \big\},
$$

where

$$
\tag{2.4}
\begin{cases}
P_{\lambda,U} = I \oplus 0 \oplus E \oplus P_U, & Q_{\lambda,U} = I \oplus 0 \oplus (I - E) \oplus Q_U, & \lambda = 1,\\
P_{\lambda,U} = I \oplus 0 \oplus I \oplus 0 \oplus P_U, & Q_{\lambda,U} = I \oplus 0 \oplus 0 \oplus I \oplus Q_U, & \lambda \in \varLambda_1
\end{cases}
$$

relative to the space decomposition (1.1). Moreover,

$$
P_U = \frac{1}{2\lambda}\begin{pmatrix} P_{11} & P_{12}U \\ P_{12}U^* & P_{22} \end{pmatrix} \quad \text{and} \quad Q_U = \frac{1}{2}\begin{pmatrix} Q_{11} & Q_{12}U \\ Q_{12}U^* & Q_{22} \end{pmatrix},
$$

where

$$
\begin{cases}
P_{11} = B^{-1}\big(B + \frac{1+\lambda}{2} I\big)\big(B - \frac{1-\lambda}{2} I\big),\\
P_{22} = -B^{-1}\big(B - \frac{1+\lambda}{2} I\big)\big(B + \frac{1-\lambda}{2} I\big),\\
Q_{11} = B^{-1}\big(B + \frac{1+\lambda}{2} I\big)\big(B + \frac{1-\lambda}{2} I\big),\\
Q_{22} = -B^{-1}\big(B - \frac{1+\lambda}{2} I\big)\big(B - \frac{1-\lambda}{2} I\big),\\
P_{12} = B^{-1}\sqrt{-\big[B^2 - \frac{(1+\lambda)^2}{4} I\big]\big[B^2 - \frac{(1-\lambda)^2}{4} I\big]}, \quad Q_{12} = -P_{12}
\end{cases}
$$

if $\lambda \in \varLambda_1$, and

$$
\begin{cases}
P_{11} = I + B, & P_{22} = I - B,\\
Q_{11} = I + B, & Q_{22} = I - B,\\
P_{12} = \sqrt{I - B^2}, & Q_{12} = -P_{12}
\end{cases}
$$

if $\lambda = 1$. From (2.4), we only need to consider the minimal geodesic problem in the spaces $N(T - I)$ and $\mathcal{H}_0$ when $\lambda = 1$. In $N(T - I)$, it is the same as in the Grassmann manifold $\mathcal{P}(N(T - I))$. In addition, we just have to take the minimal geodesic problem into account in $N(T - \lambda I) \oplus N(T - I)$ and $\mathcal{H}_0$ when $\lambda \in \varLambda_1$. In $N(T - \lambda I) \oplus N(T - I)$, one can see that there exists a unique minimal geodesic if and only if $\dim N(T-\lambda I) = \dim N(T-I)$ [2, Theorem 3.1]. From what has been discussed above, we first focus on T in generic position with respect to $\lambda \in \varLambda = \mathbb{R} \setminus \{0,-1\}$.

Set

$$\mathcal{A} = \{T\}' = \{A \in \mathcal{B}(\mathcal{H}) : AT = TA\},$$

which is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ because T is self-adjoint, and its operators clearly commute with T_λ. Firstly, we give a characterization of self-adjoint operators which are pencils of pairs of projections in generic position.

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})_{\mathrm{s}}$ be in generic position with respect to $\lambda \in \varLambda$. Then $T \in \mathcal{P}_\lambda$ if and only if there is a symmetry W on $\mathcal{H}$ such that $WT_\lambda = -T_\lambda W$.*

Proof. Suppose $(P,Q) \in \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H})$ with $T = \lambda P + Q$. It is not hard to check that the nullspace of $f(P,Q)$ defined in (2.1) is trivial, thus the polar decomposition

$$f(P,Q) = |f(P,Q)|W = W|f(P,Q)|$$

yields a symmetry W on $\mathcal{H}$. Obviously, $f(P,Q)T_\lambda = -T_\lambda f(P,Q)$, and so $f(P,Q)^2$ commutes with P and Q, which implies that $|f(P,Q)| = (f(P,Q)^2)^{1/2}$ also commutes with both projections. Hence $WT_\lambda = -T_\lambda W$.

Conversely, given a symmetry W which anti-commutes with T_λ, put

$$\begin{cases} P_W = \frac{1}{2\lambda}\{T_\lambda^{-1}T(T-I) \\ \qquad\quad + |T_\lambda|^{-1}[-T(T-1)(T-\lambda)(T-(1+\lambda))]^{1/2}W\} \\ Q_W = \frac{1}{2}\{T_\lambda^{-1}T(T-\lambda) \\ \qquad\quad - |T_\lambda|^{-1}[-T(T-1)(T-\lambda)(T-(1+\lambda))]^{1/2}W\} \end{cases} \quad (\lambda \in \varLambda_1),$$

$$\begin{cases} P_W = \frac{1}{2\lambda}\{T_\lambda + [-T(T-(1+\lambda))]^{1/2}W\} \\ Q_W = \frac{1}{2}\{T_\lambda - [-T(T-(1+\lambda))]^{1/2}W\} \end{cases} \quad (\lambda = 1).$$

Straightforward computations show that $P_W, Q_W \in \mathcal{P}(\mathcal{H})$ satisfy $T = \lambda P_W + Q_W$. ■

For a symmetry W which anti-commutes with T_λ, one can obtain $T = \lambda P_W + Q_W$, where $P_W, Q_W \in \mathcal{P}(\mathcal{H})$ as in the proof of Proposition 2.2. Then the following result holds:

Corollary 2.3. *Let $T \in \mathcal{P}_\lambda$ be in generic position with respect to $\lambda \in \varLambda$. Then the mapping*

$$S : \mathcal{P}_{T,\lambda} \to E = \{W \in \mathcal{B}(\mathcal{H}) : W \text{ is a symmetry and } WT_\lambda = -T_\lambda W\}$$

is a bijection.

Corollary 2.3 yields a 1-1 correspondence between pairs of projections and such symmetries in generic position. For any fixed $(P,Q) \in \mathcal{P}_{T,\lambda}$, let $U_{(P,Q)}$ be the unitary orbit of (P,Q):

$$U_{(P,Q)} = \{(UPU^*, UQU^*) : U \in \mathcal{U}_{\mathcal{A}}\},$$

where $\mathcal{U}_{\mathcal{A}}$ is the unitary group of $\mathcal{A}$. Consider the mapping

$$\pi_{(P,Q)} : \mathcal{U}_{\mathcal{A}} \to U_{(P,Q)} \subseteq \mathcal{P}_{T,\lambda}, \qquad \pi_{(P,Q)}(U) = (UPU^*, UQU^*).$$

It is evident that $\mathcal{P}_{T,\lambda} = U_{(P,Q)}$ in generic position for any pair (P,Q) in $\mathcal{P}_{T,\lambda}$ from (2.4). Indeed, for any pair $(P_1,Q_1) \in \mathcal{P}_{T,\lambda}$, there exists a unitary $U \in \{B\}'$ such that $(P_1,Q_1) = (P_U,Q_U)$, that is,

$$P_U = \frac{1}{2\lambda}\begin{pmatrix} P_{11} & P_{12}U \\ P_{12}U^* & P_{22} \end{pmatrix} \quad \text{and} \quad Q_U = \frac{1}{2}\begin{pmatrix} Q_{11} & Q_{12}U \\ Q_{12}U^* & Q_{22} \end{pmatrix}.$$

Set

$$\tilde{U} = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}.$$

It is easily seen that $\tilde{U} \in \mathcal{U}_{\mathcal{A}}$ and $(\tilde{U}P\tilde{U}^*, \tilde{U}Q\tilde{U}^*) = (P_1,Q_1)$. Next, we shall prove that the action $\pi_{(P,Q)}$ of $\mathcal{U}_{\mathcal{A}}$ on $\mathcal{P}_{T,\lambda}$ is transitive when T is in generic position with respect to $\lambda \in \Lambda$.

THEOREM 2.4. *Suppose $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$. Then $\pi_{(P,Q)}$ is transitive.*

Proof. For any two pairs (P_0,Q_0) and (P_1,Q_1) in $\mathcal{P}_{T,\lambda}$, we have

$$T = \lambda P_0 + Q_0 = \lambda P_1 + Q_1.$$

Consider the self-adjoint operator $f(P_0,Q_1)$ defined in (2.1) and the space decomposition

$$\mathcal{H} = N(f(P_0,Q_1)) \oplus N(f(P_0,Q_1))^\perp.$$

Firstly, we claim that $N(f(P_0,Q_1))$ reduces both (P_0,Q_0) and (P_1,Q_1). Indeed, it is not hard to verify that

$$\begin{cases} f(P_0,Q_1)\big[Q_1 - \frac{1-\lambda^2}{8}T_\lambda^{-1}\big] \\ \quad = \big[-\lambda P_0 + \frac{1+\lambda}{2}I - \frac{1-\lambda^2}{8}T_\lambda^{-1}\big]f(P_0,Q_1), \quad \lambda \in \Lambda_1, \\ f(P_0,Q_1)Q_1 = [I - P_0]f(P_0,Q_1), \quad \lambda = 1, \end{cases} \tag{2.5}$$

which implies

$$\begin{cases} \big[Q_1 - \frac{1-\lambda^2}{8}T_\lambda^{-1}\big]N(f(P_0,Q_1)) \subseteq N(f(P_0,Q_1)), \quad \lambda \in \Lambda_1, \\ Q_1 N(f(P_0,Q_1)) \subseteq N(f(P_0,Q_1)), \quad \lambda = 1, \end{cases}$$

and

$$\begin{cases} \left[-\lambda P_0 + \frac{1+\lambda}{2} I - \frac{1-\lambda^2}{8} T_\lambda^{-1}\right] N(f(P_0, Q_1)) \subseteq N(f(P_0, Q_1)), \quad \lambda \in \Lambda_1 \\ (I - P_0) N(f(P_0, Q_1)) \subseteq N(f(P_0, Q_1)), \quad \lambda = 1. \end{cases}$$

Moreover, since $T_\lambda f(P_0, Q_1) = -f(P_0, Q_1) T_\lambda$, it follows that

$$T_\lambda N(f(P_0, Q_1)) \subseteq N(f(P_0, Q_1)).$$

From the above, we can see that $N(f(P_0, Q_1))$ reduces P_0 and Q_1. Similarly also $N(f(P_1, Q_0))$ reduces P_1 and Q_0. Since

$$\lambda P_0 + Q_0 - \frac{1+\lambda}{2} I = \lambda P_1 + Q_1 - \frac{1+\lambda}{2} I,$$

we find that $N(f(P_0, Q_1)) = N(f(P_1, Q_0))$, which proves the claim.

In $N(f(P_0, Q_1))$, one can obtain

$$\begin{cases} Q_1 - \lambda P_0 - \frac{1-\lambda}{2} I - \frac{1-\lambda^2}{4} T_\lambda^{-1} = 0, \quad \lambda \in \Lambda_1, \\ Q_1 - P_0 = 0, \quad \lambda = 1, \end{cases}$$

thus

$$\begin{cases} T_\lambda \big(Q_0 - \frac{1-\lambda^2}{8} T_\lambda^{-1}\big)\big|_{N(f(P_0,Q_1))} = \big(Q_1 - \frac{1-\lambda^2}{8} T_\lambda^{-1}\big)\big|_{N(f(P_0,Q_1))} T_\lambda, \ \lambda \in \Lambda_1, \\ T_\lambda Q_0|_{N(f(P_0,Q_1))} = Q_1|_{N(f(P_0,Q_1))} T_\lambda, \ \lambda = 1. \end{cases}$$

Similarly,

$$\begin{cases} T_\lambda (\lambda P_0 + \frac{1-\lambda}{2} I + \frac{1-\lambda^2}{8} T_\lambda^{-1})|_{N(f(P_0,Q_1))} \\ \quad = \big(\lambda P_1 + \frac{1-\lambda}{2} I + \frac{1-\lambda^2}{8} T_\lambda^{-1}\big) T_\lambda|_{N(f(P_0,Q_1))}, \quad \lambda \in \Lambda_1, \\ T_\lambda P_0|_{N(f(P_0,Q_1))} = P_1 T_\lambda|_{N(f(P_0,Q_1))}, \quad \lambda = 1, \end{cases}$$

which forces

$$T_\lambda P_0|_{N(f(P_0,Q_1))} = P_1 T_\lambda|_{N(f(P_0,Q_1))}, \quad T_\lambda Q_0|_{N(f(P_0,Q_1))} = Q_1 T_\lambda|_{N(f(P_0,Q_1))}$$

in view of (2.6). Since T_λ is a self-adjoint injective operator, there exists a symmetry U_1 in the polar decomposition of $T_\lambda|_{N(f(P_0,Q_1))}$ satisfying

$$U_1 P_0|_{N(f(P_0,Q_1))} U_1 = P_1|_{N(f(P_0,Q_1))}, \quad U_1 Q_0|_{N(f(P_0,Q_1))} U_1 = Q_1|_{N(f(P_0,Q_1))}.$$

In the second subspace $N(f(P_0, Q_1))^\perp$, the operator $f(P_0, Q_1)$ is self-adjoint and has trivial nullspace. Therefore, the operator W in the polar decomposition

$$f(P_0, Q_1) = W|f(P_0, Q_1)| = |f(P_0, Q_1)|W$$

is a symmetry satisfying

$$\begin{cases} W\big(Q_0 - \frac{1-\lambda^2}{8} T_\lambda^{-1}\big)\big|_{N(f(P_0,Q_1))^\perp} W \\ \quad = \big(-\lambda P_1 + \frac{1+\lambda}{2} I - \frac{1-\lambda^2}{8} T_\lambda^{-1}\big)\big|_{N(f(P_0,Q_1))^\perp}, \quad \lambda \in \Lambda_1, \\ W Q_0|_{N(f(P_0,Q_1))^\perp} W = (I - P_1)|_{N(f(P_0,Q_1))^\perp}, \quad \lambda = 1, \end{cases}$$

by the same argument as in the proof of Proposition 2.2. The fact that

$$\lambda P_0 + Q_0 - \frac{1+\lambda}{2} I = \lambda P_1 + Q_1 - \frac{1+\lambda}{2} I$$

implies that the operator W satisfies

$$\begin{cases} W\big(-\lambda P_0 + \frac{1+\lambda}{2} I - \frac{1-\lambda^2}{8} T_\lambda^{-1}\big)\big|_{N(f(P_0,Q_1))^\perp} W \\ \quad = \big(Q_1 - \frac{1-\lambda^2}{8} T_\lambda^{-1}\big)\big|_{N(f(P_0,Q_1))^\perp}, \quad \lambda \in \Lambda_1, \\ W(I - P_0)|_{N(f(P_0,Q_1))^\perp} W = Q_1|_{N(f(P_0,Q_1))^\perp}, \quad \lambda = 1. \end{cases}$$

Then

$$W T_\lambda|_{N(f(P_0,Q_1))^\perp} W = -T_\lambda|_{N(f(P_0,Q_1))^\perp}.$$

Let W_0 be the symmetry as in (2.2) corresponding to (P_0, Q_0), which anti-commutes with T_λ and is also reduced by $N(f(P_0, Q_1))$. Clearly, the restriction $W_0|_{N(f(P_0,Q_1))^\perp}$ also anti-commutes with $T_\lambda|_{N(f(P_0,Q_1))^\perp}$. Put

$$U_2 = W W_0|_{N(f(P_0,Q_1))^\perp}.$$

We see that U_2 defined in $N(f(P_0, Q_1))^\perp$ commutes with $T_\lambda|_{N(f(P_0,Q_1))^\perp}$. Moreover,

$$\begin{aligned} U_2\bigg(Q_0 - \frac{1-\lambda^2}{8} T_\lambda^{-1}\bigg)\bigg|_{N(f(P_0,Q_1))^\perp} U_2^* \\ &= W\bigg[W_0\bigg(Q_0 - \frac{1-\lambda^2}{8} T_\lambda^{-1}\bigg) W_0\bigg]\bigg|_{N(f(P_0,Q_1))^\perp} W \\ &= W\bigg(-\lambda P_0 + \frac{1+\lambda}{2} I - \frac{1-\lambda^2}{8} T_\lambda^{-1}\bigg)\bigg|_{N(f(P_0,Q_1))^\perp} W \\ &= \bigg(Q_1 - \frac{1-\lambda^2}{8} T_\lambda^{-1}\bigg)\bigg|_{N(f(P_0,Q_1))^\perp} \end{aligned}$$

when $\lambda \in \Lambda_1$, and

$$\begin{aligned} U_2 Q_0|_{N(f(P_0,Q_1))^\perp} U_2^* &= W[W_0 Q_0 W_0]|_{N(f(P_0,Q_1))^\perp} W \\ &= W(I - P_0|_{N(f(P_0,Q_1))^\perp} W = Q_1|_{N(f(P_0,Q_1))^\perp} \end{aligned}$$

when $\lambda = 1$. Similarly,

$$\begin{cases} U_2\big(-\lambda P_0 + \frac{1+\lambda}{2} I - \frac{1-\lambda^2}{8} T_\lambda^{-1}\big)\big|_{N(f(P_0,Q_1))^\perp} U_2^* \\ \quad = \big(-\lambda P_1 + \frac{1+\lambda}{2} I - \frac{1-\lambda^2}{8} T_\lambda^{-1}\big)\big|_{N(f(P_0,Q_1))^\perp}, \quad \lambda \in \Lambda_1, \\ U_2(I - P_0)|_{N(f(P_0,Q_1))^\perp} U_2^* = (I - P_1)|_{N(f(P_0,Q_1))^\perp}, \quad \lambda = 1. \end{cases}$$

The above results imply

$$U_2 P_0|_{N(f(P_0,Q_1))^\perp} U_2^* = P_1|_{N(f(P_0,Q_1))^\perp},$$

$$U_2 Q_0|_{N(f(P_0,Q_1))^\perp} U_2^* = Q_1|_{N(f(P_0,Q_1))^\perp}.$$

Let $U = U_1 \oplus U_2$. It acts on $\mathcal{H} = N(f(P_0, Q_1)) \oplus N(f(P_0, Q_1))^{\perp}$. We know that U is a unitary operator which commutes with T_λ [8, Corollary 2.7] and satisfies $(UP_0U^*, UQ_0U^*) = (P_1, Q_1)$. ∎

The construction of U is determined by (P_0, Q_0) and (P_1, Q_1). However, U need not be continuous in (P_1, Q_1) when (P_0, Q_0) is fixed. Naturally, we are interested in when U is continuous. Set

$$\Omega = \left\{ \frac{1+\lambda}{2}, -\frac{1+\lambda}{2}, \frac{1-\lambda}{2}, -\frac{1-\lambda}{2} \right\}.$$

Obviously,

$$\prod_{k\in\Omega} (T - kI) = T_\lambda^2 f(P, Q)^2$$

when $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$, which implies that $R(\prod_{k\in\Omega}(T - kI))$ is closed if and only if $f(P, Q)$ is invertible for any $(P, Q) \in \mathcal{P}_{T,\lambda}$. Next, we shall construct a concrete continuous local unitary cross section for the mapping $\pi_{(P,Q)}$.

PROPOSITION 2.5. *Suppose $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$, and $R(\prod_{k\in\Omega}(T - kI))$ is closed. Then the mapping $\pi_{(P,Q)}$ has a continuous local unitary cross section for any fixed $(P, Q) \in \mathcal{P}_{T,\lambda}$.*

Proof. Set

$$\mathcal{B} = \{(P_1, Q_1) \in \mathcal{P}_{T,\lambda} : f(P, Q_1) \text{ is invertible in } \mathcal{H}\}.$$

It is well known that the set of invertible operators is an open subset in $\mathcal{B}(\mathcal{H})$, and thus $\mathcal{B}$ is also an open subset of $\mathcal{P}_{T,\lambda}$ in the relative topology of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. Since $f(P, Q)$ is invertible, we know $(P, Q) \in \mathcal{B}$, which means that $\mathcal{B}$ is a neighbourhood of (P, Q). Consider the map $s_{(P,Q)} : \mathcal{B} \to \mathcal{U}_{\mathcal{A}}$ defined by

$$s_{(P,Q)}(P_1, Q_1) = S(P, Q_1)S(P, Q), \tag{2.6}$$

where $S(\cdot,\cdot)$ is defined in (2.2). We easily see that $s_{(P,Q)}$ is continuous on $\mathcal{B}$. Moreover, as seen in the proof of Theorem 2.4, we have

$$\begin{aligned}(\pi_{(P,Q)} \circ s_{(P,Q)})(P_1, Q_1) &= \pi_{(P,Q)}(s_{(P,Q)}(P_1, Q_1)) \\ &= s_{(P,Q)}(P_1, Q_1) \cdot (P, Q) \cdot (s_{(P,Q)}(P_1, Q_1))^* \\ &= (P_1, Q_1).\end{aligned}$$

One may obtain local continuous unitary cross sections at other pairs in $\mathcal{P}_{T,\lambda}$ by translating this one. In other words, $\pi_{(P,Q)}$ has a local continuous unitary cross section defined on a neighbourhood of any pair in $\mathcal{P}_{T,\lambda}$. ∎

Suppose that $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$ and $(P, Q) \in \mathcal{P}_{T,\lambda}$. A direct computation leads to the form of operators in $\mathcal{A}$

relative to (1.4):

$$\mathcal{A} = \left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} : x, y, z \text{ commute with } c, (x - z)cs = 2y\left(c^2 + \frac{\lambda - 1}{2} I\right) \right\}.$$

In order to prove the next proposition, we need some facts. Put

$$I_{(P,Q)} = \{U \in \mathcal{U}_{\mathcal{A}} : UPU^* = P \text{ and } UQU^* = Q\},$$

which is a Banach–Lie subgroup of $\mathcal{U}_{\mathcal{A}}$ and the isotropy subgroup of (P, Q) under the action of $\mathcal{U}_{\mathcal{A}}$. By a simple computation, the Banach–Lie algebra of $I_{(P,Q)}$ is

$$(TI_{(P,Q)})_1 = \{X \in \mathcal{A}_{\mathrm{ah}} : XP = PX \text{ and } XQ = QX\}. \tag{2.7}$$

The tangent map of $\pi_{(P,Q)}$ at 1 is given by

$$\delta_1^{(P,Q)} := d(\pi_{(P,Q)})_1 : (T\mathcal{U}_{\mathcal{A}})_1 \, (= \mathcal{A}_{\mathrm{ah}}) \to (T\mathcal{P}_{T,\lambda})_{(P,Q)} \subseteq \mathcal{B}(\mathcal{H})_{\mathrm{s}} \times \mathcal{B}(\mathcal{H})_{\mathrm{s}}$$

such that

$$\delta_1^{(P,Q)}(X) = (XP - PX, XQ - QX),$$

where $\mathcal{A}_{\mathrm{ah}}$ stands for the anti-hermitian operators of $\mathcal{A}$. The tangent space of $\mathcal{P}_{T,\lambda}$ at (P, Q) is

$$\begin{aligned}
&(T\mathcal{P}_{T,\lambda})_{(P,Q)} \\
&= \left\{ \left(\begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix}, \begin{pmatrix} 0 & (x - z)cs + (s^2 - c^2)y \\ -(x - z)cs - (s^2 - c^2)y & 0 \end{pmatrix} \right) : \right. \\
&\qquad\qquad \left. x, y, z \in \mathcal{A}_{\mathrm{ah}} \text{ commute with } c, (x - z)cs = 2y\left(c^2 + \frac{\lambda - 1}{2} I\right) \right\} \\
&= \left\{ \left(\begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda y \\ -\lambda y & 0 \end{pmatrix} \right) : y \in \mathcal{A}_{\mathrm{ah}} \text{ and } yc = cy \right\}.
\end{aligned}$$

Clearly, for all $(Y^{(1)}, Y^{(2)}) \in (T\mathcal{P}_{T,\lambda})_{(P,Q)}$ we have $Y^{(2)} = -\lambda Y^{(1)}$.

PROPOSITION 2.6. *Suppose $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$, and $R(\prod_{k \in \Omega}(T - kI))$ is closed. Then the set $\mathcal{P}_{T,\lambda}$ is an analytic embedded Banach submanifold, and for any fixed $(P, Q) \in \mathcal{P}_{T,\lambda}$, the mapping $\pi_{(P,Q)}$ is a C^∞ submersion.*

Proof. Proposition 2.5 tells us that for any fixed $(P, Q) \in \mathcal{P}_{T,\lambda}$, the mapping $\pi_{(P,Q)}$ has a continuous local unitary cross section, which implies that the mapping $\pi_{(P,Q)} : \mathcal{U}_{\mathcal{A}} \to U_{(P,Q)}$ is open [5, Theorem 16.38]. Next, we shall prove that the differential $\delta_1^{(P,Q)}$ splits, i.e., $N(\delta_1^{(P,Q)})$ and $R(\delta_1^{(P,Q)})$ are closed complemented subspaces in $\mathcal{A}_{\mathrm{ah}}$ and $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ respectively, which completes the proof by [18, Proposition 1.5].

Clearly, the set $N(d(\pi_{(P,Q)})_1)$ is

$$(TI_{(P,Q)})_1 = \{X \in \mathcal{A}_{\mathrm{ah}} : XP = PX \text{ and } XQ = QX\},$$

whose operators have the form $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ in the space decomposition (1.4), where $x^* = -x$ and $xc = cx$. Apparently, $N(d(\pi_{(P,Q)})_1)$ is complemented in $\mathcal{A}_{\mathrm{ah}}$. Since $R(\prod_{k\in\Omega}(T-kI))$ is closed, a continuous local unitary cross section on a neighbourhood of (P,Q) was defined in (2.7) by

$$s_{(P,Q)} : \mathcal{B} \to \mathcal{U}_{\mathcal{A}}, \quad s_{(P,Q)}(P_1, Q_1) = S(P, Q_1)S(P,Q),$$

where $\mathcal{B} = \{(P_1, Q_1) \in \mathcal{P}_{T,\lambda} : f(P, Q_1) \text{ is invertible in } \mathcal{H}\}$, and we get

$$\pi_{(P,Q)} \circ s_{(P,Q)} \circ \pi_{(P,Q)} = \pi_{(P,Q)}. \tag{2.8}$$

To show that $R(\delta_1^{(P,Q)})$ is a closed complemented subspace in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$, let us extend $s_{(P,Q)}$ to

$$\tilde{s}_{(P,Q)} : \tilde{\mathcal{B}} \to \mathcal{U}_{\mathcal{A}}, \quad \tilde{s}_{(P,Q)}(A, D) = S(P, D)S(P,Q),$$

where $\tilde{\mathcal{B}} = \{(A, D) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) : f(P, D) \text{ is invertible in } \mathcal{H}\}$ is an open subset containing (P,Q) and $\tilde{s}_{(P,Q)}$ is C^∞. A natural extension of $\delta_1^{(P,Q)}$ is defined by

$$\tilde{\delta}_1^{(P,Q)} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}), \quad \tilde{\delta}_1^{(P,Q)}(X) = (XP - PX, XQ - QX).$$

Set $\tilde{\mathfrak{s}}_{(P,Q)} = d(\tilde{s}_{(P,Q)})_{(P,Q)}$. It is immediate that (2.8) is equivalent to

$$\pi_{(P,Q)} \circ \tilde{s}_{(P,Q)} \circ \pi_{(P,Q)} = \pi_{(P,Q)}.$$

Differentiating it at 1, we obtain

$$\tilde{\delta}_1^{(P,Q)} \circ \tilde{\mathfrak{s}}_{(P,Q)} \circ \tilde{\delta}_1^{(P,Q)} = \tilde{\delta}_1^{(P,Q)}, \tag{2.9}$$

which shows that $\tilde{\delta}_1^{(P,Q)} \circ \tilde{\mathfrak{s}}_{(P,Q)}$ is an idempotent and

$$R(\tilde{\delta}_1^{(P,Q)} \circ \tilde{\mathfrak{s}}_{(P,Q)}) = R(\tilde{\delta}_1^{(P,Q)}),$$

hence

$$R(\delta_1^{(P,Q)}) = \tilde{\delta}_1^{(P,Q)}(\mathcal{A}_{\mathrm{ah}})$$

is complemented in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. ∎

In the general case, when $R(\prod_{k\in\Omega}(T-kI))$ is not necessarily closed, we can use the transitive action of $\mathcal{U}_{\mathcal{A}}$ to induce a differential structure in $\mathcal{P}_{T,\lambda}$. Some results on quotients of unitary groups (see [6]) will be used to prove the following:

THEOREM 2.7. *Suppose that $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$. Then $\mathcal{P}_{T,\lambda}$ is an analytic Banach homogeneous subspace in the product $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$, and for any fixed $(P,Q) \in \mathcal{P}_{T,\lambda}$, the mapping $\pi_{(P,Q)}$ is a C^∞ submersion.*

Proof. Since the proof of Proposition 2.6 tells us that $I_{(P,Q)}$ is a Banach–Lie subgroup of $\mathcal{U}_{\mathcal{A}}$ by [6, Definition 4.1], it follows that $\mathcal{U}_{\mathcal{A}}/I_{(P,Q)}$ is an analytic Banach manifold from [6, Theorem 4.19]. Invoking [6, Theorem 4.33], we know that $\mathcal{P}_{T,\lambda}$ is homeomorphic to $\mathcal{U}_{\mathcal{A}}/I_{(P,Q)}$, and hence [6, Theorem 4.34] shows that the unital C*-subalgebra of $\mathcal{A}$ generated by P and Q is finite-dimensional, which implies that $\pi_{(P,Q)}$ has a continuous local unitary cross section and $\pi_{(P,Q)}$ is an open mapping by [5, Theorem 16.38]. Similar to the proof of Proposition 2.6, one concludes that $\mathcal{P}_{T,\lambda}$ is also an analytic Banach homogeneous space in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. ∎

From Remark 2.1 and Theorem 2.7, we get the result below.

THEOREM 2.8. *Suppose $T \in \mathcal{P}_\lambda$ with $\lambda \in \Lambda$. Then $\mathcal{P}_{T,\lambda}$ is an analytic Banach homogeneous space in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$.*

3. A reductive structure for $\mathcal{P}_{T,\lambda}$ in generic position. In this section, we still consider the case when $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$. Theorem 2.8 tells us that $\mathcal{P}_{T,\lambda}$ is a Banach homogeneous space. Next, we consider a reductive structure of $\mathcal{P}_{T,\lambda}$. Set

$$\begin{aligned} V_1^{(P,Q)} &:= (TI_{(P,Q)})_1 = \{X \in \mathcal{A}_{\mathrm{ah}} : XP = PX \text{ and } XQ = QX\} \\ &= \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathcal{B}(\mathcal{K})_{\mathrm{ah}} \text{ and } xc = cx \right\} \end{aligned}$$

and

$$\begin{aligned} H_1^{(P,Q)} := \bigg\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \mathcal{B}(\mathcal{H}) : x, y, z \in \mathcal{A}_{\mathrm{ah}} &\text{ commute with } c, \\ (x-z)cs &= 2y\Big(c^2 + \frac{\lambda-1}{2} I\Big)\bigg\} \\ = \left\{ \begin{pmatrix} y\sigma & y \\ y & -y\sigma \end{pmatrix} \in \mathcal{B}(\mathcal{H}) : y^* = -y,\ y\sigma = \sigma y \text{ and } y\sigma \in \mathcal{B}(\mathcal{K}) \right\}&, \end{aligned}$$

where $c = \cos\Gamma$, $s = \sin\Gamma$ and

$$\sigma = \left[(\cos\Gamma)^2 + \frac{\lambda - 1}{2} \right] \sec\Gamma \csc\Gamma, \qquad \|\Gamma\| \le \pi/2. \tag{3.1}$$

It is a simple matter to verify that $\mathcal{A}_{\mathrm{ah}} = V_1^{(P,Q)} \oplus H_1^{(P,Q)}$ and

$$\mathrm{ad}(U)(H_1^{(P,Q)}) = H_1^{(P,Q)},\ \forall U \in I_{(P,Q)}, \quad \text{where} \quad \mathrm{ad}(U)(X) = UXU^*,$$

which induces a reductive structure of $\mathcal{P}_{T,\lambda}$, and hence makes $\mathcal{P}_{T,\lambda}$ a reductive homogeneous space. In this case, $H_1^{(P,Q)}$ and $V_1^{(P,Q)}$ are called the *horizontal space* and the *vertical space* at 1 respectively. Based on this, we shall give a connection on $\mathcal{P}_{T,\lambda}$ in order to discuss geometric characteristics. Define a map

$$\Sigma_1^{(P,Q)} : (T\mathcal{P}_{T,\lambda})_{(P,Q)} \to H_1^{(P,Q)}$$

by

$$\begin{aligned}\Sigma_1^{(P,Q)}(Y_1, Y_2) &= T_\lambda K_{(P,Q)}^{-1}\left(Y_1 + \frac{2}{\lambda} P Y_2\right) \\ &= T_\lambda K_{(P,Q)}^{-1}(1 - 2P)Y_1,\end{aligned}$$

where $K_{(P,Q)} = PQ + QP - 2PQP$ is invertible when $y\delta \in \mathcal{B}(\mathcal{K})$. One may find that

$$\begin{aligned}\delta_1^{(P,Q)} \circ \Sigma_1^{(P,Q)} \circ \delta_1^{(P,Q)}(X) &= \delta_1^{(P,Q)}(\Sigma_1^{(P,Q)}(XP + PX^*, XQ + QX^*)) \\ &= (XP + PX^*, XQ + QX^*) \\ &= \delta_1^{(P,Q)}(X), \quad \forall X \in \mathcal{A}_{\mathrm{ah}},\end{aligned}$$

which tells us that

$$R(\Sigma_1^{(P,Q)} \circ \delta_1^{(P,Q)}) = H_1^{(P,Q)} \quad \text{and} \quad N(\Sigma_1^{(P,Q)} \circ \delta_1^{(P,Q)}) = N(\delta_1^{(P,Q)}).$$

Thus the map $\Sigma_1^{(P,Q)} \circ \delta_1^{(P,Q)}$ is an idempotent in $\mathcal{B}(\mathcal{A}_{\mathrm{ah}})$. So we can see that $\delta_1^{(P,Q)}$ is a linear isomorphism between $H_1^{(P,Q)}$ and $(T\mathcal{P}_{T,\lambda})_{(P,Q)}$, which provides a way to introduce a connection.

Remark 3.1. For every $U \in \mathcal{U}_{\mathcal{A}}$, set the *horizontal space* and the *vertical space* at U to be

$$H_U^{(P,Q)} = UH_1^{(P,Q)} \quad \text{and} \quad V_U^{(P,Q)} = UV_1^{(P,Q)}$$

respectively. Thus $(T\mathcal{U}_{\mathcal{A}})_U = H_U^{(P,Q)} \oplus V_U^{(P,Q)}$. To get the parallel transport of tangent spaces of $\mathcal{P}_{T,\lambda}$, we need the following results. Given $U \in \mathcal{U}_{\mathcal{A}}$, the differential map of $\pi_{(P,Q)}$ at U,

$$\delta_U^{(P,Q)} := d(\pi_{(P,Q)})_U : (T\mathcal{U}_{\mathcal{A}})_U \to (T\mathcal{P}_{T,\lambda})_{(P_1,Q_1)},$$

is given by

$$\delta_U^{(P,Q)}(X) = (XPU^* + UPX^*, XQU^* + UQX^*),$$

where $(P_1, Q_1) = (UPU^*, UQU^*)$. Moreover,

$$\Sigma_U^{(P,Q)} : (T\mathcal{P}_{T,\lambda})_{(P_1,Q_1)} \to H_U^{(P,Q)}$$

is given by

$$\Sigma_U^{(P,Q)}(Y_1, Y_2) = T_\lambda K_{(P,Q)}^{-1}\left(Y_1 + \frac{2}{\lambda} P Y_2\right)U.$$

It is easily seen that $N(\delta_U^{(P,Q)}) = V_U^{(P,Q)}$, and so

$$\delta_U^{(P,Q)}|_{H_U^{(P,Q)}} : H_U^{(P,Q)} \to (T\mathcal{P}_{T,\lambda})_{(P_1,Q_1)}$$

is a linear isomorphism.

Next, we shall compute the horizontal lift differential equation of the connection above. A *lift* in $\mathcal{U}_{\mathcal{A}}$ of a given pair of smooth curves $\gamma = (\gamma_1, \gamma_2) \subseteq \mathcal{P}_{T,\lambda}$ is a smooth curve $\Gamma \subseteq \mathcal{U}_{\mathcal{A}}$ such that

$$\gamma = \pi_{(P,Q)}(\Gamma) = (\Gamma P \Gamma^*, \Gamma Q \Gamma^*).$$

If $\dot{\Gamma} \in H_\Gamma^p$, then Γ is a horizontal lift of γ. Hereafter, we shall examine when there is a horizontal lift of (γ_1, γ_2).

REMARK 3.2. Set $(P_1, Q_1) = (UPU^*, UQU^*)$ and consider a pair of smooth curves

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) \subseteq \mathcal{P}_{T,\lambda}, \quad t \in [0,1],$$

with $\gamma(0) = (P,Q)$. Suppose there exists a horizontal lift $\Gamma(t) \subseteq \mathcal{U}_{\mathcal{A}}$ of $\gamma(t)$, i.e.

$$\pi_{(P,Q)}(\Gamma(t)) = (\Gamma(t) P \Gamma(t)^*, \Gamma(t) Q \Gamma(t)^*) = \gamma(t) \tag{3.2}$$

and $[\Gamma(t)]^\cdot \in H_{\Gamma(t)}^{(P,Q)}$, $t \in [0,1]$. Differentiating (3.2), we have

$$[\Gamma(t)]^\cdot = \Sigma_{\Gamma(t)}^{(P,Q)}([\gamma(t)]^\cdot), \tag{3.3}$$

that is,

$$\dot{\Gamma} = \Sigma_\Gamma^{(P,Q)}(\dot{\gamma}) = T_\lambda K_{(P,Q)}^{-1}\left(\dot{\gamma}_1 + \frac{2}{\lambda} P \dot{\gamma}_2\right)\Gamma,$$

where we omit the variable t in (3.3), and $\dot{\gamma} = (\dot{\gamma}_1, \dot{\gamma}_2)$. Since Γ lifts γ, (3.3) can be changed to

$$\dot{\Gamma} = T_\lambda K_{(P,Q)}^{-1}\left(\dot{\gamma}_1 + \frac{2}{\lambda} P \dot{\gamma}_2\right)\Gamma, \tag{3.4}$$

and then we know that the solutions with given initial conditions exist and are unique.

Based on Remark 3.2, we are going to discuss the existence of a horizontal lift of (γ_1, γ_2). By referring to the existence of solutions of linear differential equations [16, Theorem 31.A], the lemma below can be obtained.

LEMMA 3.3. *Let $\gamma = (\gamma_1, \gamma_2)$ be a pair of smooth curves in $\mathcal{P}_{T,\lambda}$. Then $T_\lambda K_{(P,Q)}^{-1}(\dot{\gamma}_1 + \frac{2}{\lambda} P \dot{\gamma}_2) \in \mathcal{A}_{\mathrm{ah}}$.*

PROPOSITION 3.4. *Let $\gamma = (\gamma_1, \gamma_2)$ be a smooth curve in $\mathcal{P}_{T,\lambda}$ with $\gamma(0) = (P,Q)$. Suppose Γ is the unique solution of* (3.4) *with initial condition $\Gamma(0) = 1$. Then Γ is the horizontal lift of γ in $\mathcal{U}_{\mathcal{A}}$.*

Proof. Lemma 3.3 tells us that Γ lies in $\mathcal{U}_{\mathcal{A}}$ since Γ is the unique solution of (3.4). Differentiating

$$\Gamma^* \cdot (\gamma) = (\Gamma^*\gamma_1\Gamma, \Gamma^*\gamma_2\Gamma),$$

we obtain

$$\begin{aligned}
&([\Gamma^*\gamma_1\Gamma]^\cdot, [\Gamma^*\gamma_2\Gamma]^\cdot)\\
&\qquad = ([\Gamma^*]^\cdot\gamma_1\Gamma + \Gamma^*\dot{\gamma}_1\Gamma + \Gamma^*\gamma_1\dot{\Gamma}, [\Gamma^*]^\cdot\gamma_2\Gamma + \Gamma^*\dot{\gamma}_2\Gamma + \Gamma^*\gamma_2\dot{\Gamma})\\
&\qquad = (\Gamma^*(-\Delta\gamma_1 + \dot{\gamma}_1 + \gamma_1\Delta)\Gamma, \Gamma^*(-\Delta\gamma_2 + \dot{\gamma}_2 + \gamma_2\Delta)\Gamma),
\end{aligned}$$

where $\Delta = T_\lambda K_{(P,Q)}^{-1}\big(\dot{\gamma}_1 + \frac{2}{\lambda}P\dot{\gamma}_2\big)$. Since $\gamma = (\gamma_1, \gamma_2) \subseteq \mathcal{P}_{T,\lambda}$, it follows that

$$(\gamma_1^*, \gamma_2^*) = (\gamma_1^2, \gamma_2^2),$$

which implies that

$$(\dot{\gamma}_1, \dot{\gamma}_2) = ([\gamma_1^2]^\cdot, [\gamma_2^2]^\cdot) = (\dot{\gamma}_1\gamma_1 + \gamma_1\dot{\gamma}_1, \dot{\gamma}_2\gamma_2 + \gamma_2\dot{\gamma}_2).$$

It is easily seen that $(\Delta\gamma_1 - \gamma_1\Delta, \Delta\gamma_2 - \gamma_2\Delta) = (\dot{\gamma}_1, \dot{\gamma}_2)$, which implies $(\Gamma^*\gamma_1\Gamma, \Gamma^*\gamma_2\Gamma) = (0, 0)$, and so

$$([\Gamma^*(0)\gamma_1(0)\Gamma(0)]^\cdot, [\Gamma^*(0)\gamma_2(0)\Gamma(0)]^\cdot) = (P, Q).$$

Consequently, $(\Gamma P\Gamma^*, \Gamma Q\Gamma^*) = \gamma$ and Γ is horizontal by reversing the argument of Remark 3.2. ∎

The transport maps between the tangent spaces of $\mathcal{U}_{\mathcal{A}}$ and $\mathcal{P}_{T,\lambda}$ at different points, the covariant derivative etc. can be calculated by the corresponding formulas, so we will not list them all here. Let the map $\exp_{(P,Q)} : \{X \in \mathcal{B}(\mathcal{H}) : X^* = -X\} \to \mathcal{P}_{T,\lambda}$ satisfy

$$\exp_{(P,Q)}(X) = e^X \cdot (P, Q) = (e^X Pe^{-X}, e^X Qe^{-X}).$$

In order to compute the geodesic curves in $\mathcal{P}_{T,\lambda}$, we need the following result.

Theorem 3.5. *Suppose $(P, Q) \in \mathcal{P}_{T,\lambda}$. Then the map*

$$\begin{aligned}
&\exp_{(P,Q)} : \{X \in H_1^{(P,Q)} : \|X\| < \pi/2\}\\
&\qquad\qquad\qquad \to \exp_{(P,Q)}(\{X \in H_1^{(P,Q)} : \|X\| < \pi/2\})
\end{aligned}$$

is a bijection, and $\exp_{(P,Q)}(\{X \in H_1^{(P,Q)} : \|X\| < \pi/2\})$ *is an open dense subset of $\mathcal{P}_{T,\lambda}$.*

Proof. Clearly, $\exp_{(P,Q)}$ is a surjection, and it remains to prove that it is injective. Suppose

$$X_1, X_2 \in \{X \in H_1^{(P,Q)} : \|X\| < \pi/2\}$$

are such that $\exp_{(P,Q)}(X_1) = \exp_{(P,Q)}(X_2)$, that is, $e^{X_1}Pe^{-X_1} = e^{X_2}Pe^{-X_2}$ and $e^{X_1}Qe^{-X_1} = e^{X_2}Qe^{-X_2}$. It is easily seen that

$$e^{-X_2}e^{X_1}Pe^{-X_1}e^{X_2} = P \quad \text{and} \quad e^{-X_2}e^{X_1}Qe^{-X_1}e^{X_2} = Q,$$

which makes us set

$$e^{-X_2}e^{X_1} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \qquad uc = cu \text{ and } u \in \mathcal{U}(\mathcal{K}).$$

Since X_k has the form

$$\begin{pmatrix} -y_k\sigma & y_k \\ y_k & y_k\sigma \end{pmatrix}, \qquad k = 1, 2,$$

where y_k is anti-hermitian and σ is defined in (3.1), it follows that

$$\begin{aligned}
e^{X_k} &= \cosh(X_k) + \sinh(X_k) \\
&= \cosh\begin{pmatrix} -y_k\sigma & y_k \\ y_k & y_k\sigma \end{pmatrix} + \sinh\begin{pmatrix} -y_k\sigma & y_k \\ y_k & y_k\sigma \end{pmatrix} \\
&= \cosh(y_k\sqrt{1+\sigma^2})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + \sinh(y_k\sqrt{1+\sigma^2})\frac{1}{\sqrt{\lambda c^2 + \frac{(\lambda-1)^2}{4}}}\begin{pmatrix} c^2 + \frac{\lambda-1}{2} & cs \\ cs & -\big[c^2 + \frac{\lambda-1}{2}\big] \end{pmatrix} \\
&= \cosh(iD_k)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + \sinh(iD_k)\frac{1}{\sqrt{\lambda c^2 + \frac{(\lambda-1)^2}{4}}}\begin{pmatrix} c^2 + \frac{\lambda-1}{2} & cs \\ cs & -\big[c^2 + \frac{\lambda-1}{2}\big] \end{pmatrix} \\
&= \cos D_k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + i\sin D_k\frac{1}{\sqrt{\lambda c^2 + \frac{(\lambda-1)^2}{4}}}\begin{pmatrix} c^2 + \frac{\lambda-1}{2} & cs \\ cs & -\big[c^2 + \frac{\lambda-1}{2}\big] \end{pmatrix} \quad \text{for } k = 1, 2,
\end{aligned}$$

according to the Taylor expansion formula, where $D_k = -iy_k\sqrt{1+\sigma^2}$. From this, we deduce

$$\begin{pmatrix} \cos D_1 & 0 \\ 0 & \cos D_1 \end{pmatrix} = \begin{pmatrix} \cos D_2u & 0 \\ 0 & \cos D_2u \end{pmatrix},$$

$$\begin{pmatrix} \sin D_1 & 0 \\ 0 & \sin D_1 \end{pmatrix} = \begin{pmatrix} \sin D_2u & 0 \\ 0 & \sin D_2u \end{pmatrix},$$

which imply that

$$\cos D_1 = \cos D_2u \quad \text{and} \quad \sin D_1 = \sin D_2u,$$

and hence $\tan D_1 = \tan D_2$. As seen above,

$$\|D_k\| = \|(1+\sigma^2)y_k^2\|^{1/2} = \|X_k^2\|^{1/2} = \|X_k\| < \pi/2,$$

so $D_1 = D_2$ since $\tan x$ is strictly increasing over $(-\pi/2, \pi/2)$, and finally $X_1 = X_2$.

Corollary 2.3 tells us that the map $S : \mathcal{P}_{T,\lambda} \to E$ is a bijection, hence we next claim that $S(R(\exp_{(P,Q)}))$ is an open dense subset in E which implies that $R(\exp_{(P,Q)})$ is an open dense subset in $\mathcal{P}_{T,\lambda}$, which finishes the proof. Let $W = \operatorname{sgn}(f(P,Q))$. From a direct computation, one can obtain

$$H_1^{(P,Q)} = \{X \in \mathcal{A} : X^* = -X \text{ and } XW = -WX\}. \tag{3.5}$$

In general, any two unitary operators W', W satisfy $\|W' - W\| \le 2$, which means that

$$\begin{aligned}&\{W \in \mathcal{U}(\mathcal{H}) : W^* = W,\ WT_\lambda = -T_\lambda W\}\\ &\qquad = \{W \in \mathcal{U}(\mathcal{H}) : W^* = W,\ WT_\lambda = -T_\lambda W,\ \|W' - W\| \le 2\}.\end{aligned}$$

Set $\mathcal{O} = \{W' \in \mathcal{U}(\mathcal{H}) : (W')^* = W',\ W'T_\lambda = -T_\lambda W',\ \|W' - W\| < 2\}$. We deduce that the map

$$\varphi : \exp_{(P,Q)}(\{X \in H_1^{(P,Q)} : \|X\| < \pi/2\}) \to \mathcal{O} \tag{3.6}$$

is a bijection. Indeed, since S is bijective and $\exp_{(P,Q)}$ is injective, it follows that $\varphi = S \circ \exp_{(P,Q)}$ is injective. Conversely, for any $W' \in \mathcal{O}$, we know $\|W' - W\| < 2$, so there exists a unique $X \in \mathcal{B}(\mathcal{H})_{\mathrm{ah}}$ satisfying $XW = -WX$ and $\|X\| < \pi/2$ such that $W' = e^X W e^{-X}$ [17], and thus

$$W' = e^{2X} W \quad \text{and} \quad e^{2X} = W'W.$$

Based on this, we have the unique logarithm of the unitary $W'W$ satisfying $X = \frac{1}{2}\log(W'W)$ combined with $\|2X\| < \pi$. Since $W'W = -WW'$, we conclude $W'WT_\lambda = T_\lambda W'W$, so $W'W \in \mathcal{A}$, and thus $X \in \mathcal{A}$. Obviously, one can obtain $X \in H_1^{(P,Q)}$ from (3.4), which implies φ is surjective. ∎

Naturally, we are also interested in the properties of the map $\exp_{(P,Q)}$ on the closure of the set $\{X \in H_1^{(P,Q)} : \|X\| < \pi/2\}$. The next result concerns this issue.

THEOREM 3.6. *Suppose $(P,Q) \in \mathcal{P}_{T,\lambda}$. Then the map*

$$\exp_{(P,Q)} : \{X \in H_1^{(P,Q)} : \|X\| \le \pi/2\} \to \mathcal{P}_{T,\lambda}$$

is surjective.

Proof. Take any pair $(P,Q)\in\mathcal{P}_{T,\lambda}$, and consider the Halmos space decomposition of the pair $\left(\frac{W+1}{2},\frac{W_1+1}{2}\right)$:

$$\mathcal{H}_{11}=R\Big(\frac{W+1}{2}\Big)\cap R\Big(\frac{W_1+1}{2}\Big),\qquad \mathcal{H}_{00}=N\Big(\frac{W+1}{2}\Big)\cap N\Big(\frac{W_1+1}{2}\Big),$$
$$\mathcal{H}_{10}=R\Big(\frac{W+1}{2}\Big)\cap N\Big(\frac{W_1+1}{2}\Big),\qquad \mathcal{H}_{01}=N\Big(\frac{W+1}{2}\Big)\cap R\Big(\frac{W_1+1}{2}\Big),$$

where $W_1=\operatorname{sgn}(f(P_1,Q_1))$ and $W=\operatorname{sgn}(f(P,Q))$ defined in (2.2). Denote $J=\operatorname{sgn}(T_\lambda)$. Since T_λ anti-commutes with W and W_1, we see that the symmetry J anti-commutes with W and W_1.

Next, we assert that

$$J(\mathcal{H}_{11})=\mathcal{H}_{00}\quad\text{and}\quad J(\mathcal{H}_{10})=\mathcal{H}_{01}.$$

For any $0\neq\xi\in\mathcal{H}_{11}$, we have

$$\frac{W+1}{2}J\xi=-J\frac{W}{2}\xi+\frac{1}{2}J\xi=0,\qquad \frac{W_1+1}{2}J\xi=-J\frac{W_1}{2}\xi+\frac{1}{2}J\xi=0,$$

hence $J\xi\mathcal{H}_{11}\subseteq\mathcal{H}_{00}$ and vice versa. The same happens with $\mathcal{H}_{10}$ and $\mathcal{H}_{01}$. Since there exists a geodesic in $\mathcal{P}(\mathcal{H})$ joining $\frac{W+1}{2}$ to $\frac{W_1+1}{2}$ [2, Theorem 3.1] if and only if

$$\dim\mathcal{H}_{10}=\dim\mathcal{H}_{01},$$

it follows that there exists an anti-hermitian operator X with $\|X\|<\pi/2$ which is co-diagonal with respect to W such that $e^XWe^{-X}=W_1$, and X may not be unique if $\mathcal{H}_{10}$ is non-zero-dimensional.

We shall prove $X\in\mathcal{A}$, which forces $X\in H_1^{(P,Q)}$ from (3.6). Set

$$\mathcal{H}_1=\mathcal{H}_{11}\oplus\mathcal{H}_{00},\qquad \mathcal{H}_2=\mathcal{H}_{10}\oplus\mathcal{H}_{01},\qquad \mathcal{H}_0=\mathcal{H}\ominus[\mathcal{H}_1\oplus\mathcal{H}_2].$$

It is easily seen that J, T_λ, and such X are reduced by the spaces $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_0$, which implies that W and W_1 are reduced by the spaces $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_0$. From [2, Remark 2.2], we can choose

$$X=\begin{pmatrix}0&0&0\\0&\frac{i\pi}{2}J|_{\mathcal{H}_2}&0\\0&0&\log^{(SW|_{\mathcal{H}_0})}\end{pmatrix},\quad\text{where}\quad S=\operatorname{sgn}\big(\tfrac{1}{2}\{W+W_1\}|_{\mathcal{H}_0}\big).$$

Moreover, we obtain $ST_\lambda=-T_\lambda S$, which combined with $(W+W_1)T_\lambda=-T_\lambda(W+W_1)$ implies $SWT_\lambda|_{\mathcal{H}_0}=T_\lambda|_{\mathcal{H}_0}SW$, thus $X\in\mathcal{A}$. ■

4. Hopf–Rinow theorem for $\mathcal{P}_{T,\lambda}$. For a C*-algebra $\mathcal{A}$ and a C*-subalgebra $\mathcal{B}\subseteq\mathcal{A}$, consider the quotient map

$$\pi:\mathcal{U}_\mathcal{A}\to\mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B},$$

and for any tangent vector $Y\in T(\mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B})_1$, which identifies with $\mathcal{A}_{\mathrm{ah}}/\mathcal{B}_{\mathrm{ah}}$, there exists some $X\in\mathcal{A}_{\mathrm{ah}}$ such that $Y=(T\pi)_1(X)$. Durán, Mata-Lorenzo and Recht [11] defined the Finsler norm of Y by

$$|Y|_{[1]} = \inf\{\|X\| : X \in \mathcal{A}_{\mathrm{ah}} \text{ and } Y = (T\pi)_1(X)\}, \tag{4.1}$$

which is equivalent to

$$|Y|_{[1]} = \inf\{\|X_0 + D\| : D \in \mathcal{B}_{\mathrm{ah}}\},$$

where X_0 is an arbitrary element satisfying $Y = (T\pi)_1(X_0)$. Generally, the element achieving the lower bound above may not exist [7] in $\mathcal{A}_{\mathrm{ah}}$, and may not be unique [4]. However, such a minimal element exists when $\mathcal{B}$ and $\mathcal{A}$ are both von Neumann algebras [11]. Since $\mathcal{A}$ is a von Neumann algebra, we will consider this theory on $\mathcal{P}_{T,\lambda}$.

THEOREM 4.1. *Suppose $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$. If $(P,Q) \in \mathcal{P}_{T,\lambda}$ and $Y \in (T\mathcal{P}_{T,\lambda})_{(P,Q)}$, then there exists $X \in H_1^{(P,Q)}$ such that*

$$\gamma(t) = (e^{tX}Pe^{-tX}, e^{tX}Qe^{-tX})$$

is a minimal geodesic for the Finsler metric (4.1), *in the time interval*

$$\left[0, \frac{\pi}{2\|y^2(1+\sigma^2)\|^{1/2}}\right],$$

where σ is defined in (3.1).

Proof. Remark 3.1 tells us that the map $\Sigma_1^{(P,Q)}$ is an isomorphism between $(T\mathcal{P}_{T,\lambda})_{(P,Q)}$ and $H_1^{(P,Q)}$, which implies that there exists $X \in H_1^{(P,Q)}$ such that $X = \Sigma_1^{(P,Q)}(Y)$ for any $Y \in (T\mathcal{P}_{T,\lambda})_{(P,Q)}$. Recall that every X in $H_1^{(P,Q)}$ has the form

$$\begin{pmatrix} y\sigma & y \\ y & -y\sigma \end{pmatrix},$$

where $y^* = -y$. Since X is anti-hermitian, we have

$$X^2 = \begin{pmatrix} y\sigma & y \\ y & -y\sigma \end{pmatrix}^2 = \begin{pmatrix} y^2(1+\sigma^2) & 0 \\ 0 & y^2(1+\sigma^2) \end{pmatrix} \le 0,$$

which gives

$$y^2(1+\sigma^2) \le 0 \quad \text{and} \quad \|X\|^2 = \|y^2(1+\sigma^2)\|.$$

From [13, Proposition 4.3.3], one infers that there exists a state ϕ in $\mathcal{B}(\mathcal{K})$ such that

$$\phi(y^2(1+\sigma^2)) = -\|y^2(1+\sigma^2)\|.$$

Let

$$\varphi : \mathcal{B}(\mathcal{K}) \times \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$$

be a positive unital linear map satisfying

$$\varphi\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right) = \frac{1}{2}(A_{11} + A_{22}).$$

It is easily seen that $\rho = \phi \circ \varphi$ is a state in $\mathcal{B}(\mathcal{H})$ such that

$$\rho(X^2) = (\phi \circ \varphi)(X^2) = \phi(y^2(1+\sigma^2)) = -\|y^2(1+\sigma^2)\|.$$

Moreover, an easy computation shows that

$$\varphi(Xm) = \varphi\left(\begin{pmatrix} y\sigma m' & ym' \\ ym' & -y\sigma m' \end{pmatrix}\right) = 0$$

for $m^* = -m$ and $m = \left(\begin{smallmatrix} m' & 0 \\ 0 & m' \end{smallmatrix}\right)$. Then

$$\rho(Xm) = (\phi \circ \varphi)(Xm) = \phi(0) = 0.$$

From [11, Proposition 5.2] and [4, Theorem 2.2], we have $\|X\| = |Y|_{[1]}$, which implies that

$$\gamma(t) = (e^{tX} P e^{-tX}, e^{tX} Q e^{-tX})$$

is a minimal geodesic with

$$|t| \le \frac{\pi}{2|Y|_{[1]}} = \frac{\pi}{2\|X_Y\|} = \frac{\pi}{2\|y^2 + (y\sigma)^2\|^{1/2}}$$

in view of [11, Theorem II]. ∎

Corollary 4.2. *Suppose $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$. If $(P_0, Q_0), (P, Q) \in \mathcal{P}_{T,\lambda}$ are such that*

$$\|W - W_0\| < 2, \quad \textit{where} \quad W = \operatorname{sgn}(f(P,Q)), W_0 = \operatorname{sgn}(f(P_0,Q_0)),$$

then there exists a unique $X \in H_1^{(P_0,Q_0)}$ with $\|X\| < \pi/2$ such that the curve

$$\gamma(t) = (e^{tX} P_0 e^{-tX}, e^{tX} Q_0 e^{-tX})$$

is a minimal geodesic in $\mathcal{P}_{T,\lambda}$ joining (P_0, Q_0) and (P, Q).

Proof. Since $\|W - W_0\| < 2$, it follows that there exists a unique $X \in H_1^{(P_0,Q_0)}$ with $\|X\| < \pi/2$ such that

$$e^X W_0 e^{-X} = W$$

from (3.6). According to Theorem 4.1, $\gamma(t) = (e^{tX} P_0 e^{-tX}, e^{tX} Q_0 e^{-tX})$ is a minimal geodesic joining (P_0, Q_0) and (P, Q). ∎

Using Theorem 3.6, one can show that any two pairs of elements in $\mathcal{P}_{T,\lambda}$ can be joined by a minimal geodesic when we drop the uniqueness condition in Corollary 4.2.

Theorem 4.3. *Suppose $T \in \mathcal{P}_\lambda$ is in generic position with respect to $\lambda \in \Lambda$. If $(P, Q), (P_0, Q_0) \in \mathcal{P}_{T,\lambda}$, then there exists a minimal geodesic of $\mathcal{P}_{T,\lambda}$ of length $\le \pi/2$ joining (P, Q) and (P_0, Q_0).*

Proof. Theorems 3.6 and 4.1 tell us that there exists a geodesic γ, parametrized in $[0,1]$, joining (P,Q) and (P_0,Q_0), and the norm of the exponent X which is equal to the length of the geodesic γ is less than or equal to $\pi/2$.

Next we shall prove that γ is minimal. Obviously, we only need to consider the minimality of the case of $l(\gamma)=\pi/2$ in view of Corollary 4.2. Assume that there exists a curve $\gamma_0 \in \mathcal{P}_{T,\lambda}$ that starts at (P_0,Q_0) and ends at (P,Q) with $l(\gamma_0)=\pi/2-\varepsilon$ for some $\varepsilon>0$. Then there exists $t_0\in(0,1)$ such that $l(\gamma|_{[t_0,1]})=(1-t_0)\pi/2$, which leads to a curve joining (P,Q) and $\gamma(t_0)$ whose length is strictly less than $t_0\pi/2$ obtained by adjoining to γ_0 the curve $\gamma|_{[t_0,1]}$ reversed. This contradicts the curve $\gamma|_{[0,t_0]}$ having minimal length $t_0\pi/2$ from Corollary 4.2. ∎

Remark 4.4. For $X\in H_1^{(P,Q)}$, we know

$$X=\begin{pmatrix} -y\sigma & y \\ y & y\sigma \end{pmatrix},$$

where x,y,z commute with c and $(x-z)cs=2y\big(c^2+\frac{\lambda-1}{2}I\big)$. It is immediate that $y(cs)^{-1}$ is bounded and

$$\begin{aligned}(x-z)^2c^2s^2 &= 4y^2\bigg[c^4+(\lambda-1)c^2+\Big(\frac{\lambda-1}{2}\Big)^2\bigg]\\ &= 4y^2\bigg[c^2-c^2s^2+(\lambda-1)c^2+\Big(\frac{\lambda-1}{2}\Big)^2\bigg],\end{aligned}$$

that is,

$$\bigg[\Big(\frac{x-z}{2}\Big)^2+y^2\bigg]c^2s^2=\bigg[\lambda c^2+\Big(\frac{\lambda-1}{2}\Big)^2\bigg]y^2,$$

where $\big(\frac{x-z}{2}\big)^2+y^2$ is bounded. It is easy to see that $\big[\lambda c^2+\big(\frac{\lambda-1}{2}\big)^2\big]y^2c^{-2}s^{-2}$ is bounded, thus $y^2c^{-2}s^{-2}$ is bounded because $\lambda c^2+\big(\frac{\lambda-1}{2}\big)^2$ is bounded, and hence also $yc^{-1}s^{-1}$ is bounded. Moreover,

$$\begin{aligned}\|z\| = \|y^2\sigma^2+y^2\|^{1/2} &= \bigg\|\bigg(1+\frac{c^2+\frac{\lambda-1}{2}}{cs}\bigg)^2y^2\bigg\|^{1/2}\\ &= \bigg\|\frac{c^2s^2+c^4+(\lambda-1)c^2+\frac{(\lambda-1)^2}{4}}{c^2s^2}y^2\bigg\|^{1/2}\\ &= \bigg\|\frac{\lambda c^2+\big(\frac{\lambda-1}{2}\big)^2}{c^2s^2}y^2\bigg\|^{1/2} = \bigg\|\frac{\sqrt{\big|\lambda c^2+\big(\frac{\lambda-1}{2}\big)^2\big|}}{cs}y\bigg\|.\end{aligned}$$

Therefore, if $\|z\|\le\pi/2$, the distance between (P_0,Q_0) and $e^z\cdot(P_0,Q_0)$ equals $\Big\|\frac{\sqrt{|\lambda c^2+(\frac{\lambda-1}{2})^2|}}{cs}y\Big\|$.

The Hopf–Rinow theorem in $\mathcal{P}_{T,\lambda}$ in generic position has been discussed above, we now consider the general case. From Corollary 4.2 and (2.4), one can obtain the following result.

COROLLARY 4.5. *Suppose $T \in \mathcal{P}_\lambda$ and $(P_0, Q_0), (P_1, Q_1) \in \mathcal{P}_{T,\lambda}$ with $\lambda \in \Lambda$.*

(1) *If $\|W - W_0\| < 2$ when $\lambda = 1$, then there exists a unique $X \in H_1^{(P_0,Q_0)}$ with $\|X\| < \pi/2$ such that the curve*
$$\gamma(t) = (e^{tX} P_0 e^{-tX}, e^{tX} Q_0 e^{-tX})$$
is a minimal geodesic in $\mathcal{P}_{T,\lambda}$ joining (P_0, Q_0) and (P, Q).

(2) *If*
$$\|W - W_0\| < 2 \quad \text{and} \quad \dim N(T - \lambda I) = \dim N(T - I)$$
when $\lambda \in \Lambda_1$, then there exists $X \in H_1^{(P_0,Q_0)}$ with $\|X\| < \pi/2$ such that the curve
$$\gamma(t) = (e^{tX} P_0 e^{-tX}, e^{tX} Q_0 e^{-tX})$$
is a minimal geodesic in $\mathcal{P}_{T,\lambda}$ joining (P_0, Q_0) and (P, Q), where
$$W = \operatorname{sgn}(f(P, Q)) \quad \text{and} \quad W_0 = \operatorname{sgn}(f(P_0, Q_0)).$$

Proof. (1) The equalities (2.4) yield
$$P_i = I \oplus 0 \oplus E_i \oplus P_{U_i}, \qquad Q_i = I \oplus 0 \oplus (I - E_i) \oplus Q_{U_i}, \qquad i = 0, 1.$$
If $\|W - W_0\| < 2$ when $\lambda = 1$, we have
$$\|(1 - 2E_1) - (1 - 2E_0)\| < 2 \quad \text{and} \quad \big\|W_1|_{\mathcal{H}_0} - W_0|_{\mathcal{H}_0}\big\| < 2,$$
which implies that there exist $X \in H_1^{(P_0,Q_0)}$ with $\|X\| < \pi/2$ such that $\gamma(t) = (e^{tX} P_0 e^{-tX}, e^{tX} Q_0 e^{-tX})$ is a unique minimal geodesic in $\mathcal{P}_{T,\lambda}$ joining (P_0, Q_0) and (P_1, Q_1) in view of Theorem 3.6 and [17], where
$$X = 0 \oplus 0 \oplus x_1 \oplus x_2$$
with $x_1 \in H_1^{(P_0,Q_0)}|_{(\mathcal{H}_{10} \oplus \mathcal{H}_{01})}$ and $x_2 \in H_1^{(P_0,Q_0)}|_{\mathcal{H}_0}$ with $\|x_i\| < \pi/2$.

(2) From equalities (2.4), we know that
$$P_i = 1 \oplus 0 \oplus 1 \oplus 0 \oplus P_{U_i}, \qquad Q_i = 1 \oplus 0 \oplus 0 \oplus 1 \oplus Q_{U_i}, \qquad i = 0, 1.$$
Since $\dim N(T - \lambda I) = \dim N(T - I)$ and $\|W - W_0\| < 2$ when $\lambda \in \Lambda_1$, it follows that there exists $X \in H_1^{(P,Q)}$ such that $\gamma(t) = (e^{tX} P_0 e^{-tX}, e^{tX} Q_0 e^{-tX})$ is a minimal geodesic in $\mathcal{P}_{T,\lambda}$ joining (P_0, Q_0) and (P_1, Q_1) in view of Theorem 3.6 and [17], where
$$X = 0 \oplus 0 \oplus \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \oplus x_2,$$
where $x_1 \in \mathcal{B}(N(T - I), N(T - \lambda I))_{\mathrm{ah}}$ and $x_2 \in H_1^{(P_0,Q_0)}|_{\mathcal{H}_0}$ with $\|x_i\| < \pi/2$. Moreover, x_1 can be chosen arbitrarily. ∎

Combining Theorem 4.3 and (2.4), we have

THEOREM 4.6. *Suppose $T \in \mathcal{P}_\lambda$ and $(P_0, Q_0), (P, Q) \in \mathcal{P}_{T,\lambda}$ with $\lambda \in \Lambda$. Then there exists a minimal geodesic of $\mathcal{P}_{T,\lambda}$ of length $\le \pi/2$ joining (P, Q) and (P_0, Q_0) if and only if*

$$\begin{cases} \dim[R(P|_{N(T-I)}) \cap N(P_0|_{N(T-I)})], \\ \quad = \dim[N(P|_{N(T-I)}) \cap R(P_0|_{N(T-I)})], \quad \lambda = 1, \\ \dim N(T - \lambda I) = \dim N(T - I), \quad \lambda \in \Lambda_1. \end{cases}$$

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References

[1] E. Andruchow, *Operators which are the difference of two projections*, J. Math. Anal. Appl. 420 (2014), 1634–1653.

[2] E. Andruchow, *Pairs of projections: geodesics, Fredholm and compact pairs*, Complex Anal. Oper. Theory 8 (2014), 1435–1453.

[3] E. Andruchow, G. Corach and L. Recht, *Projections with fixed difference: A Hopf–Rinow theorem*, Differential Geom. Appl. 66 (2019), 155–180.

[4] E. Andruchow, L. E. Mata-Lorenzo, L. Recht, A. Mendoza and A. Varela, *Infinitely many minimal curves joining arbitrarily close points in a homogeneous space of the unitary group of a C^*-algebra*, Rev. Un. Mat. Argentina 46 (2005), 113–120.

[5] C. Apostol, L. A. Fialkow, D. A. Herrero and D. Voiculescu, *Approximation of Hilbert Space Operators*, Vol. II, Pitman, 1984.

[6] D. Beltiţă, *Smooth Homogeneous Structures in Operator Theory*, Chapman Hall/CRC Monogr. Surveys Pure Appl. Math. 137, Chapman Hall/CRC, Boca Raton, FL, 2006.

[7] T. Bottazzi and A. Varela, *Minimal length curves in unitary oribits of a Hermitian compact operator*, Differential Geom. Appl. 45 (2016), 1–22.

[8] M. M. Cui and G. X. Ji, *Pencils of pairs of projections*, Studia Math. 249 (2019), 117–141.

[9] C. Davis, *Separation of two linear subspaces*, Acta Sci. Math. (Szeged) 19 (1958), 172–187.

[10] J. Dixmier, *Position relative de deux variétés linéaires fermées dans un espace de Hilbert*, Revue Sci. 86 (1948), 387–399.

[11] C. E. Durán, L. E. Mata-Lorenzo and L. Recht, *Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra: Part I – minimal curves*, Adv. Math. 184 (2004), 342–366.

[12] P. R. Halmos, *Two subspaces*, Trans. Amer. Math. Soc. 144 (1969), 381–389.

[13] R. V. Kadison and J. R. Ringose, *Fundamentals of the Theory of Operator Algebras*, Vol. I, Academic Press, 1983.

[14] J. J. Koliha and V. Rakočević, *Fredholm properties of the difference of orthogonal projections in a Hilbert space*, Integral Equations Operator Theory 52 (2005), 125–134.

[15] A. S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Amer. Math. Soc., 1986.

[16] J. L. Massera and J. J. Schäffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.

[17] H. Porta and L. Recht, *Minimality of geodesics in Grassmann manifolds*, Proc. Amer. Math. Soc. 100 (1987), 464–466.

[18] I. Raeburn, *The relationship between a commutative Banach algebra and its maximal ideal space*, J. Funct. Anal. 25 (1977), 366–390.

[19] I. Raeburn and A. M. Sinclair, *The C^*-algebra generated by two projections*, Math. Scand. 65 (1989), 278–290.

[20] W. J. Shi, G. X. Ji and H. K. Du, *Pairs of orthogonal projections with a fixed difference*, Linear Algebra Appl. 489 (2016), 288–297.

Miaomiao Cui
School of Mathematics
Northwest University
Xi'an, 710119, P.R. China
E-mail: cuiye_mafei@nwu.edu.cn