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Real operator spaces and operator algebras

by

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Abstract. We verify that a large portion of the theory of complex operator spaces and operator algebras (as represented by the 2004 book by the author and Le Merdy for specificity) transfers to the real case. We point out some of the results that do not work in the real case. We also discuss how the theory and standard constructions interact with the complexification, which is often as important, but sometimes much less obvious. For example, we develop the real case of the theory of operator space multipliers and the operator space centralizer algebra, and discuss how these topics connect with complexification. This turns out to differ in some important details from the complex case. We also characterize real structure in complex operator spaces and give 'real' characterizations of some of the most important objects in the subject.

1. Introduction. Complex operator spaces are an important umbrella category containing C^*-algebras, operator systems, operator algebras, von Neumann algebras, and many other objects of interest in modern analysis and also in modern quantum physics (such as quantum information theory). They have an extensive theory (see e.g. [25, 40, 38, 7]) and have very important applications in all of these subjects.

Ruan initiated the study of real operator spaces in [42, 43], and this study was continued in [44, 15]. A real operator space may be viewed either as a real subspace of $B(H)$ for a real Hilbert space H, or abstractly as a vector space with a norm $\|\cdot\|_n$ on $M_n(X)$ for each $n \in \mathbb{N}$, satisfying the conditions of Ruan's characterization in [42]. Real structure occurs naturally and crucially in many areas of mathematics, as is also mentioned for example in the first paragraphs of [15] and [44], or in [41]; it also shows up in places in modern mathematical quantum physics (see e.g. [23] and references therein). Unfortunately there is not much literature on real operator spaces; the works

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cited at the start of the paragraph do not add up to a lot of pages. Even for Hilbert spaces, very rough comparisons with the amount of literature in the complex case mentioned in the introductions of [18, 15] are startling. A researcher working on a problem which involves real operator spaces or systems would have to reconstruct a large amount of the theory from scratch. Thus for contemporary applications in the areas of mathematics and physics mentioned above, it is of interest to understand the real case of the important results in (complex) operator space theory—what works and how it is connected to the complex case.

This is one goal of the present paper, to supply in some fashion such a resource. Since this is a daunting and not well-defined task we restrict ourselves in the last part of our paper mostly to the more modest target of checking the real case of the most important chapters in [7], and how the facts and structures there relate to the complexification, which is sometimes quite nontrivial. To not try the reader's patience we have attempted to be brief. Our proofs are often deceptively short, frequently referencing deep results and arguments. Some complementary results can be found in the companion paper [5].

We will begin our paper with several fundamental applications. Section 2 establishes the real case of some of the most important characterizations of objects of particular interest in operator spaces: operator algebras, operator modules, unital operator spaces, and operator systems. In Section 3 we characterize when a real operator space may be given a complex structure. For example, as we observe, the quaternions are a real operator space and a complex Banach space, but are not a complex operator space. We explain why this happens and what is needed to remedy it. We show also in this section that in contrast to the Banach space case (see e.g. [26]), every complex operator space X has a unique complex operator space structure up to complete isometry, and moreover such structures have a very simple classification.

Section 4 is devoted to extending to real operator spaces the deeper aspects of the theory of operator space multipliers and operator space centralizers (see [17, Section 7] for the latter in the complex case). This is one of the more profound parts of the 'completely isometric theory' of operator spaces. Some of this is needed in [6] which uses such multipliers and operator space centralizers in the real case. Indeed, it will be applicable, and probably critical, in situations in the future involving the real case of operator modules in the sense of Christensen and Sinclair, or more generally involving subspaces of $B(H)$ that are invariant under left multiplication by various operators on H. The real theory differs in a few important details from the complex case. It does not apply to some operator spaces (some spaces have no interesting M-ideals or multipliers), but is a very powerful tool in spaces that do possess some 'operator algebraic structure' in a loose sense. For example, Section 3 shows how multipliers are a main ingredient in the complex structure of real operator spaces. The centralizer is a generalization of the center of a C^*-algebra, and is also often key to understanding the ideal structure (or M-ideal structure in general settings). Thus one may expect the centralizer to play a role in future generalizations of ideal structure and centers, and possibly in directions such as Bryder's result relating the intersection property to the action on the center of the injective envelope [19].

Although there are other things there, Sections 5–9 mainly verify in a very economical format the real case of the remaining theory in Chapters 1–4 and 8 of [7], and establish how the basic constructions there interact with the complexification. We also check some selected results of Chapter 5 there. The Appendix of [7] consists of standard facts in functional analysis, the real case of almost all of which are well-known. The few that are unclear in the real case we have discussed in scattered locations below. Since the chapters in [7] build on each other, we will systematically start from the beginning (avoiding of course results already in the literature). As one would expect, many results in the real theory are proved just as in the complex case. We say almost nothing about such results, although we may sometimes mention a potentially confusing step in the proof. Similarly, many of the results follow swiftly by complexification, but sometimes we give additional details on how this should be done. Then, some results do require new proofs, usually because the complex argument involves facts that fail in the real case. Indeed, in addition to the 'real issues' listed in the introductions to [15, 5], arguments that involve selfadjoint or positive elements and their span, or the polarization identity, often fail in the real case. Some things are considerably more difficult in the real case. For example a hard question in the real case is the characterization of M-ideals in real TRO's; see [12]. We also point out the results that cannot be made to work in the real case.

Thus for example Section 9 may be viewed in some sense as a (very economical) complete theory of real C^*-modules, and in particular the real operator space aspects of that subject. Sections 5–7 verify in the real case aspects of the general theory of operator spaces (and in particular their duality and tensor products), operator algebras, and operator modules. These sections also establish functoriality of the complexification (see the next paragraph) for many important constructions (such as their tensor products), and develop some other new aspects of the complexification. Section 8 mostly concerns the real case of the few remaining topics from Chapter 4 of [7].

As we have alluded to earlier, not only do we want to check that the real versions of the complex theory work, but we also want to know what the complexifications of standard constructions are, and this is often as important but much less obvious. For example, it is important to know that the complexification of a particular operator space tensor product is a particular tensor product of the complexifications. More generally, it is important to know for which ‘constructions’ F in the theory we have $F(X)_{\rm c} = F(X_{\rm c})$ canonically completely isometrically. In some cases one has to be careful with the identifications. For example, just because one has proved that a complex space W is complex linearly completely isometrically isomorphic to $X_{\rm c}$, and that W is a reasonable complexification of a real space Y, one cannot conclude that then $Y \cong X$. In fact Y may not be isometric to X. This can happen even if X, Y, W are finite-dimensional C^*-algebras (even if $X = M_2(\mathbb{R}), W = M_2(\mathbb{C})$). Or as another example, if we have a complex space (such as a von Neumann algebra) with a unique operator space predual, and that predual is a reasonable complexification of a real space, then that real space need not be unique up to complete isometry (or even isometry).

We now turn to notation. The reader will need to be familiar with the basics of complex operator spaces and von Neumann algebras as may be found in early chapters of [7, 25, 38, 40], and e.g. [39] respectively. It would be helpful to also browse the (small) existing real operator space theory [42, 43, 44, 15]. Some basic real C^*-algebra theory may be found in [35, 2, 28]. The letters H, K are reserved for real Hilbert spaces. Every complex Hilbert space is a real Hilbert space with the ‘real part’ of the inner product. We sometimes write the complex number i as ι to avoid confusion with matrix subscripting. For us a *projection* in an algebra is always an orthogonal projection (so $p = p^2 = p^*$). A normed algebra A is *unital* if it has an identity 1 of norm 1, and a map T is unital if $T(1) = 1$. We say that A is *approximately unital* if it has a contractive approximate identity (cai). We write $X_{\rm sa}$ for the selfadjoint operators in X.

An operator space X comes with a norm $\| \cdot \|_n$ on $M_n(X)$. Sometimes the sequence $(\| \cdot \|_n)_{n \in \mathbb{N}}$ of norms is called the *operator space structure.* If $T : X \to Y$, we write $T^{(n)}$ for the canonical ‘entrywise’ amplification taking $M_n(X)$ to $M_n(Y)$. The completely bounded norm is $\|T\|_{\rm cb} = \sup_n \|T^{(n)}\|$, and T is completely contractive if $\|T\|_{\rm cb} \leq 1$. A map T is said to be *positive* if it takes positive elements to positive elements, and *completely positive* if $T^{(n)}$ is positive for all $n \in \mathbb{N}$. A *UCP map* is unital and completely positive.

An *operator space complexification* of a real operator space X is a pair $(X_{\rm c}, \kappa)$ consisting of a complex operator space $X_{\rm c}$ and a real linear complete isometry $\kappa : X \to X_{\rm c}$ such that $X_{\rm c} = \kappa(X) \oplus i\kappa(X)$ as a vector space. For simplicity, we usually identify X and $\kappa(X)$ and write $X_{\rm c} = X + iX$. We say that the complexification is *reasonable* if the map $\theta_X(x + iy) = x - iy$ on $X_{\rm c}$ (that is, $\kappa(x) + i\kappa(y) \mapsto \kappa(x) - i\kappa(y)$ for $x, y \in X$) is a complete isometry. Ruan [43] proved that a real operator space has a unique reasonable complexification $X_c = X + iX$ up to complete isometry. Sometimes we will call this a 'completely reasonable complexification', or a 'reasonable operator space complexification'. Conversely, if X is a real operator space with a complete isometry $\kappa : X \to Y$ into a complex operator space, then (Y, κ) is a reasonable operator space complexification of X if and only if Y possesses a conjugate linear completely isometric period 2 automorphism whose fixed points are $\kappa(X)$ [43, Theorem 3.2]. We will use the latter result repeatedly, as well as the notation θ_X.

We recall that the complexification may be identified up to real complete isometry with the operator subspace V_X of $M_2(X)$ consisting of matrices of the form

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \tag{1.1}$$

for $x, y \in X$.

We will need the fact that $M_{n,m}(X)_c = M_{n,m}(X_c)$ completely isometrically for an operator space X. This may be seen in several ways, for example by using the identification of X_c with V_X above. Or it may be proved by noting that it is sufficient to assume $m = n$ and $X = B(H)$. For a C^*-algebra B, we have $M_n(B)_c \cong M_n(B_c)$ $*$-isomorphically, and so isometrically. If $T : X \to Y$ is a real completely bounded map then $T_c(x + iy) = T(x) + iT(y)$ for $x, y \in X$. Ruan shows in [43, Theorem 2.1] that $\|T_c\|_{cb} = \|T\|_{cb}$. We include a short proof of this.

Proposition 1.1. *If $T : X \to Y$ is a real completely bounded* (*resp. completely isometric, complete quotient*) *linear operator between real operator spaces, then T_c is completely bounded* (*resp. a complete isometry, complete quotient*), *with $\|T_c\|_{cb} = \|T\|_{cb}$.*

Proof. Clearly $\|T\|_{cb} \leq \|T_c\|_{cb}$ since $\|T\|_{cb} = \|T_c|_X\|_{cb}$. For the other inequality, by the discussion around (1.1) we can identify $\|T_c^{(n)}([x_{ij} + \iota\, y_{ij}])\|$ with the norm of the matrix

$$\begin{bmatrix} T(x_{ij}) & -T(y_{ij}) \\ T(y_{ij}) & T(x_{ij}) \end{bmatrix},$$

for $[x_{ij} + \iota\, y_{ij}] \in M_n(X_c)$. This quantity is dominated by $\|T\|_{cb}$ times the norm of the matrix in (1.1), which is $\|[x_{ij} + \iota\, y_{ij}]\|$. Hence T_c is completely bounded and $\|T_c\|_{cb} \leq \|T\|_{cb}$. So $\|T_c\|_{cb} = \|T\|_{cb}$. If T is a complete isometry, then the matrix in the displayed equation has the same norm as the matrix in (1.1), so that T_c is a complete isometry. The 'complete quotient' assertion is generalized in [6, Proposition 5.5]. ∎

We showed in [5, Section 2] that $CB(X_c, Y_c)$ is a reasonable complexification of $CB(X, Y)$. In places, the reader will also need to be familiar with the theory of the injective envelope $I(X)$. In the complex case this may be found in e.g. [7, 25, 38]. The real case was initiated in [44], and continued in [5] (see also Section 8, although this is not used much in sections before that).

A real *unital operator space* is an operator space X with a distinguished element $u \in X$ and a real complete isometry $T : X \to B(H)$ with $T(u) = I_H$. A real *operator system* is a unital operator space X with an involution and a real complete isometry $T : X \to B(H)$ with $T(u) = I_H$, which is selfadjoint (that is, $T(x^*) = T(x)^*$). The diagonal $\Delta(X) = X \cap X^*$ of a unital real operator space X is a well-defined real operator system independent of representation, as in the complex case. This follows e.g. by the discussion after Corollary 2.5 in [15]: Suppose that $T : X \to Y$ is a surjective unital complete isometry between real unital operator spaces with $I_H \in X \subset B(H)$ and $I_K \in Y \subset B(K)$. Then the canonical extension $\tilde{T} : X + X^* \to B(K) : x + y^* \mapsto T(x) + T(y)^*$ is well-defined for $x, y \in X$, is selfadjoint and is a completely isometric complete order embedding onto $Y + Y^*$. (Whether this is isometric when T is a unital isometry was asked in [15] when X and Y are in addition operator algebras. In fact, this is false even in the complex case. Simple counterexamples may be manufactured using the $\mathcal{U}(X)$ construction from [7, Section 2.2].) We see that $T(X \cap X^*) = Y \cap Y^*$, and that T is selfadjoint on $X \cap X^*$. Indeed, the results in [7, 1.3.4–1.3.7] hold in the real case, as does Choi's Propositions 1.3.11 and 1.3.12 and the Choi–Effros Theorem 1.3.13 there (by going to the complexification if necessary). Most of this was established in [44, 15] and [42, Section 4]. Example 5.10 in [15] shows that 1.3.8 in [7] fails in the real case, although it is true for unital complete contractions (by e.g. [15, 2.4, 2.5]). For the Paulsen system we have $\mathcal{S}(X)_{\mathrm{c}} \cong \mathcal{S}(X_{\mathrm{c}})$; indeed, this is inherited from the relation $M_2(B(H))_{\mathrm{c}} \cong M_2(B(H)_{\mathrm{c}})$.

Perhaps shockingly, the complexification A_c of an (even unital) operator algebra A need not be well-defined up to isometric (as opposed to complete isometric) isomorphism. For that reason operator algebras and their complexification are almost always treated here in the operator space setting. The diagonal $\Delta(A) = A \cap A^*$ of a real operator algebra A (with possibly no kind of identity) is a well-defined operator algebra independent of representation, as in the complex case. Indeed, by [15, Theorem 2.6], a contractive homomorphism $\pi : A \to B$ between operator algebras takes the diagonal $\Delta(A)$ $*$-isomorphically onto a closed C^*-subalgebra of $\Delta(B)$. Thus the results in [7, 2.1.2] hold in the real case. For an operator algebra in $B(H)$ or a unital operator space we have $\Delta(X_{\mathrm{c}}) = X_{\mathrm{c}} \cap X_{\mathrm{c}}^* = \Delta(X)_{\mathrm{c}}$ in $B(H)_{\mathrm{c}}$. Because the selfadjoint elements do not necessarily span a real C^*-algebra (or may be all of the C^*-algebra), one needs to be careful in places. Thus $\Delta(A)$

need not be the span of the selfadjoint elements (nor of the projections if A is also weak*-closed and not commutative), unlike in the complex case.

2. Abstract characterizations. Complex operator space theory began with Ruan's abstract characterization of complex operator spaces [25, Theorem 2.3.5]. Ruan also gave the matching characterization of real operator spaces in [42]. Similarly, there is a well-known abstract characterization of complex operator algebras with contractive approximate identity (cai) [7, Theorem 2.3.2], and its matching real version is in [44]. Real 'dual operator algebras' are characterized in Theorem 6.3 below. Nonunital operator algebras are characterized up to completely bounded isomorphism in [7, Section 5.2], and the real case follows immediately by complexification. Nonunital real operator algebras may be characterized up to complete isometry as follows.

THEOREM 2.1 (Real version of the Kaneda–Paulsen theorem). *Let X be a real operator space. The real algebra products on X for which there exists a real linear completely isometric homomorphism from X onto a real operator algebra, are in a bijective correspondence with the elements $z \in \mathrm{Ball}(I(X))$ such that $Xz^*X \subset X$ in the ternary product of $I(X)$ (recall that the injective envelope $I(X)$ is a real ternary subsystem of $I(X_c)$* [5, Section 4]*). For such z the associated operator algebra product on X is xz^*y.*

Proof. As in [9, proof of Theorem 5.2], one direction, and the last statement, follow from [14, Remark 2, p. 194], viewing $I(X)$ as a ternary system in $B(H)$, and taking V there to be z^*. For the other direction, suppose that X is a real operator algebra, and that (by the complex version of the present theorem) the canonical operator algebra product on X_c is given by xz^*y for some $z \in \mathrm{Ball}(I_{\mathbb{C}}(X_c)) = \mathrm{Ball}(I_{\mathbb{C}}(X)_c)$ (see [5, Section 4] for the last identification). Write $z = z_1 + iz_2$ with $z_i \in I(X)$. Since $Xz^*X \subset X$ it follows that $Xz_2^*X = 0$. Therefore $X_c z_2^* X_c = 0$ and $z_2 = 0$ by [7, Theorem 4.4.12]. So $z \in \mathrm{Ball}(I(X))$. The bijectivity follows similarly from [7, Theorem 4.4.12] as in [9, proof of Theorem 5.2]. ∎

Real unital operator spaces. Appropriate variants of the characterizations of unital operator spaces from [8, 9] hold in the real case.

THEOREM 2.2. *Let X be a real operator space and $u \in X$ with $\|u\| = 1$. The following are equivalent:*

(1) *(X, u) is a real unital operator space.*
(2) *$\|[u_n \ \ x]\| \geq \sqrt{2}$ and $\left\|\begin{bmatrix} u_n \\ x \end{bmatrix}\right\| \geq \sqrt{2}$ for all $n \in \mathbb{N}$ and $x \in M_n(X)$ with $\|x\| = 1$.*
(3) *$\left\|\begin{bmatrix} u_n & -x \\ x & u_n \end{bmatrix}\right\| \geq \sqrt{1 + \|x\|}$, for all $n \in \mathbb{N}$ and $x \in M_n(X)$.*

Here, u_n is the diagonal matrix $u \otimes I_n$ in $M_n(X)$ with u in each diagonal entry.

Proof. (1)⇒(2) This is the easy direction, and follows from the real C^*-identity (as in [8]).

(1)⇒(3) Assuming (1), $(X_{\rm c}, u)$ is a complex unital operator space. The left side of (3) is unchanged if we replace x by $-x$ (as may be seen be pre- and post-multiplying by the diagonal matrix with entries $1, -1$). Thus by [8, Lemma 2.2] the left side of (3) (with respect to $X_{\rm c}$) is $\max\{\|u_n + i^k x\| : k = 0, 1, 2, 3\}$. This is $\geq \sqrt{1 + \|x\|}$ by [8, Theorem 1.1].

(2)⇒(1) Let $v : X_{\rm c} \to M_2(X)$ be the complete isometry yielding the matrix in (1.1). If $x \in M_n(X_{\rm c}) = M_n(X)_{\rm c}$ with $\|x\| = 1$, then $\|v^{(n)}(x)\| = 1$. Applying v we get

$$\|[u_n \;\; x]\| = \|[u_{2n} \;\; v^{(n)}(x)]\| \geq \sqrt{2}.$$

Thus by the characterization in [8] there exists a linear complete isometry $T : X_{\rm c} \to B(H)$ with $T(u) = I_H$. Restricting to X we get the result. (We are also using the fact that complex Hilbert spaces are real Hilbert spaces, as we do in many places in this paper.)

(3)⇒(1) This is somewhat similar. Let $x \in M_n(X_{\rm c})$ with $x = y + iz$ for $y, z \in M_n(X)$. By reversing the argument for (1)⇒(3) above we have

$$\max\{\|u_n + i^k x\| : k = 0, 1, 2, 3\} = \left\| \begin{bmatrix} u_n & -x \\ x & u_n \end{bmatrix} \right\|.$$

Applying v as in the last paragraph, this equals

$$\left\| \begin{bmatrix} u_{2n} & -v^{(n)}(x) \\ v^{(n)}(x) & u_{2n} \end{bmatrix} \right\| \geq \sqrt{1 + \|v^{(n)}(x)\|} = \sqrt{1 + \|x\|}.$$

Thus $(X_{\rm c}, u)$ is a complex unital operator space by [8, Theorem 1.1], and we finish as in the last paragraph. ∎

Most of the other characterizations in the papers [8, 9] hold in the real case, for example the characterization of isometries, coisometries and unitaries. The characterization of these objects in [8, Theorem 2.4] in the real case follows by a modification of an argument in the last proof. Also, Lemma 5.1, Theorem 5.5 and Corollary 5.6 of [9] hold in the real case. We plan to give more details elsewhere (see "Added in proof"). Thus unital real operator algebras can be characterized as real operator spaces X with a coisometry u and a bilinear map $m : X \times X \to X$ such that $m(x, u) = x$ for $x \in X$, and such that

$$\left\| \begin{bmatrix} m(x, a_{ij}) \\ b_{ij} \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} a_{ij} \\ b_{ij} \end{bmatrix} \right\|, \qquad [a_{ij}], [b_{ij}] \in M_n(X),$$

for $n \in \mathbb{N}$ and $x \in \mathrm{Ball}(X)$. This 'improves' the characterization of unital operator algebras mentioned in the first paragraph of this section.

Real operator systems. The metric characterizations of operator systems in e.g. [8, Theorem 3.4] and [9, Proposition 4.2] are valid in the real case by the same proofs. We state only the latter characterization in the real case. We note that this uses the real case of the proof of the former characterization (i.e. [8, Theorem 3.4]), which does not even involve an involution on the space.

THEOREM 2.3. *Let X be a real operator space which possesses a real linear period 2 involution $*$ with $\|x^*\| = \|x\|$ for $x = [x_{ij}] \in M_n(X)$, $n \in \mathbb{N}$. Here $x^* = [x_{ji}^*]$. Let $u \in \mathrm{Ball}(X)$ with $u = u^*$. Then there exists a real $*$-linear complete isometry from X onto a real operator system with $T(u) = 1$ if and only if*

$$\left\| \begin{bmatrix} u_n & x \\ -x^* & u_n \end{bmatrix} \right\| = \sqrt{1 + \|x\|^2}, \qquad n \in \mathbb{N},\ x \in M_n(X).$$

Here $u_n = u \otimes I_n$ as in Theorem 2.2.

Proof. We adapt the complex proof of [9, Proposition 4.2]. This proof is quite complicated, so that even utilizing the hints below in conjunction with a careful study of each step of [9, Proposition 4.2] it may take the reader some time to master this adaption. As stated above, this adaption uses the real version of the proof of the characterization [8, Theorem 3.4], but setting $y = y_n = -x^*$ there, where $*$ is the involution on X. This proof involves suprema over states on $M_2(B(H))$. We may use vector states here, using the fact that vector states norm elements with $a = a^*$ even in the real case. (The latter adjoint is in $M_2(B(H))$, and is not yet related to the involution in the statement of the present theorem.) We also need to appeal to Theorem 2.2(2) above in place of its complex variant in the proof of [9, Proposition 4.2]. ∎

There are characterizations of complex operator systems in terms of the positive cones in $M_n(X)$ in the literature, most notably the Choi–Effros characterization [24] (but see also e.g. [34, Section 7] for a recent characterization in terms of noncommutative convexity/quasistates). We will not discuss these here, except to say that in the real case an operator system may have no nontrivial positive elements so that the obvious formulations of such a result in the real case fail. For this reason, no doubt some authors study a subclass of the operator systems which do have large positive cones; however, this would exclude some of the most interesting real unital C^*-algebras.

Real operator modules. The real version of the Christensen–Effros–Sinclair theorem characterizing operator modules and bimodules (the complex version may be found e.g. in [7, Theorem 3.3.1]) holds easily by complexification:

THEOREM 2.4. *Let A and B be approximately unital real operator algebras, and let X be a real operator space which is a nondegenerate bimodule over A and B, satisfying*

$$\|\alpha x\beta\| \leq \|\alpha\| \, \|x\| \, \|\beta\|, \qquad n \in \mathbb{N}, \, x \in M_n(X), \, \alpha \in M_n(A), \, \beta \in M_n(B).$$

Then there exist real Hilbert spaces H and K, a real linear completely isometric map $\Phi\colon X \to B(K,H)$, and completely contractive nondegenerate real linear representations θ of A on H, and π of B on K, such that

$$\theta(a)\Phi(x) = \Phi(ax) \quad \text{and} \quad \Phi(x)\pi(b) = \Phi(xb), \qquad a \in A, \, b \in B, \, x \in X.$$

Therefore X is completely A-B-isometric to the concrete operator A-B-bimodule $\Phi(X)$. Moreover, Φ, θ, π may all be chosen to be completely isometric, and such that $H = K$. If $A = B$ then one may choose, in addition to all the above, $\pi = \theta$.

Proof. It is easy to see that $X_{\rm c}$ is an $A_{\rm c}$-$B_{\rm c}$-bimodule algebraically. We claim that $\|\alpha x\| \leq \|\alpha\| \, \|x\|$ for $x \in M_n(X_{\rm c})$ and $\alpha \in M_n(A_{\rm c})$. This follows by a matrix calculation whose details we omit since it is similar to the first five lines of the proof that (ii) implies (iii) in [44, Theorem 3.2]. Similarly, $\|x\beta\| \leq \|x\| \, \|\beta\|$. Hence the displayed inequality as stated holds in the complexified spaces. Thus $X_{\rm c}$ is a nondegenerate bimodule over $A_{\rm c}$ and $B_{\rm c}$ satisfying the conditions of the complex version of the theorem. Our result follows from that version, viewing all complex spaces as real spaces, and restricting the ensuing Φ, θ, π to real linear maps on X, A, B respectively, as in earlier proofs. ∎

3. Complex structure in operator spaces. This section could be read after Section 4 in one sense, since the proofs (not the theorem statements) here use the perspective of that theory. However, we have placed it here in view of its importance, and because of the sheer length and technicality of Section 4. Before the first theorem, we will summarize in a more or less selfcontained way the few results we need from [44] for the proofs, but some readers unfamiliar with the existing complex theory of multipliers will possibly need to look at [44] or Section 4 at times during the proofs.

By a *complex operator space structure* (resp. *complex Banach space structure*) on a real operator (resp. Banach) space X we mean an action of i on X by a real linear map $J : X \to X$ such that X (with unchanged norms) is a complex operator (resp. Banach) space with scalar product $(s + it)x = sx + tJ(x)$ for $x \in X$ and $s, t \in \mathbb{R}$. Any complex operator space of course has a canonical complex operator space structure corresponding to

$J(x) = ix$ for $x \in X$, in the notation above. It is well-known and obvious that a map $J : X \to X$ on a real Banach space corresponds to a complex Banach space structure on X (with $ix = Jx$) if and only if $J^2 = -I$ and $x \mapsto sx + tJ(x)$ is an isometry whenever $s = \cos\theta$, $t = \sin\theta$, that is, whenever $s^2 + t^2 = 1$. Write $u_\theta = I\cos\theta + J\sin\theta$. Since u_θ has inverse $u_{-\theta} = \cos\theta - J\sin\theta$, it follows that u_θ is an isometry for all θ if and only if it is a contraction for all θ.

A real operator space which is a complex Banach space need not be a complex operator space. Indeed, a real C^*-algebra which is a complex Banach space need not be a complex C^*-algebra. A counterexample is the quaternions, for which it is easy to see the C^*-algebra statement, for example from the fact that all two-dimensional complex C^*-algebras are commutative. We argue in a remark below that the quaternions are not a complex operator space, although they are obviously a complex Banach space because $\mathbb{C}$ is embedded as a subalgebra. In this section we explain why this happens and what is needed to remedy it. In particular, what more is needed for a real operator space X which is also a complex Banach space to be a complex operator space?

Sharma defines in [44, Section 5] (see also Section 4 below for more details if desired) the real version $\mathcal{M}_\ell^{\mathbb{R}}(X)$ of the left operator space multiplier algebra. We suppress the superscript here if the real setting is understood. We have $\mathcal{M}_\ell(X) \subset CB(X)$, but with a generally different norm. Sharma also defines there the C^*-algebra of 'left adjointable multipliers' $\mathcal{A}_\ell(X)$, which coincides with the diagonal $\Delta(\mathcal{M}_\ell(X)) = \mathcal{M}_\ell(X) \cap \mathcal{M}_\ell(X)^*$ (see Section 1) inside the C^*-algebra $pI(\mathcal{S}(X))p$ in the notation of [44] (more details on this are given in Section 4 below). Similarly, one may define the real right multiplier algebra and 'right adjointable multipliers' $\mathcal{M}_r(X)$ and $\mathcal{A}_r(X)$. We define the real operator space centralizer algebra $Z(X)$ (or $Z_{\mathbb{R}}(X)$ when we want to emphasize that this is the real case) to be the set $\mathcal{A}_\ell(X) \cap \mathcal{A}_r(X)$ in $CB(X)$. The centralizer is a real commutative C^*-algebra. This is because $\mathcal{A}_\ell(X)$ and $\mathcal{A}_r(X)$ commute (as may be seen by their canonical representations in the space $I(\mathcal{S}(X))$ mentioned above). Finally, X is an operator bimodule over $Z(X)$ in the sense of Theorem 2.4 (for basically the same reason as in the last line; more details on any of this if needed are given in Section 4 below).

Theorem 3.1. *A real linear map $J : X \to X$ on a real operator space corresponds to a complex operator space structure on X (with $\iota x = Jx$) if and only if*

$$J^2 = -I \quad \textit{and} \quad \|[d_i x_{ij}]\|_n \le \|[x_{ij}]\|_n \quad \textit{and} \quad \|[d_j x_{ij}]\|_n \le \|[x_{ij}]\|_n,$$

for all $[x_{ij}] \in M_n(X)$ and $d_1, \ldots, d_n$ maps of form u_θ defined in the second *paragraph of this section (that is, $d_k = u_{\theta_k}$ for real θ_k), and for all $n \in \mathbb{N}$. Indeed, in the $n > 1$ case of the above displayed inequalities only the values $\theta = 0$ and $\pi/2$ are needed in this characterization; that is, $d_i = 1$, or $d_i = J = \iota$.*

Proof. The necessity of the condition is obvious. Conversely, given such a map J, from the case $n = 1$ we see that X is a complex Banach space. We may therefore write $Jx = ix$ henceforth. We will not need this here, but since u_θ has inverse $u_{-\theta}$ it is easy to see that the displayed inequalities are equivalent to

$$\|[d_i x_{ij}]\| = \|[d_j x_{ij}]\| = \|[x_{ij}]\|, \qquad [x_{ij}] \in M_n(X).$$

From the displayed inequalities it follows that if $x, y \in M_n(X)$ and

$$z = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad w = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$$

then by the case of the displayed inequalities where the d_k are i or 1 we have

$$\left\| \begin{bmatrix} ix \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} ix & 0 \\ y & 0 \end{bmatrix} \right\| \leq \|w\| = \|z\|.$$

Thus by [44, Theorem 5.4] we see that the map $Jx = ix$ is a contraction in $\mathcal{M}_\ell^{\mathbb{R}}(X)$. A similar argument with $[xi \;\; y]$ shows that $J \in \mathcal{M}_r^{\mathbb{R}}(X)$. Since J is a contraction with $J^2 = -I$ it follows that J is a unitary in the 'diagonal' of $\mathcal{M}_\ell^{\mathbb{R}}(X)$ mentioned above, with adjoint $-J$. Thus, J is left adjointable, and similarly it is right adjointable. Thus $J \in Z_{\mathbb{R}}(X)$. The map $\lambda : \mathbb{C} \to Z_{\mathbb{R}}(X)$ with $\lambda(s+it) = sI + tJ$ is a homomorphism. Indeed, λ is a $*$-homomorphism, so it is contractive, and hence completely contractive [15]. So X is a real operator $\mathbb{C}$-bimodule since X is an operator $Z_{\mathbb{R}}(X)$-bimodule (see the discussion above Proposition 4.3 if necessary). In particular, X satisfies Ruan's condition characterizing complex operator spaces: $\|\alpha x \beta\| \leq \|\alpha\| \, \|x\| \, \|\beta\|$ for $\alpha, \beta \in M_n(\mathbb{C})$ and $x \in M_n(X)$. ∎

Remark. 1. The quaternions $\mathbb{H}$ are a complex Banach space and a real operator space, but not a complex operator space with the 'canonical action' coming from the copy of $\mathbb{C}$ as a real subalgebra. Indeed, it is well-known that the complexification of $\mathbb{H}$ is $M_2(\mathbb{C})$, which has no nontrivial (complex or real) ideals. A nonzero real ideal would have to contain one of the matrix units e_{ij} (since $I = e_{11} + e_{22}$ and $x = IxI$), but $M_2 e_{ij} M_2 = M_2$. If $\mathbb{H}$ was a complex operator space then its complexification would be $\mathbb{H} \oplus^\infty \overline{\mathbb{H}}$ (see Lemma 5.1), which has a nontrivial ideal.

We make a stronger claim: the real operator space $\mathbb{H}$ has no complex operator space structure at all, that is, there is no action $\mathbb{C} \times \mathbb{H} \to \mathbb{H}$ extending the action of the reals and making $\mathbb{H}$ a complex operator space. Indeed, according to the examples above Theorem 4.4, $Z_{\mathbb{R}}(\mathbb{H})$ is the center $\mathbb{R}\,1$ of $\mathbb{H}$, whereas if $\mathbb{H}$ had a complex structure then $Z_{\mathbb{R}}(\mathbb{H})$ would have real dimension > 1 by Corollary 3.2 below, or by the last assertion in Theorem 4.5.

The quaternions are however a real 'operator bimodule' over the complex numbers, in the sense of operator space theory (see Theorem 2.4 above for example). However, unlike the situation for complex operator spaces, the left or right actions of $\mathbb{C}$ are not as 'centralizers', and $iw \neq wi$ in general, so things are quite subtle and different to what we are used to for complex spaces.

2. One may rephrase the theorem as: if X is a real operator space and a complex Banach space, then X is a complex operator space if and only if $\|dx\|_n \leq \|x\|_n$ and $\|xd\|_n \leq \|x\|_n$, for all $x \in M_n(X)$ and diagonal matrices d with entries in $\{1, i\}$.

One may also prove part of this theorem using matrix theory to verify Ruan's condition in the last line of the proof. Any complex matrix may be written via the polar decomposition and diagonalization as a product UEU' for complex unitary matrices U, U' and a diagonal matrix E. It is known that any complex unitary matrix U may be written as KDV where K and V are real orthogonal matrices and D is diagonal and unitary (see e.g. [27, Theorem 5.1]). Putting these facts together gives the theorem (except for its final assertion about 1 and i).

Corollary 3.2. *A real operator space X is a complex operator space if and only if it possesses a real linear antisymmetry $J : X \to X$ (antisymmetry means that $J^2 = -I$) which is also a real operator space centralizer.*

It follows that a real operator space X has a complex operator space structure if and only if its real centralizer algebra $Z(X)$ has a complex Banach space structure, because this happens if and only if $Z(X)$ possesses an antisymmetry. We have some interesting and at present exciting-looking applications of these results that we hope to present in a future work. We mention a sample result:

Corollary 3.3. *Let X be a real operator space such that the real operator space bidual X^{**} has a complex operator space structure. Then X has a complex operator space structure.*

There is a similar result for the dual in place of the bidual, but it is more complicated to state.

A second question one may ask about complex structure is whether we can classify all of the scalar multiplications $\mathbb{C} \times X \to X$ on a real operator space that make X a complex operator space. Note that if X and Y are complex operator spaces, then $X \oplus^{\infty} Y$ is a complex operator space with a 'twisted' complex multiplication given by $i \cdot (x, y) = (ix, -iy)$. Surprisingly, it turns out that the latter *is the only* example of a complex structure on a complex operator space space besides the given one:

THEOREM 3.4. *A complex operator space X has a unique complex operator space structure up to complete isometry. Moreover, for any other complex operator space structure on X there exist two complex* (*with respect to the first complex structure*) *subspaces X_+ and X_- of X with $X = X_+ \oplus^\infty X_-$ completely isometrically such that the second complex multiplication of $\lambda \in \mathbb{C}$ with $x_+ + x_-$ is $(\lambda x_+) + (\bar{\lambda} x_-)$ for $x_+ \in X_+$ and $x_- \in X_-$.*

Proof. Let X be a complex operator space with a second scalar multiplication making it a complex operator space. Let $u : X \to X$ be multiplication by i in the second scalar multiplication. As we saw in the last theorem, multiplication by i in the first product is in $Z_{\mathbb{R}}(X)$, and similarly $u \in Z_{\mathbb{R}}(X)$. Now $Z_{\mathbb{R}}(X)$ is a real commutative C^*-algebra. In such an algebra an element with $f^2 = -1$ may be viewed as a function (see [35, end of Section 5.1, Example (3)]). Then $f(x) = \pm i$, so that the involution/adjoint of f is $-f$. Thus in $Z_{\mathbb{R}}(X)$ the element $s = -iu$ is selfadjoint with square I.

Therefore $p = (I + s)/2$ is a projection in $Z_{\mathbb{R}}(X)$. Let $X_+ = pX$ and $X_- = (1 - p)X$. We claim that $X = X_+ \oplus^\infty X_-$ completely isometrically and as complex operator spaces. To see this recall that p commutes with i, so it is $\mathbb{C}$-linear. By [44, Theorem 5.4] the map τ_p in [7, Theorem 4.5.15] is completely contractive on $C_2(X)$. By the latter theorem, p is a complex left M-projection. Similarly it is a right M-projection, and a (complex) complete M-projection by [7, Proposition 4.8.4]. This proves our claim.

Now $up = is(I + s)/2 = ip$, and similarly $up^\perp = -ip^\perp$. We may define a real complete isometry $v : X \to X$ by $v(px + (1-p)y) = px - (1-p)y$. Then

$$v(uz) = v(u(px + (1 - p)y)) = v(ipx - i(1 - p)y) = ipx + i(1 - p)y = iz$$

for $z = px + (1 - p)y$. ∎

REMARK. In particular, if X is the complexification of a real operator space Y, then $Z_{\mathbb{C}}(X) = Z_{\mathbb{R}}(X) = Z_{\mathbb{R}}(Y)_{\mathrm{c}}$ by Lemma 4.2 and Theorem 4.5 below. If θ is the multiplication by i in a different complex operator space scalar multiplication, then by the last theorem we obtain a projection in $Z(X) = Z(Y)_{\mathrm{c}}$. If $p = q_{\mathrm{c}}$ for projection $q \in Z(Y)$ then we may obtain an M-summand decomposition in Y corresponding to the adjusted complex structure as in Theorem 3.4. However, there may be projections in $Z(X)$ that are not of this form.

4. Operator space multipliers and the operator space centralizer. This section is a little more technical and requires a familiarity with the theory of complex operator space multipliers and centralizers (as in e.g. [7, 17]). We will also use the theory of the real injective envelope $I(X)$ initiated in [44] and continued in [5]. As pointed out in [44], the important Construction 4.4.2 in [7] relating the injective envelope $I(X)$ to the injective envelope $I(\mathcal{S}(X))$ of the Paulsen system is valid in the real case. If $p = 1 \oplus 0$ and $q = 0 \oplus 1$ then as usual I_{11} and I_{22} are the corners $pI(\mathcal{S}(X))p$ and $p^{\perp}I(\mathcal{S}(X))p^{\perp}$, and $I(X) = pI(\mathcal{S}(X))p^{\perp}$. It is shown in [5, Section 4] that $I(X)_{\mathrm{c}} = I(X_{\mathrm{c}})$. A similar relation holds for the other two important 'corners' of $I(\mathcal{S}(X))$:

LEMMA 4.1. *For a real operator space X we have $I_{11}(X_{\mathrm{c}}) \cong (I_{11}(X))_{\mathrm{c}}$ as unital C^*-algebras. Similarly, $I_{22}(X_{\mathrm{c}}) \cong (I_{22}(X))_{\mathrm{c}}$.*

Proof. Indeed, $I(\mathcal{S}(X))_{\mathrm{c}} \cong I(\mathcal{S}(X)_{\mathrm{c}}) \cong I(\mathcal{S}(X_{\mathrm{c}}))$ as unital C^*-algebras, via a $*$-homomorphism that preserves the projection $p = 1 \oplus 0 \in \mathcal{S}(X)$. Then

$$I_{11}(X_{\mathrm{c}}) = pI(\mathcal{S}(X_{\mathrm{c}}))p \cong pI(\mathcal{S}(X))_{\mathrm{c}}p \cong (pI(\mathcal{S}(X))p)_{\mathrm{c}} = (I_{11}(X))_{\mathrm{c}}.$$

Similarly, $I_{22}(X_{\mathrm{c}}) \cong (I_{22}(X))_{\mathrm{c}}$. ∎

We defined the left and right real operator space multiplier algebras $\mathcal{M}_{\ell}(X)$ and $\mathcal{M}_r(X)$ just above Theorem 3.1, as well as the real C^*-algebras of left and right adjointable multipliers $\mathcal{A}_{\ell}(X)$ and $\mathcal{A}_r(X)$, and the real operator space centralizer algebra $Z(X)$. Giving a little more detail from [44], we define matrix norms as in the complex case (see [7, 4.5.3]), i.e. so that $M_n(\mathcal{M}_{\ell}(X)) \cong \mathcal{M}_{\ell}(C_n(X))$ isometrically. Also, $T \in \mathrm{Ball}(\mathcal{M}_{\ell}(X))$ if and only if there exists a real linear complete isometry $j : X \to B(H)$ for a real Hilbert space H and an operator $S \in \mathrm{Ball}(B(H))$ with $j(Tx) = Sj(x)$ for all $x \in X$. Complexifying this, if $T \in \mathcal{M}_{\ell}(X)$ we see that $T_{\mathrm{c}} \in \mathcal{M}_{\ell}(X_{\mathrm{c}})$. The most useful characterization, however, is the τ_T criterion which in the real case is [44, Theorem 5.4]. (Sharma does not state this, but the real versions of all statements in [7, 4.5.1–4.5.9 and 4.5.12–4.5.13] are valid by the same arguments, with the possible exception of [7, Theorem 4.5.2(v)]. Items 4.5.14–4.5.15 of [7] are in [44], and we will check 4.5.10 and 4.5.11 below. We will discuss Banach–Stone theorems such as 4.5.13 briefly in Section 8.)

As in the complex case, we obtain a completely contractive one-to-one homomorphism $\mathcal{M}_{\ell}(X) \to CB(X)$, and a completely isometric homomorphism $\mathcal{M}_{\ell}(X) \to I_{11}(X)$. Indeed, we have

$$\mathcal{M}_{\ell}(X) \cong \{a \in I_{11}(X) : aj(X) \subset j(X)\},$$

where $j : X \to I(X)$ is the canonical inclusion (that is, $(I(X), j)$ is an injective envelope of X).

LEMMA 4.2. *For a real operator space X we have $\mathcal{M}_{\ell}(X_{\mathrm{c}}) \cong \mathcal{M}_{\ell}(X)_{\mathrm{c}}$ completely isometrically as operator algebras, and $T \mapsto T_{\mathrm{c}}$ is the canonical map of $\mathcal{M}_{\ell}(X)$ into this complexification. Similarly for the spaces of* *right multipliers. Moreover, $\mathcal{A}_\ell(X_c) \cong \mathcal{A}_\ell(X)_c$ $*$-isomorphically. The operator space centralizer algebra $Z(X)$ is a commutative unital real C^*-algebra, and $Z(X_c) \cong Z(X)_c$ $*$-isomorphically.*

Proof. We may identify $\mathcal{M}_\ell(X_c)$ with

$$\{a \in I_{11}(X_c) : aj_c(X_c) \subset j_c(X_c)\} \cong \{b + ic \in (I_{11}(X))_c : (b + ic)j_c(X_c) \subset j_c(X_c)\}.$$

The latter inclusion is equivalent to b and c being in $\mathcal{M}_\ell(X)$. It follows (if necessary, thinking of $b+ic$ as the usual 2×2 matrix as in (1.1)) that the right side of the last displayed equation is $\mathcal{M}_\ell(X)_c$. That is, $\mathcal{M}_\ell(X_c) \cong \mathcal{M}_\ell(X)_c$ completely isometrically as operator algebras. It is easy to see the assertion regarding T_c: if $Tj(x) = aj(x)$ for $a \in I_{11}(X)$ then $T_c\,(j_c)(x+iy) = aj_c(x) + iaj_c(y)$. Similar arguments hold in the $\mathcal{M}_r(X)$ case.

Using the fact $\Delta(A_c) = \Delta(A)_c$ from the introduction concerning the diagonal operator algebra, we have

$$\mathcal{A}_\ell(X)_c = \Delta(\mathcal{M}_\ell(X))_c \cong \Delta(\mathcal{M}_\ell(X)_c) \cong \Delta(\mathcal{M}_\ell(X_c)) = \mathcal{A}_\ell(X_c),$$

and

$$Z(X_c) = \mathcal{A}_\ell(X_c) \cap \mathcal{A}_r(X_c) = \mathcal{A}_\ell(X)_c \cap \mathcal{A}_r(X)_c = Z(X)_c.$$

Moreover, for $T \in Z(X)$ the involution of T in $\mathcal{A}_\ell(X)$ coincides with its involution in $\mathcal{A}_r(X)$, since the analogous statement is true in $Z(X_c)$ [17]. So $Z(X)$ is a real C^*-algebra. ∎

We have $\mathcal{M}_\ell(X) = \{T \in B(X) : T_c \in \mathcal{M}_\ell(X_c)\}$. Indeed, for example τ_T is clearly completely contractive if and only if $\tau_{T_c} = (\tau_T)_c$ is completely contractive. The map $T \mapsto T_c$ restricts to a faithful contractive homomorphism, hence $*$-homomorphism, $\mathcal{A}_\ell(X) = \Delta(\mathcal{M}_\ell(X)) \to \mathcal{A}_\ell(X_c) = \Delta(\mathcal{M}_\ell(X))$. Thus if $T \in \mathcal{A}_\ell(X)$ then $(T_c)^* = (T^*)_c$. It is easy to see that $\mathcal{A}_\ell(X) = \{T \in B(X) : T_c \in \mathcal{A}_\ell(X_c)\}$, and a similar relation holds for $Z(X)$. Indeed, $T_c \in Z(X_c) = \mathcal{A}_\ell(X_c) \cap \mathcal{A}_r(X_c)$ iff $T \in \mathcal{A}_\ell(X) \cap \mathcal{A}_r(X)$, and iff $T \in Z(X)$.

As in the complex case, X is an operator bimodule (or h-bimodule) over all of these spaces in the sense of [7, Section 3.1]. Indeed, X is an operator $\mathcal{M}_\ell(X)$-$\mathcal{M}_\ell(X)$-bimodule, which in turn can be viewed as an $\mathcal{M}_\ell(X)$-$\mathcal{M}_\ell(X)$-subbimodule of X_c. The actions of $\mathcal{A}_\ell(X)$, $\mathcal{A}_r(X)$ and $Z(X)$ on X are restrictions of the $\mathcal{M}_\ell(X)$-$\mathcal{M}_\ell(X)$-bimodule action. These bimodules can as usual all be concretely viewed at any time as subspaces of a C^*-algebra, e.g. of the C^*-algebra which is the injective envelope of the Paulsen system. For example, X regarded as an operator $\mathcal{M}_\ell(X)$-$\mathcal{M}_\ell(X)$-bimodule may be viewed in the 1-2 corner of $I(\mathcal{S}(X))$, with $\mathcal{M}_\ell(X)$ a subalgebra of the 1-1 corner I_{11} as above. The action $T(x)$, for $T \in \mathcal{M}_\ell(X)$ and $x \in X$, is identified in the usual way with the product of the two 2×2 matrices in $I(\mathcal{S}(X))$ corresponding to the copy of T in I_{11} and the copy of X in the 1-2 corner.

There is a subtlety, however, with the $Z(X)$-bimodule action. We often write $ax = xa$ for $a \in Z(X)$ and $x \in X$, and this is perfectly true if interpreted correctly as an abstract bimodule action. However, if X is represented concretely, such as above in the 1-2 corner of $I(\mathcal{S}(X))$, one should usually rather write $\rho(a)x = x\pi(a)$, where π and ρ are faithful $*$-homomorphisms with domain $Z(X)$. That is, although the right action of $Z(X)$ on X coincides with the left action, if one represents X as a concrete bimodule, as above say, $Z(X)$ usually gets represented in 'two different places'.

PROPOSITION 4.3. *Let X be a real operator space.*

(1) *A real linear map $u\colon X \to X$ is in $\mathcal{A}_l(X)$ if and only if there exist a real Hilbert space H, a linear complete isometry $\sigma\colon X \to B(H)$, and a map $v : X \to X$ with $\sigma(ux)^*\sigma(y) = \sigma(x)^*\sigma(vy)$ for all $x, y \in X$.*
(2) *Same as* (1), *but with the conditions involving v replaced by existence of $S \in B(H)$ with $\sigma(ux) = S\sigma(x)$ for all $x \in X$, and such that also $S^*\sigma(X) \subset \sigma(X)$.*
(3) *$\mathcal{A}_l(X) \cong \{a \in I_{11}(X) : aX \subset X$ and $a^*X \subset X\}$ as C^*-algebras.*
(4) *If u, S and σ are as in* (1) (*resp.* (2)), *then the involution u^* in $\mathcal{A}_l(X)$ is v* (*resp. the map $x \mapsto \sigma^{-1}(S^*\sigma(x))$ on X*).
(5) *The canonical inclusion map from $\mathcal{A}_l(X)$ into $CB(X)$ is a completely isometric homomorphism. Moreover $\|T\|_{\mathrm{cb}} = \|T_{\mathrm{c}}\| = \|T\|$ for $T \in \mathcal{A}_l(X)$, so the inclusion viewed as a map into $B(X)$ is an isometric homomorphism.*

Proof. If $S \in B(H)$ with all the properties in (2) exists, then $v = \sigma^{-1}(S^*\sigma(\cdot))$ satisfies the condition in (1). Conversely, if v satisfies the condition in (1) then $v_{\mathrm{c}} : X_{\mathrm{c}} \to X_{\mathrm{c}}$ satisfies $\sigma(u_{\mathrm{c}}x)^*\sigma(y) = \sigma(x)^*\sigma(v_{\mathrm{c}}y)$ for all $x, y \in X_{\mathrm{c}}$. By [7, Theorem 8.4.4] we deduce that $u_{\mathrm{c}} \in \mathcal{A}_l(X_{\mathrm{c}})$ so that $u \in \mathcal{A}_l(X)$ by an earlier observation. The rest of the proofs of (2)–(4) are as in the complex case and using the real cases discussed earlier of results cited in [7, Proposition 4.5.8].

For (5), if $\mathcal{A}_l(X)$ we have

$$\|T\|_{\mathcal{M}_\ell(X)} = \|T_{\mathrm{c}}\|_{\mathcal{M}_\ell(X_{\mathrm{c}})} = \|T_{\mathrm{c}}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}.$$

From the complex case this equals $\|T_{\mathrm{c}}\|$, by the same proof but using Theorem 6.1 (the real version of [7, Theorem A.5.9]). The matricial case follows similarly, as in the complex case. The last assertion follows from Theorem 6.1. ∎

Note that $Z_{\mathbb{R}}(X)$ is a real commutative C^*-algebra but need not be a real $C(K)$ space (a simple example is $X = \mathbb{C}$ considered as a real operator space).

Examples. (1) For any real approximately unital operator algebra A we have $\mathcal{M}_\ell(A) = LM(A)$ and $\mathcal{M}_r(A) = RM(A)$, just as in the proof of [7, Proposition 4.5.11]. Hence $\mathcal{A}_\ell(A) = \Delta(\mathcal{M}_\ell(A)) = \Delta(LM(A))$. If B is a real C^*-algebra then it is then easy to deduce that $\mathcal{A}_\ell(B)$ is just $M(B)$. We claim that $Z(A)$ is the diagonal of the center of $M(A)$. Indeed, if $T \in Z(A)$ then $T_{\rm c} \in Z_{\mathbb{C}}(A_{\rm c})$, so that $T_{\rm c}$ is in the diagonal of the center of $M(A_{\rm c})$. Thus $T \in M(A)$ by our earlier discussion of the multiplier algebra of A. It is easy to see that T is in the center of $M(A)$. The map $Z(A)$ into the center of $M(A)$ is a contractive homomorphism, so it maps into the diagonal of the center of $M(A)$. Conversely, if T is in the diagonal of the center of $M(A)$ then $T \in \Delta(LM(A)) = \mathcal{A}_\ell(A)$. Similarly $T \in \mathcal{A}_r(A)$, so that $T \in Z(A)$.

(2) If $X = H_{\rm col}$, a real Hilbert column space, then as in the complex case $\mathcal{M}_\ell(X) = \mathcal{A}_\ell(X) = B(H)$, $\mathcal{M}_r(X) = \mathbb{R}\, I$, so that $Z(X) = \mathbb{R}\, I$.

(3) Suppose that X is a minimal real operator space, that is, $X = \mathrm{Min}(E)$ for a real Banach space E (see [44]). Then it is known that the Banach space injective envelope is a $C(K)$ space, and [44, Proposition 4.10] shows that the injective envelope of X is a $C(K)$ space with its canonical operator space structure. By the argument in [7, 4.5.10], $\mathcal{M}_\ell(X)$ is a minimal operator space, and $\mathcal{A}_\ell(X)$ is a commutative C^*-algebra. Indeed, we have the following, which is the real case of [17, Proposition 7.6(i)].

Theorem 4.4. *For a real Banach space X we have*

$$\mathcal{M}_\ell(\mathrm{Min}(X)) = \mathcal{A}_\ell(\mathrm{Min}(X)) = Z(\mathrm{Min}(X)) = \mathrm{Cent}(X) = \mathcal{M}(X),$$

where Cent *and* $\mathcal{M}$ *are the real Banach space centralizer and multiplier algebra* [3, 30]. *This also agrees with $\{T \in B(X) : T_{\rm c} \in \mathcal{M}(X_{\rm c})\}$, and for such T we have $\|T\| = \|T_{\rm c}\|$.*

Proof. It is known (and obvious from the definitions) that $\mathrm{Cent}(X) = \mathcal{M}(X)$ in the real case [3, 30]. We have $T \in \mathcal{M}_\ell(\mathrm{Min}(X))$ if and only if

$$T_{\rm c} \in \mathcal{M}_\ell((\mathrm{Min}(X))_{\rm c}) = \mathcal{M}_\ell(\mathrm{Min}(X_{\rm c})) = \mathcal{M}(X_{\rm c}),$$

using [44, Proposition 2.1] and [7, 4.5.10]. If $T \in \mathcal{M}(X)$ then as in [4, Theorem 2.1] one sees that there is a compact space K, a linear isometry $\sigma : X \to C(K)$, and an $f \in C(K)$, such that $\sigma(Tx) = f\sigma(x)$ for $x \in X$. By a universal property of Min (see [44]) it is easy to see that σ is a complete isometry on $\mathrm{Min}(X)$. So T is in $\mathcal{A}_\ell(\mathrm{Min}(X))$ (since f is selfadjoint) and indeed in $Z(\mathrm{Min}(X))$, since the C^*-algebras concerned all commute. Thus we have shown that

$$\mathrm{Cent}(X) = \mathcal{M}(X) \subset Z(\mathrm{Min}(X)) \subset \mathcal{A}_\ell(\mathrm{Min}(X)) \subset \mathcal{M}_\ell(\mathrm{Min}(X)).$$

Now $I(\mathrm{Min}(X)) = (C(K,\mathbb{R}), j)$, a real $C(K)$ space, as stated above the theorem. From the real version of [7, Proposition 4.4.13] in conjunction with the representation of $\mathcal{M}_\ell$ above Lemma 4.2, we have $\mathcal{M}_\ell(\mathrm{Min}(X)) =$

$\{f \in C(K) : fj(X) \subset j(X)\}$. By the real version of [7, Theorem 3.7.2] we see that $\mathcal{M}_\ell(\mathrm{Min}(X)) \subset \mathcal{M}(X)$.

That $\|T\| = \|T_c\|$ follows from the above and e.g. Proposition 4.3(5). ∎

REMARK. In [3, 30] it is shown that for a complex Banach space X we have $\mathrm{Cent}_{\mathbb{C}}(X) = \mathrm{Cent}_{\mathbb{R}}(X) + i\,\mathrm{Cent}_{\mathbb{R}}(X)$.

THEOREM 4.5. *If X is a complex operator space viewed as a real operator space X_r, we have $\mathcal{M}_\ell(X_r) = \mathcal{M}_\ell^{\mathbb{C}}(X)$, $\mathcal{A}_\ell(X_r) = \mathcal{A}_\ell^{\mathbb{C}}(X)$, and $Z_{\mathbb{R}}(X_r) = Z_{\mathbb{C}}(X)$. The projections in the latter algebra are the (complex) complete M-projections (of e.g. [17, Section 7] or [7, Chapter 4]).*

Proof. If u is multiplication by i on X then $u \in \mathrm{Ball}(\mathcal{M}_\ell(X_r))$, and indeed it is in the centralizer as we saw in the proofs in Section 3. Hence u commutes with all members of $\mathcal{M}_\ell(X_r)$. It follows that operators in $\mathcal{M}_\ell(X_r)$ are $\mathbb{C}$-linear on X, and therefore are in $\mathcal{M}_\ell^{\mathbb{C}}(X)$. Thus $\mathcal{M}_\ell(X_r) = \mathcal{M}_\ell^{\mathbb{C}}(X)$. This is a completely isometric identification, since $C_n^{\mathbb{R}}(X_r) \cong C_n^{\mathbb{C}}(X)$ real completely isometrically, and thus

$$M_n(\mathcal{M}_\ell(X_r)) = \mathcal{M}_\ell(C_n(X_r)) = \mathcal{M}_\ell^{\mathbb{C}}(C_n(X)) \cong M_n(\mathcal{M}_\ell^{\mathbb{C}}(X))$$

isometrically. Taking the 'diagonal', we have $\mathcal{A}_\ell(X_r) = \mathcal{A}_\ell^{\mathbb{C}}(X)$. Here we used the fact that for a unital subalgebra A of $B(H)$ in the complex setting, if we view $B(H)$ as a real C^*-algebra $B(H)_r$ then the 'real and complex diagonal algebras' of A coincide. Then

$$Z_{\mathbb{R}}(X_r) = \mathcal{A}_\ell(X_r) \cap \mathcal{A}_r(X_r) = \mathcal{A}_\ell^{\mathbb{C}}(X) \cap \mathcal{A}_r^{\mathbb{C}}(X) = Z_{\mathbb{C}}(X)$$

as desired. The last assertion may be found in e.g. [17, Section 7]. ∎

REMARK. 1) The above is quite different to the Banach space case, where $\mathrm{Cent}_{\mathbb{R}}(X)$ need not contain $\mathrm{Cent}_{\mathbb{C}}(X)$. Indeed, if we take $X = \mathbb{C}$ then $\mathrm{Cent}_{\mathbb{F}}(X) = \mathbb{F} \cdot I$, while $Z(X) = \mathbb{C} = Z_{\mathbb{R}}(X_r) = \mathcal{A}_\ell(X)$. Nor do we have $Z_{\mathbb{R}}(X) \subset \mathrm{Cent}_{\mathbb{R}}(X)$, where the latter is the Banach space centralizer algebra, in contrast to the complex theory [17, Corollary 7.2]. Indeed, $Z_{\mathbb{R}}(A)$ is not a subset of $\mathrm{Cent}_{\mathbb{R}}(A)$ in general, even for real commutative two-dimensional C^*-algebras. Nor is $\mathrm{Cent}_{\mathbb{R}}(X)$ a subset of $Z_{\mathbb{R}}(X)$ in general. Indeed, suppose that X is a complex operator space with a complex M-projection P which is not a complete M-projection. Then P is a real M-projection so it is in $\mathrm{Cent}_{\mathbb{R}}(X)$. However, P is not in $Z_{\mathbb{R}}(X)$, for if it were then one sees that $P^{(n)}$ is an M-projection for all n, contradicting that P is not a complete M-projection. Sharma does not discuss real complete M-projections, but just as in the complex case one sees that these are just the $P \in B(X)$ for which $P^{(n)}$ is an M-projection for all n.

We recall that a real M-projection on a complex Banach space is a complex M-projection [30, Theorem I.1.23]. Similarly, each real complete

M-projection P on a complex operator space is a complex complete M-projection. This may be easily seen by the proof of the claim in the proof of Theorem 3.4.

2) If X is a complex operator space then we recall from [5, Section 4] that $I^{\mathbb{R}}(X) = I^{\mathbb{C}}(X)$. Replacing X by $I(X)$ in the last result, we see that $I^{\mathbb{R}}_{11}(X) = \mathcal{M}_\ell(I(X)_r) = \mathcal{M}^{\mathbb{C}}_\ell(I(X)) = I^{\mathbb{C}}_{11}(X)$. Similarly for I_{22}.

Proposition 4.6. *We have $T \in \mathcal{M}_\ell(X)$ if and only if $T^{**} \in \mathcal{M}_\ell(X^{**})$, and the multiplier norms of these coincide. Similarly $T \in Z(X)$ if and only if $T^{**} \in Z(X^{**})$. Also, $\mathcal{A}_\ell(X) \subset \mathcal{A}_\ell(X^{**})$ isometrically via a faithful $*$-homomorphism.*

Proof. The first statement follows by the argument in [17, Proposition 5.14]. Similarly for the second statement. For $Z(X)$ we note that $T \in Z(X)$ iff $T_c \in Z(X_c)$, and iff $(T_c)^{**} \in Z((X_c)^{**})$ by [17, Theorem 7.4]. Since $(T_c)^{**} = (T^{**})_c$ and $(X_c)^{**} = (X^{**})_c$, the latter happens iff $(T^{**})_c \in Z((X^{**})_c)$, which as before is equivalent to $T^{**} \in Z(X^{**})$. ∎

Remark. We do not in general have the 'if and only if' above in the case of $\mathcal{A}_\ell(X)$. That is, one may have $T \in \mathcal{M}_\ell(X) \setminus \mathcal{A}_\ell(X)$ with $T^{**} \in \mathcal{A}_\ell(X^{**})$.

The remaining results in [7, Section 4.6] seem unchanged in the real case. In [7, Section 4.7], the first result follows by complexification, and the rest of the results follow with unchanged proofs. As in the complex case, $\mathcal{M}_\ell(X)$ is a dual operator algebra, and its subspace $\mathcal{A}_\ell(X)$ is a real von Neumann subalgebra if X is a dual operator space. It follows that $Z(X)$ is a commutative real von Neumann subalgebra, whose projections as in the complex case are the complete M-projections. Indeed, $Z(X)$ may be viewed as the fixed points of the weak* continuous period 2 $*$-automorphism of the complex commutative von Neumann algebra $Z(X)_c \cong Z(X_c)$. Most of [7, Section 4.8] is covered in [44], with the notable exception of complete (two-sided) M-ideals. We have already mentioned in Remark 1 after Theorem 4.5 that one sees, just as in the complex case, that complete M-projections are just the $P \in B(X)$ with $P^{(n)}$ an M-projection for all $n \in \mathbb{N}$. As in the complex case these are the left M-projections that are also right M-projections, and they are precisely the projections in $Z(X)$, as in the complex case.

Note that if a real operator algebra A has both a left and a right cai then it has a cai as in e.g. [7, Proposition 2.5.8].

Theorem 4.7. *If A is an approximately unital real operator algebra then the complete M-ideals (resp. complete M-summands) are exactly the two-sided ideals in A with contractive approximate identity (resp. Ae for a central projection $e \in M(A)$).*

Proof. Let P be a complete M-projection on an approximately unital operator algebra. Then it is a left M-projection and a right M-projection. Thus $P_{\rm c}$ is a complex left M-projection on $A_{\rm c}$ by [44, Proposition 5.8], and similarly on the right, hence it is a complex complete M-projection. By the complex case in [7, Theorem 4.8.5] we have $P_{\rm c}(x) = ex$ for a central projection $e \in M(A_{\rm c})$. Since $M(A_{\rm c}) = M(A)_{\rm c}$ by Lemma 6.2, and $P_{\rm c}A = eA \subset A$, it follows that $e \in M(A)$. This proves the statement about M-summands. Similarly it follows as in [44, Corollary 5.9] that a real subspace J is a complete M-ideal (or summand) in X if and only if $J_{\rm c}$ is a complete M-ideal (or summand) in $X_{\rm c}$. From this, the statement about M-ideals follows, or it may be deduced from [44, Corollary 5.11]. ∎

It is not at all obvious if the word 'complete' in the last result can be dropped. Indeed, such a result has not previously appeared in the literature even in the real C^*-algebra case. For a commutative real C^*-algebra A the M-ideals are the closed ideals. Indeed, by [5, Theorem 3.1] A^{**} is the L^∞ sum of real and complex $C(K)$ spaces. As we said earlier, real and complex M-projections on a complex space coincide. Any real M-summand V of A^{**} corresponds to real M-summands of these $C(K)$ spaces. Thus $V = eA^{**}$ for an orthogonal projection $e \in A^{**}$. Recently in joint work with M. Neal, A. Peralta and S. Su [12] we have proved that the M-ideals in a general real C^*-algebra (and in more general objects of interest) are the closed ideals, and coincide with the complete M-ideals. See the "Added in proof" section. In particular, we have fully generalized Theorem 4.8.5 of [7] to the real case. We remark that (4) of that theorem was later improved to replace LM by the multiplier algebra $M(A)$, and this holds also in the real (even Jordan) operator algebra case.

5. Operator spaces. The remaining sections of our paper, as stated in the introduction, verify in a very economical, linear, and somewhat systematic format the real case of the remaining theory in several chapters of [7], and establish how the basic constructions there interact with the complexification. We begin with [7, Chapter 1]. A few of the facts below appear in the papers of Ruan and Sharma (see e.g. [44, end of Section 2]). Ruan showed in [43] that $(X_{\rm c})^* \cong (X^*)_{\rm c}$ completely isometrically. For linear functionals we have $\|\varphi\| = \|\varphi\|_{\rm cb}$, as follows from [42, Lemma 5.2]. It is pointed out, however, in [44, Proposition 2.8] that $(X_r)^* \neq (X^*)_r$ completely isometrically for a complex operator space (some mistakes in the proof of that proposition are easily fixed). Here X_r is the space regarded as a real operator space.

We note that a real C^*-algebra A has a unique operator space structure such that $M_n(A)$ is a real C^*-algebra for all $n \in \mathbb{N}$. This is because a real $*$-algebra has at most one C^*-algebra norm, as follows immediately by complexification, or from e.g. [44, Lemma 2.2]. Proposition 1.2.4 in [7] then follows from [15, Theorem 2.6]. The assertion there that π is a complete quotient map with closed range can easily be seen by considering π_c. It is interesting, however, that the Russo–Dye theorem holds for real von Neumann algebras [36]; the Kadison–Schwarz inequality is discussed there too. On quotient operator spaces we have $X_c/Y_c \cong (X/Y)_c$ by Proposition 1.1 (the proof of this in [44] may be corrected as in [6, Proposition 5.5]). It is easy to see that the operator space complexification commutes with c_0 and ℓ^∞ sums of operator spaces (see e.g. [44, proof of Lemma 5.14]). The rest of [7, Section 1.2] is in the small real operator space literature, or follows as in the complex case or (as in the case of e.g. 1.2.31) by complexification. The exception is 1.2.30, for obvious reasons; we will not discuss interpolation in the present paper.

In what follows, $\overline{X}$ is the 'conjugate operator space' of a complex operator space X. That is, $\overline{X}$ is the set of formal symbols $\overline{x}$ for $x \in X$, with matrix norms $\|[\overline{x}_{ij}]\| = \|[x_{ij}]\|$. With 'conjugate scalar multiplication' $\overline{X}$ is a complex operator space; the map $x \mapsto \overline{x}$ is conjugate linear (see [16, Proposition 2.1 and Remark after it]). This defines a functor on the complex operator space, with the action on morphisms being $\overline{T}(\overline{x}) = \overline{T(x)}$. In practice this construction is often extremely useful. The first reason for this is that in checking certain facts about a complexification Y_c we very frequently must use the important conjugate linear isomorphism θ_Y on $W = Y_c$. One often wants to apply results from the complex theory to θ_Y, and to do this it is often convenient to view conjugate linear maps into W as complex linear maps into $\overline{W}$. Lemma 5.3 is an example of this. The next result is also useful for similar reasons (see [21] for an example of this).

Lemma 5.1. *The operator space complexification of a complex operator space X is complex linearly completely isometric to $X \oplus^\infty \overline{X}$, where we identify X with $\{(x, \overline{x}) : x \in X\}$, and identify $x + iy$ in the complexification with $(x + iy, \overline{x - iy}) \in X \oplus^\infty \overline{X}$. If B is a complex C^*-algebra then $B_c \cong B \oplus^\infty B^\circ$ $*$-isomorphically (as complex C^*-algebras). Here B° is the 'opposite' C^*-algebra of B.*

Proof. Note that $X + iX$ becomes $\{(x, \overline{x}) + (iy, -\overline{iy}) : x, y \in X\} = X \oplus_\infty \overline{X}$. Also, $(x, \overline{x}) = (iy, -\overline{iy})$ implies that $x = iy = -iy$, so that $x = y = 0$ and $X \cap (iX) = \{0\}$. The embedding $x \mapsto (x, \overline{x})$ is clearly completely isometric (by definition of $\overline{X}$ above). Finally, if we define $\theta(x + iy, \overline{x - iy}) = (x - iy, \overline{x + iy})$, then $\theta((a, \overline{b})) = (b, \overline{a})$ for $a, b \in X$, which is clearly completely isometric, and the copy of X is the set of fixed points of θ. So by Ruan's uniqueness theorem, the operator space complexification of X is $X \oplus^\infty \overline{X}$.

If B is a complex C^*-algebra then $\overline{B}$ is a C^*-algebra with product $\overline{x}\overline{y} = \overline{xy}$. Indeed, $\overline{x} \mapsto (x^*)^\circ$ is a complex linear $*$-isomorphism $\overline{B} \to B^\circ$. ∎

REMARK. We remark that in the last result $X \oplus^\infty \overline{X}$ is real completely isometric to the set of matrices of the form (1.1) with $x, y \in X$. However, the latter set is not always complex linearly completely isometric to $X_{\rm c}$ (similarly to the fact mentioned in Section 2 that a Banach space E need not be complex isometric to $\overline{E}$), so it is not very useful for many purposes. Note also that the statement in the lemma about C^*-algebras does not imply that $A \oplus A^\circ$ is completely isometric to $A_{\rm c}$ for a (complex) operator algebra A, where A° is the opposite algebra of [7, 2.2.8].

The basics of the theory of real operator systems are much the same as in the complex case (see [43, 42, 44, 15]) in many ways, most results being proved in the same way as in the complex case, or simply following from that case by complexification. We have already indicated some of the problematic issues with real operator systems and states in other papers (see also [43, 15]), such as the selfadjoint or positive elements not necessarily spanning a real operator system $\mathcal{S}$, or that they may be all of $\mathcal{S}$. See also comments on [14, Section 2.1] below.

The facts in [7, Section 1.4] are valid with the same proofs. See e.g. [44, Proposition 2.4 and Remark on p. 104] for some examples of this.

LEMMA 5.2. *Let X be a real operator space, which is the operator space dual of another real operator space Y. Then X is completely isometrically isomorphic, via a homeomorphism for the weak* topologies, to a weak* closed subspace of $B(H)$ for a real Hilbert space H. Also, X is a weak* closed real subspace of its operator space complexification $X_{\rm c} \cong (Y_{\rm c})^*$, and the canonical maps $X \to X_{\rm c}$, $X_{\rm c} \to X$ and $\theta_X : X_{\rm c} \to X_{\rm c}$ are weak* continuous.*

Conversely, any weak closed subspace X of $B(H)$ is the operator space dual of $B(H)_*/X_\perp$.*

Proof. The first and the last statements follow perhaps most quickly from [7, proofs of Lemmas 1.4.6 and 1.4.7]. If X is a weak* closed subspace of $B(H)$ then in the setup of [5, Section 2], $B(H)_{\rm c}$ may be viewed as a weak* closed real subspace of $M_2(B(H)) \cong B(H^{(2)}) \cong B_{\mathbb{R}}(H_{\rm c})$, and the induced embedding $B(H)_{\rm c} \hookrightarrow B_{\mathbb{R}}(H_{\rm c})$ has range in $B_{\mathbb{C}}(H_{\rm c})$ and coincides with the canonical map $B(H)_{\rm c} \to B_{\mathbb{C}}(H_{\rm c})$. The latter range is weak* closed. The isomorphism $B(H^{(2)}) \cong B_{\mathbb{R}}(H_{\rm c})$ is a weak* homeomorphism. Hence the canonical maps $B(H) \to B(H)_{\rm c}$ and $B(H)_{\rm c} \to B(H)$ ('real' and 'imaginary' parts) are weak* continuous. Indeed, $S_t + iT_t \xrightarrow[t\to\infty]{} S + iT$ weak* iff $S_t \xrightarrow[t\to\infty]{} S$ weak* and $T_t \xrightarrow[t\to\infty]{} S$ weak*. It follows that $X_{\rm c}$ is weak* closed in $B(H)_{\rm c} \cong B(H_{\rm c})$, and the canonical maps $X \to X_{\rm c}$ and $X_{\rm c} \to X$ are weak* continuous, as thus is θ_X on $X_{\rm c}$. Thus X is a weak* closed real subspace of its complexification.

If X has a real operator space predual Y then $X_{\rm c}$ has operator space predual $Y_{\rm c}$, of course. By the net convergence formulation above, the associated weak* topology on $X_{\rm c}$ coincides with the one inherited from $B(H_{\rm c})$ above if the embedding $X \subset B(H)$ is weak* continuous with respect to $\sigma(X, Y)$, such as the embedding in the first statement. ∎

A real C^*-algebra is representable as a real von Neumann algebra if and only if it has a Banach space predual [32]. Such a Banach space predual is unique by [32, 6.2.5]. By [35, last part of Section 5.5] the bidual of a real C^*-algebra A is a real W^*-algebra, and $(A_{\rm c})^{**} \cong (A^{**})_{\rm c}$. Indeed, the theory of the real bidual of an operator space or an operator algebra is very similar to the complex theory (e.g. in [7, Sections 1.4 and 2.5]).

LEMMA 5.3. *The operator space* 1*-direct sum* $W = \bigoplus_\alpha^1 (X_\alpha)_{\rm c}$ *is the reasonable complexification of* $V = \oplus_\alpha^1 X_\alpha$.

Proof. The inclusion $X_\alpha \to W$ induces a complete contraction $\kappa : V \to W$, by the real variant of [7, 1.3.14]. The universal property of the 1-sum applied to the complexification of the inclusion $i_\alpha : X_\alpha \to V$ gives a complete contraction $r : W \to V_{\rm c}$ with $r \circ \kappa = I_V$. So, κ is a complete isometry. It is not hard to show that $\overline{W}$ has the universal property of $\bigoplus_\alpha^1 \overline{(X_\alpha)_{\rm c}}$, so that these operator spaces may be identified. The composition of θ_{X_α} with the canonical map $(X_\alpha)_{\rm c} \to W \to \overline{W}$ induces a linear complete contraction $u : W \to \overline{W}$. Similarly, we obtain a map $v : \overline{W} \to W$ with $vu = I_W$. Then $\overline{u(\cdot)}$ is a period 2 conjugate linear complete isometric surjection $\theta : W \to W$ fixing $\mathrm{Ran}(\kappa)$. Thus W is the reasonable complexification. ∎

Basic aspects of the real version of the theory of operator space tensor products are stated in Ruan's paper [42]. We will expand on his very terse remarks, for example discussing functoriality of the complexification for the three tensor products in [7, Section 1.5]: the minimal, projective, and Haagerup operator space tensor products. The operator space projective tensor product is designed to linearize real bilinear maps that are *jointly completely bounded* in the sense of [7, 1.5.11]. One can see in several ways that a 'jointly completely contractive' real bilinear map $u : X \times Y \to Z$ of real operator spaces extends uniquely to a jointly completely contractive complex bilinear map $u : X_{\rm c} \times Y_{\rm c} \to Z_{\rm c}$ (this follows e.g. from the next lemma). As in the complex case $(E \widehat{\oplus} F)^* \cong CB(E, Y^*)$ completely isometrically (see the last page of [42] and [7, (1.51)]).

The Haagerup tensor product of real operator spaces is defined e.g. as in [7, 1.5.4] so as to have the universal property of linearizing bilinear maps that are completely bounded in the sense of Christensen and Sinclair (called simply completely bounded in e.g. [7]). As in the complex case, if $X_i \subset Y_i$ completely isometrically then $X_1 \otimes_{\rm h} X_2 \subset Y_1 \otimes_{\rm h} Y_2$ completely isometrically (this is the 'injectivity' of the Haagerup tensor product). There are several known proof strategies for this, but it also follows from the complex case and the complete isometry beginning the third paragraph of the next proof. The proof of the CSPS theorem in [7, Theorem 1.5.7] is just as in the complex case. The other results in [7, Section 1.5], e.g. the 'selfduality' of the Haagerup tensor product, are also just as in the complex case.

LEMMA 5.4. *For real operator spaces X and Y we have $(X \otimes_{\min} Y)_c \cong X_c \otimes_{\min} Y_c$ and $(X \widehat{\otimes} Y)_c \cong X_c \widehat{\otimes} Y_c$ completely isometrically. Similarly $(X \otimes_h Y)_c \cong X_c \otimes_h Y_c$ completely isometrically. In particular, a completely contractive* (*in the sense of Christensen and Sinclair*) *real bilinear map $u : X \times Y \to Z$ of real operator spaces extends uniquely to a completely contractive complex bilinear map $u : X_c \times Y_c \to Z_c$.*

Proof. If $\otimes_{\min}$ is the uncompleted minimal operator space tensor product, we have

$$(X \otimes_{\min} Y)_c \subset CB(Y^*, X)_c = CB((Y^*)_c, X_c) = CB((Y_c)^*, X_c).$$

Thus the algebraic identification of $(X \otimes_{\min} Y)_c$ and $X_c \otimes_{\min} Y_c$ is a complete isometry. A similar principle shows that $(X \widehat{\otimes} Y)_c \cong X_c \widehat{\otimes} Y_c$ completely isometrically. Indeed,

$$(X_c \widehat{\otimes} Y_c)^* \cong CB(X_c, (Y_c)^*) \cong CB(X_c, (Y^*)_c) \cong CB(X, Y^*)_c \cong ((X \widehat{\otimes} Y)^*)_c$$

completely isometrically, so that the algebraic identification of (the uncompleted) operator space projective tensor products $X_c \widehat{\otimes} Y_c$ and $(X \widehat{\otimes} Y)_c$ is a complete isometry.

Let $u : X \times Y \to Z$ be a completely contractive (in the sense of Christensen and Sinclair) real bilinear map and let $u^c : X_c \times Y_c \to Z_c$ be the unique complex bilinear extension. Let $\theta : Z_c \to M_2(Z)$ be the map taking $x + iy$ to the matrix in (1.1). For $n \in \mathbb{N}$ and $a \in M_n(X_c), b \in M_n(Y_c)$ the reader can check that $\theta_n(u_n^c(a, b)) = u_{2n}(\theta_n(a), \theta_n(b))$. Hence

$$\begin{aligned}\|u_n^c(a, b)\| &= \|\theta_n(u_n^c(a, b))\| = \|u_{2n}(\theta_n(a), \theta_n(b))\| \\ &\le \|\theta_n(a)\| \, \|\theta_n(b)\| = \|a\| \, \|b\|.\end{aligned}$$

Thus u^c is completely contractive.

It is easy to deduce from the latter and the universal property that the canonical map $X \otimes_h Y \to X_c \otimes_h Y_c$ is a complete isometry. To show that $(X \otimes_h Y)_c \cong X_c \otimes_h Y_c$ completely isometrically it suffices by the 'injectivity' of the Haagerup tensor product, and of $X \mapsto X_c$, to assume that X and Y are finite-dimensional. Clearly $\otimes : X \times Y \to X \otimes_h Y$ is completely contractive, so by the facts above we obtain a completely contractive map $X_c \times Y_c \to (X \otimes_h Y)_c$, and we see that the canonical map $X_c \otimes_h Y_c \to (X \otimes_h Y)_c$ is completely contractive. Hence the canonical map $X_c^* \otimes_h Y_c^* \to (X^* \otimes_h Y^*)_c$

is completely contractive. Dualizing this we get a complete contraction

$$(X^* \otimes_{\mathrm{h}} Y^*)^*_{\mathrm{c}} \to (X^*_{\mathrm{c}} \otimes_{\mathrm{h}} Y^*_{\mathrm{c}})^* \cong X_{\mathrm{c}} \otimes_{\mathrm{h}} Y_{\mathrm{c}},$$

the latter by the 'selfduality' above the lemma. Also by this 'selfduality' we have $X \otimes_{\mathrm{h}} Y \cong (X^* \otimes_{\mathrm{h}} Y^*)^*$. Complexifying this last isomorphism, and composing with the map in the last displayed equation, one may see that the canonical map $(X \otimes_{\mathrm{h}} Y)_{\mathrm{c}} \to X_{\mathrm{c}} \otimes_{\mathrm{h}} Y_{\mathrm{c}}$ is a complete contraction. Thus $(X \otimes_{\mathrm{h}} Y)_{\mathrm{c}} \cong X_{\mathrm{c}} \otimes_{\mathrm{h}} Y_{\mathrm{c}}$ completely isometrically. ∎

Nearly all of the real versions of the results in [7, Section 1.6] are valid by the same arguments. We end this section by discussing the real versions of a couple of results there that are not so clear.

If X, Y are weak* closed real subspaces of $B(H)$ and $B(K)$, we define the normal spatial tensor product $X \mathbin{\overline{\otimes}} Y$ to be the weak* closure in $B(H \otimes K)$ of the span of the operators $x \otimes y$ for $x \in X$ and $y \in Y$. Define the *normal Fubini tensor product* $X \otimes_{\mathcal{F}} Y$ to be the subspace of $B(H \otimes K)$ of elements whose 'left slices' are in X and 'right slices' are in Y (see [25, bottom of p. 134]).

Lemma 5.5. *If X and Y are weak* closed real subspaces of $B(H)$ and $B(K)$ then there is a canonical weak* continuous completely isometric isomorphism*

$$(X_* \mathbin{\widehat{\otimes}} Y_*)^* \cong X \otimes_{\mathcal{F}} Y,$$

which carries the weak closure of the canonical copy of $X \otimes Y$ in $(X_* \mathbin{\widehat{\otimes}} Y_*)^*$ onto the normal spatial tensor product $X \mathbin{\overline{\otimes}} Y$.*

In addition, $X_{\mathrm{c}} \mathbin{\overline{\otimes}} Y_{\mathrm{c}}$ is a reasonable complexification of $X \mathbin{\overline{\otimes}} Y$; indeed, $X \mathbin{\overline{\otimes}} Y$ may be identified with the fixed points of the period 2 automorphism $\theta_X \otimes \theta_Y$ on $X_{\mathrm{c}} \mathbin{\overline{\otimes}} Y_{\mathrm{c}}$.

Proof. The proof of the complex version of the first assertion, which is [25, Theorem 7.2.3], works in the real case.

Consider the canonical map

$$X \mathbin{\overline{\otimes}} Y \xrightarrow{\kappa} X_{\mathrm{c}} \mathbin{\overline{\otimes}} Y_{\mathrm{c}}.$$

Here $\kappa(x \otimes y) = x \otimes y$ for $x \in X, y \in Y$. Note that κ is weak* continuous and completely isometric since it is a restriction of the canonical map

$$B(H) \mathbin{\overline{\otimes}} B(K) = B(H \otimes K) \xrightarrow{\kappa} B(H_{\mathrm{c}}) \mathbin{\overline{\otimes}} B(K_{\mathrm{c}}) \;=\; B(H_{\mathrm{c}} \otimes K_{\mathrm{c}}).$$

(One may use Lemma 5.2 here and below too if desired.) Note also that $\theta_X \otimes \theta_Y$ is the weak* continuous restriction of $\theta_{B(H)} \otimes \theta_{B(K)}$ to $X_{\mathrm{c}} \mathbin{\overline{\otimes}} Y_{\mathrm{c}}$, and is thus easily seen to be a well-defined period 2 automorphism on $X_{\mathrm{c}} \mathbin{\overline{\otimes}} Y_{\mathrm{c}}$. Let

$$Q(z) = \tfrac{1}{2}(z + (\theta_X \otimes \theta_Y)(z)), \qquad z \in X_{\mathrm{c}} \mathbin{\overline{\otimes}} Y_{\mathrm{c}}.$$

This is a weak* continuous idempotent, and $\mathrm{Ran}(Q)$ is the set of fixed points of $\theta_X \otimes \theta_Y$. It is easy to check that $Q \circ \kappa = \kappa$ first on elementary tensors, and then on $X \mathbin{\overline{\otimes}} Y$ by continuity and density. On the other hand, $Q((x+iy)\otimes(x'+iy'))$ is easily seen to be in $\mathrm{Ran}(\kappa)$, so since the latter is weak* closed we have $\mathrm{Ran}(Q) \subset \mathrm{Ran}(\kappa)$ again by continuity and density. Thus $\mathrm{Ran}(Q) = \mathrm{Ran}(\kappa)$ is the set of fixed points of $\theta_X \otimes \theta_Y$. The result follows from this and [5, Proposition 2.1]. ∎

The next result is quite important, and contains the real case of a celebrated result of Effros and Ruan:

Theorem 5.6. *If M and N are real von Neumann algebras, then the operator space projective tensor product $M_* \widehat{\otimes} N_*$ is the operator space predual of the von Neumann algebra $M \mathbin{\overline{\otimes}} N$. In addition, $M \mathbin{\overline{\otimes}} N$ is real $*$-isomorphic to the fixed points of the period 2 automorphism $\theta_M \otimes \theta_N$ on $M_{\mathrm{c}} \mathbin{\overline{\otimes}} N_{\mathrm{c}}$.*

Proof. That $M_* \widehat{\otimes} N_*$ is the operator space predual of the normal spatial tensor product $M \mathbin{\overline{\otimes}} N$ follows exactly as in the complex case [25, Theorem 7.2.4], but using Lemma 5.5 above and the real version of the commutation theorem for tensor products in [35, Theorem 4.4.3]. The last assertion follows from the previous result. ∎

Finally, we consider the formula $L^\infty(\Omega,\mu) \mathbin{\overline{\otimes}} Y = L^\infty(\Omega,\mu;Y)$ in [7, 1.6.6], for a dual operator space Y with separable predual. In this case again the proof referenced there (from Sakai's book) works in the real case, since it relies on the Dunford–Pettis theorem and the well-known fact that $L^1(\Omega,\mu) \hat{\otimes} Y = L^1(\Omega,\mu;Y)$ (due to Grothendieck), both of which are valid in the real case. (Actually, we only need the first ten lines of Sakai's proof, the rest is covered by our earlier discussion.) This concludes our discussion of Chapter 1 in [7].

6. Operator algebras. Before we discuss technical issues arising in [7, Chapter 2] we say a few things about real Banach algebras. The first point is that the Neumann lemma and its variants, and the norm formulae often accompanying the Neumann lemma, are valid verbatim for unital real operator algebras via complexification. This is because of the uniqueness of the operator space complexification, and the fact that the canonical map $A_{\mathrm{c}} \to A$ is a contraction. The basic spectral theory of real Banach algebras may be found e.g. in [28, 35]. Nonzero multiplicative linear functionals (characters) χ on a real operator algebra A need to be complex-valued in general to get a sensible theory (see [28, 35]). If A is unital then χ is contractive. If A is not unital then χ extends to a unital character on A^1. So χ is (completely) contractive. Thus the Gelfand transform is a contraction, and the characters on commutative unital real operator algebras are in bijective correspondence with the maximal ideals [28, 35]. There is a natural involutive homeomorphism on the spectrum of a commutative real C^*-algebra (see [35, end of Section 5.1, Example (3)]). There is therefore a good functional calculus for real C^*-algebras that works in the expected way provided that one uses continuous functions on the spectrum that are involutive with respect to the just mentioned involution (the real C^*-algebra generated by a normal element a corresponds via the Gelfand transform to such involutive functions on the spectrum). See e.g. [36] for a recent survey on some aspects of real Banach algebras.

It seems to be well-known that Cohen's factorization theorem works for real Banach modules over real approximately unital Banach algebras, with the same proof. Namely, for a nondegenerate real left Banach A-module X over a real approximately unital Banach algebra, if $x \in X$ has norm < 1 there exist $a \in \mathrm{Ball}(A)$ and $y \in X$ of norm < 1 with $x = ay$. Since we do not have a reference on hand, we also mention that the basic proof of Cohen's theorem uses the Neumann lemma, which as we said works in the real case with the same norm inequality consequences, and builds the element a from convex combinations of the cai. Similarly, the element y is also constructed using real methods, so that we still have $a \in \mathrm{Ball}(A)$ and $y \in X$ with desired norm inequality.

The following result is somewhat well-known in the complex case, and has various known proofs, for example using a result of Kaplansky on minimal algebra norms on $C_0(K)$, or see e.g. [7, Theorem A.5.9]. The real case seems to be new, but may be proved similarly using Kaplansky's result applied to the algebra generated by x^*x. We give a longer but selfcontained and novel route.

Theorem 6.1. *Let $\theta : A \to B$ be a contractive homomorphism from a real C^*-algebra into a real Banach algebra. Then $\pi(A)$ is norm closed, and it possesses an involution with respect to which it is a real C^*-algebra. Moreover, π is then a $*$-homomorphism into this C^*-algebra. If π is one-to-one then it is an isometry.*

Proof. Assume first that A is commutative. By replacing B by $\overline{\pi(B)}$ we may assume that B is commutative, so Arens regular, although this is not necessary. By extending to the bidual we may assume that $A = M$ is a commutative real W^*-algebra and π is weak* continuous. (Note that if π^{**} is a $*$-homomorphism into a C^*-algebra then so is π, and its range is closed and is a C^*-algebra.) Quotienting by the kernel, a weak* closed ideal, we may assume that π is one-to-one. By [5, Theorem 3.1] we may assume that $M = L^\infty(X, \mathbb{C}) \oplus^\infty L^\infty(Y, \mathbb{R})$. Let p be the projection in M corresponding to the first summand. We claim that $\pi(pM) = \pi(L^\infty(X, \mathbb{C}))$ and $\pi((1-p)M) = \pi(L^\infty(Y, \mathbb{R}))$ are closed and are C^*-algebras, so that $\pi(pM) \oplus^\infty \pi((1-p)M)$ is a C^*-algebra.

If e and f are mutually orthogonal projections in M then $\pi(e)$ and $\pi(f)$ are contractive nonzero idempotents with zero product, and are of norm 1. Hence it is easy to see that π is isometric on real linear combinations of mutually orthogonal projections in M. Since such real linear combinations are norm dense in $L^\infty(Y,\mathbb{R})$, it follows that π is isometric on $(1-p)M$. So $\pi((1-p)M)$ is closed and has an involution making it a commutative real W^*-algebra with its original norm, and π is a $*$-homomorphism.

We now consider the complementary space $pM = L^\infty(X,\mathbb{C})$. Let D be the closure of $\pi(L^\infty(X,\mathbb{C}))$. By the argument above, π is isometric on $(pM)_{\rm sa}$. A similar argument will show it is isometric on all of pM. If $J=\pi(i)$ then $J^2=-I$, and since J and $\pi(-i)$ are contractive with product 1, they have norm 1. Note that

$$\begin{aligned}\|(\cos\varphi I + J\sin\varphi)x\| &= \|\pi(\cos\varphi\cdot 1 + i\cdot\sin\varphi)x\| \\ &\le \|\pi(\cos\varphi\cdot 1 + i1\cdot\sin\varphi)\|\|x\| \le \|x\|\end{aligned}$$

since π is contractive and $|\cos\varphi + i\sin\varphi| = 1$. So D is a complex Banach space with $ix = Jx$ for $x\in D$ (by the simple criterion in the second paragraph of Section 3). Indeed, it clearly is a complex Banach algebra (see [36, Proposition 6.2]). Now $\pi(if) = J\pi(f) = i\pi(f)$, so that π is a complex linear contractive homomorphism. By the argument in the last paragraph, but taking complex combinations, π is isometric on pM, and $\pi(pM)$ is closed and possesses an involution with respect to which it is a complex, hence real, commutative C^*-algebra, and π is a $*$-homomorphism. Let $q=\pi(p)$, the identity of the latter C^*-algebra.

For $x,y\in M$ we have

$$\|q\pi(x)\| \le \|q\pi(x)+(1-q)\pi(y)\| \le \|px+(1-p)y\| = \max\{\|px\|,\|(1-p)y\|\}.$$

Since θ is isometric on the two pieces, the latter equals $\max\{\|q\pi(x)\|, \|(1-q)\pi(y)\|\}$. Thus $\pi(M) = \pi(pM)\oplus^\infty \pi((1-p)M)$ isometrically, and is norm closed and possesses an involution with respect to which it is a real C^*-algebra. Also, π is an isometric $*$-homomorphism.

In the general case, again $\theta^{**} : A^{**}\to B^{**}$ is a unital contraction, and it is easy to see that it is a homomorphism with respect to, say, the left Arens product on B^{**}. So we may assume that A is a real von Neumann algebra. Indeed, if the range of θ^{**} is a C^*-algebra then θ is a $*$-homomorphism by [15, Theorem 2.6], so it has closed range which is a C^*-algebra. As above we may assume that θ is one-to-one. We claim that θ is an isometry. By the commutative case, θ is an isometry on selfadjoint elements since they generate a commutative unital C^*-algebra. For $a\in A$ let $a=v|a|$ be the polar decomposition in A (see [35, Proposition 4.3.4(2)]). Then

$$\|\theta(a)\| = \|\theta(v|a|)\| = \|\theta(|a|)\| = \big\||a|\big\|,$$

since $\theta(v), \theta(v^*)$ are contractions and $|a| = v^*v|a|$. So θ is an isometry on A. Hence $\pi(A)$ is closed, and has an involution with respect to which it is a C^*-algebra. ∎

We now survey the real case of the remaining results in [7, Chapter 2] (see also the last paragraphs of the introduction). As said elsewhere, the selfadjoint elements in a real operator algebra are not necessarily the hermitian elements, although a selfadjoint element a is positive iff $\varphi(a) \geq 0$ for real states φ. States are thoroughly discussed in [15]. One may add to that that the real states φ on an approximately unital real operator algebra A are precisely the real parts of complex states on $A_{\rm c}$ (such as $\varphi_{\rm c}$), or of a complex C^*-algebra generated by $A_{\rm c}$ (using [15, Lemma 4.13] in the approximately unital case). However, the real parts of two different such complex states may coincide on A. A similar fact holds in the Jordan operator algebra case.

The results of [7, 2.1.5–2.1.8 and Appendix A.6] rely in places on Cohen factorization, but as we mentioned above, the latter is valid in the real case. Lemmas 2.1.9 and 2.1.18 of [7] in the real case appear as Proposition 4.3 and Lemma 4.12 in [15]. Indeed, most results in [7, Section 2.1] are just as in the complex case, or are explicitly in [44, 15]. For example, the unitization A^1 was studied in [44, Section 3] where it was checked that $(A^1)_{\rm c} \cong (A_{\rm c})^1$ completely isometrically. For the $\mathcal{U}(X)$ construction from [7, Section 2.2] it is clear that $\mathcal{U}(X)_{\rm c} \cong \mathcal{U}(X_{\rm c})$ completely isometrically and as operator algebras (just as we saw the analogous result for $\mathcal{S}(X)_{\rm c}$ in the introduction). Sharma abstractly characterized real operator algebras in [44]. The rest of the results in [7, Sections 2.2–2.5] are virtually unchanged in the real case, with the exception (as in Chapter 1) being the complex interpolation results in [7, Section 2.3]. We mention in particular the fact that the quotient of a real operator algebra A by a closed two-sided ideal is a real operator algebra, which follows as in [7, Proposition 2.3.4] or from the complex case and the earlier complete isometry $A/I \subset (A/I)_{\rm c} \cong A_{\rm c}/I_{\rm c}$ [44, Lemma 5.12].

The theory of the left, right and two-sided multiplier algebras of an approximately unital operator algebra A in [7, Section 2.6] is just as in the complex case. The next lemma follows from Lemma 4.2 but we include a short direct proof.

Lemma 6.2. *For a real approximately unital operator algebra A, a reasonable complexification of $LM(A)$ is $(LM(A_{\rm c}), T \mapsto T_{\rm c})$. Similarly $M(A)_{\rm c} = M(A_{\rm c})$ and $RM(A)_{\rm c} = RM(A_{\rm c})$ completely isometrically isomorphically.*

Proof. We have (completely isometrically as operator algebras)

$$LM(A_{\rm c}) = \{\eta \in (A_{\rm c})^{**} \cong (A^{**})_{\rm c} : \eta A_{\rm c} \subset A_{\rm c}\},$$

which is $\{\zeta + i\xi \in (A^{**})_{\rm c} : \zeta A \subset A, \xi A \subset A\}$. Since the latter is reasonable, it is $LM(A)_{\rm c}$. It follows that $(LM(A_{\rm c}), T \mapsto T_{\rm c})$ is a reasonable complexification of $LM(A)$; also $T \in LM(A)$ iff $T_c \in LM(A_c)$. The others are similar. ∎

Moreover, 2.6.14–17 (for real selfadjoint UCP maps) and the real version of Section 2.7 hold in the real case. The real version of [7, Theorem 2.7.9] holds by complexification. For the reader's convenience we walk quickly through the details as being representative of such arguments:

THEOREM 6.3. *Let M be a real operator algebra which is a dual operator space. Then the product on M is separately weak* continuous, and M is a dual operator algebra. That is, there exists a real Hilbert space H and a weak* continuous completely isometric homomorphism $\pi : M \to B(H)$.*

Proof. Indeed, if A is a real operator algebra with an operator space predual Y, then A_c is a complex operator algebra with an operator space predual Y_c. Also A is a weak* closed real subalgebra of its complexification by our comments above on [7, Section 1.4]. Thus by [7, Theorem 2.7.9] there exists a (complex, hence) real Hilbert space H and a weak* continuous completely isometric homomorphism $\pi : A \to B(H)$ with $\pi(A)$ a weak* closed real operator algebra on H. ∎

We call these the *real dual operator algebras*. We remark that some of the real theory of dual operator algebras also follows from the involutive approach as in [16, Section 4].

We end this section with a few words about the completely isomorphic version of the theory. As mentioned in the introduction, we only check some selected results in [7, Chapter 5]. The results in [7, Section 5.1] are valid in the real case, with the usual exception of complex interpolation [7, (5.1.10)]. As we said in Section 2 above, the main result in [7, Section 5.2] follows from the complex case by complexification. Similarly, the matching result for modules [7, 5.2.17] is valid in the real case with unchanged proof.

7. Operator modules. The real analogues of nearly all results in [7, Chapter 3] are unproblematic, although some of these results need some facts about operator space multipliers from Section 4 above in place of their complex variants. The complexification of what are called operator modules, h-modules, Hilbert modules, and matrix normed modules in that chapter are complex operator modules, h-modules, and matrix normed modules, respectively (using for example facts mentioned in Section 5 about the complexification of tensor products in reference to [7, Section 1.5]). For example, using this principle, the proof in Section 2 of the real version of the Christensen–Effros–Sinclair theorem characterizing operator modules in [7, Theorem 3.3.1] may be rephrased in terms of complexifying all spaces and using the last assertion of Lemma 5.4.

Items [7, 3.1.11 and 3.5.4–3.5.5] use Cohen factorization, but as we mentioned above, the latter and several accompanying results in [7, Appendix A.6] are valid in the real case. Theorem 3.2.14 there uses Appendix A.1.5. However, the latter result follows in the real case by complexification. Indeed, if W is as in Appendix A.1.5 and T is as in the fifth line of the appendix then $W_{\rm c}$ is weak* closed in $B(H_{\rm c})$, and clearly $(W_{\rm c})^{(\infty)} = (W^{(\infty)})_{\rm c}$. We also clearly have $T_{\rm c}\zeta \in [(W^{(\infty)})_{\rm c}\,\zeta]$ for $\zeta \in (H_{\rm c})^{(\infty)} = (H^{(\infty)})_{\rm c}$. Since $(W_{\rm c})^{(\infty)}$ is reflexive by [7, A.1.5] in the complex case, we have $T_{\rm c} \in (W_{\rm c})^{(\infty)} = (W^{(\infty)})_{\rm c}$, so that $T \in W^{(\infty)}$.

Much of the real version of [7, Section 3.7] is classical, is in the literature in some form, and is largely due to E. Behrends and his collaborators [3, 30]. Some of this is explicitly discussed in Section 4, mostly in Theorem 4.4 and its proof. Some other results in Section 3.7 follow immediately from the complex case by complexification. The real versions of results in Section 3.8 in [7] follow with the same proofs or by complexification.

8. Operator spaces and injectivity, etc. We turn to the few remaining parts of Chapter 4 in [7]. The motivational Section 4.1 is largely a review of the classical Shilov and Choquet boundary of complex function spaces. There is a literature of these boundaries for real function spaces, but we will say no more here on this topic, since our goal is the more general real operator space case. Early results in Section 4.2 and on the real injective envelope and C^*-envelope are covered in [44] (some are mentioned without proof in Ruan's papers on real operator spaces). Indeed, items up to 4.2.7 and 4.2.11 are in [44] or are obvious in the real case. Corollary 4.2.8 there is handled in [5, Theorem 4.2], and Corollary 4.2.9 follows by complexification. The first part of 4.2.10 is as in the complex case. That $I(M_{m,n}(X)) \cong M_{m,n}(I(X))$ is generalized in [7, 4.6.12], whose proof is unchanged in the real case. It is proved in [5] that a complex operator space is real injective if and only if it is complex injective.

The real C^*-envelope is discussed in [44] and [5, Section 4]. The real versions of the important facts in 4.3.2 and 4.3.6 in [7] about the C^*-envelope are true, but some of these use the real versions of facts from Chapters 1 and 2 of [7] that are discussed above. Examples (1) and (2) in 4.3.7 are essentially unchanged, except that one must use the classification of finite-dimensional real C^*-algebras in [35, Theorem 5.7.1]. We have not checked 4.3.8–4.3.11, although we would guess that 4.3.8 and the Dirichlet algebra results are unchanged. These results seem like a possibly interesting project. Theorem 4.4.3 in [7] is valid, and its Corollary 4.4.4 was already in [42]. The remaining results in Section 4.4 in [7] are unchanged, or follow by complexification as in the case of Youngson's Theorem 4.4.9 and the Corollary after that.

In view of the importance to real operator spaces and their complexifications of the space $\overline{X}$ discussed in and around Lemma 5.1, the following is often quite useful.

PROPOSITION 8.1. *If X (resp. A) is a complex operator space (resp. algebra), then*

$$\overline{(A^1)} = (\overline{A})^1, \quad I(\overline{X}) = \overline{I(X)}, \quad \mathcal{T}(\overline{X}) = \overline{\mathcal{T}(X)}, \quad C_e^*(\overline{A}) = \overline{C_e^*(A)}.$$

Proof. If $(I(X), j)$ is a (complex operator space) injective envelope of a complex operator space X, then $(\overline{I(X)}, \overline{j})$ is an injective envelope of $\overline{X}$. Indeed, as indicated in [16, Proposition 2.1], a routine diagram chase (i.e. applying the functor above the proposition to the 'universal injectivity and rigidity diagrams') shows that $\overline{I(X)}$ is injective, and has the 'rigidity property', so is an injective envelope of $\overline{X}$.

In complicated situations it is sometimes very useful to view the above in terms of a containing C^*-algebra B, where it is much easier to see what the 'bar' is doing in terms of the C^*-algebra adjoint and opposite (indeed recall from the proof of Lemma 5.1 that $\overline{B} = B^\circ$, the 'opposite' C^*-algebra). We take the time to spell this out in a bit more detail. Namely, consider the C^*-algebra $B = I(\mathcal{S}(X))$ which has $I(X)$ as its 1-2 corner Z (see e.g. [7, 4.4.2]). Viewed in this way, it is clear that $I(X)$ is a ternary system or TRO (that is, $ZZ^*Z \subset Z$). In terms of the 'opposite' X° and 'adjoint' $X^\star$ constructions from [7, 1.2.25] and [16, Proposition 2.1] we have $\overline{X} = (X^\circ)^\star$. We view X° and $I(X)^\circ$ as the appropriate subspaces of the 'opposite' C^*-algebra B°. Then $\overline{X}$ and $\overline{I(X)}$ are the 'adjoints' of the latter subspaces of B° (by adjoint we mean the C^*-algebra involution of the C^*-algebra B°). Similarly, for any complex subTRO Z of B the canonical map $Z \to \overline{Z}$ is therefore a conjugate linear TRO morphism and 'complete isometry'. Viewed in this way, it is easy to see that $\overline{I(X)}$ is again a TRO, and indeed $I(\overline{X}) = \overline{I(X)}$. Similarly, $(\overline{\mathcal{T}(X)}, \overline{j})$ is a ternary envelope of $\overline{X}$. That is, $\overline{\mathcal{T}(X)}$ is the subTRO of $\overline{I(X)}$ generated by $\overline{j}(\overline{X})$. This can also be deduced from [16, Proposition 2.1]:

$$\mathcal{T}(\overline{X}) = \mathcal{T}((X^\circ)^\star) = \mathcal{T}(X^\circ)^\star = (\mathcal{T}(X)^\circ)^\star = \overline{\mathcal{T}(X)}.$$

Similarly, if $X = A$ is a complex operator algebra then one may deduce in the same way from [16, Proposition 2.1] that $\overline{(A^1)} = (\overline{A})^1$, and $(\overline{C_e^*(A)}, \overline{j})$ is a C^*-envelope of $\overline{A}$. ∎

It follows for example from the last result and Lemma 5.1 that for a complex operator space X we have $I(X_{\rm c}) \cong I(X \oplus^\infty \overline{X}) \cong I(X) \oplus^\infty \overline{I(X)} \cong I(X)_{\rm c}$.

Concerning the Banach–Stone type Theorem 4.5.13, in the case that A is unital there are many proofs of this in the literature which still work in the real case. For example, the method in 8.3.13, whose proof is unchanged in the real case. The unital case also follows from the real unital C^*-algebra Banach–Stone type theorem [43, Theorem 4.4] (which generalizes the simple [7, Corollary 1.3.10]), by passing to the injective or C^*-envelope. We have several more general complex Banach–Stone theorems in e.g. [10, 11], and it would be interesting to see which of these are true in the real case (see also the paragraph after Proposition 2.10 in [15]). We remark that Ph.D. student Dylan Phelps is pursuing some of these directions under our co-direction with Labuschagne.

9. C^*-modules and TROs. Real Hilbert C^*-modules over a real C^*-algebra are defined almost exactly as in the complex case—see [29, Definition 2.4]. Let A be a real C^*-algebra and V a real right C^*-module over A. Then it is known (see e.g. [29]) that there is an A_c-valued inner product on $V_c = V + iV$,

$$\langle v + iw, x + iy\rangle = \langle v, x\rangle + \langle w, y\rangle + i(\langle v, y\rangle - \langle w, x\rangle),$$

extending the original A-valued inner product on V, such that V_c is a complex C^*-module over A_c. This complexification is 'reasonable' with respect to the canonical norm: $\|v - iw\| = \|v + iw\|$. Similarly, $M_n(V)$ is a real right C^*-module over $M_n(A)$, and its complexification is reasonable. With these matrix norms V is a real operator space, a real subspace of V_c with the latter regarded as a real operator space. Thus the C^*-module V_c with its canonical operator space structure is a (completely) reasonable complexification. It follows from Ruan's theorem that this canonical operator space structure coincides with the unique operator space complexification of V.

If $\{Y_t\}$ is a family of real right C^*-modules over A then the C^*-module sum $\bigoplus_t Y_t$ is a real right C^*-module over A as in the complex case and

$$\Big(\bigoplus_t Y_t\Big)_c \cong \bigoplus_t (Y_t)_c.$$

Indeed, one may define $\bigoplus_t Y_t$ to be the closure of the real span of the copies of Y_t in $\bigoplus_t (Y_t)_c$. Then the inherited inner product on this closed real subspace lies in A. Viewing the latter as an operator space as in [7, Section 8.2], it is clear that $\bigoplus_t (Y_t)_c$ is a reasonable operator space complexification of $\bigoplus_t Y_t$. Indeed, since A_c is a reasonable complexification of A it is easy to see that

$$\begin{aligned}\|(y_t) + i(z_t)\|^2 &= \Big\|\sum_t \langle y_t + iz_t, y_t + iz_t\rangle\Big\| = \Big\|\sum_t \langle y_t - iz_t, y_t - iz_t\rangle\Big\| \\ &= \|(y_t) - i(z_t)\|^2\end{aligned}$$

for $y_t, z_t \in Y$. We leave it to the reader to check the matricial case of this computation, that is, that $\bigoplus_t (Y_t)_c$ is a completely reasonable complexification. By Ruan's theorem we deduce that $(\bigoplus_t Y_t)_c \cong \bigoplus_t (Y_t)_c$.

If $T : Y \to Z$ is adjointable then it is easy to check that $T_{\rm c}$ is adjointable with $(T_{\rm c})^* = (T^*)_{\rm c}$. The set $\mathbb{B}_A(Y)$ of such adjointable $T : Y \to Y$ is therefore $*$-isomorphic to a real $*$-subalgebra of the complex C^*-algebra $\mathbb{B}_{A_{\rm c}}(Y_{\rm c})$. The following is useful:

Lemma 9.1. *If Y, Z are real right C^*-modules over A and $T : Y \to Z$ is an A-module map. Then $T_{\rm c} : Y_{\rm c} \to Z_{\rm c}$ is adjointable if and only if T is adjointable.*

Proof. The converse direction is obvious. If $T_{\rm c} : Y_{\rm c} \to Z_{\rm c}$ is adjointable then

$$\langle y, (T_{\rm c})^*(z)\rangle = \langle T_{\rm c} y, z\rangle = \langle Ty, z\rangle \in A, \qquad y \in Y,\ z \in Z.$$

Writing $(T_{\rm c})^*(z) = y_1 + iy_2$ we easily see, by examining the part of the left side of the last equation that is in iA, that $\langle y, y_2\rangle = 0$. Setting $y = y_2$ gives $\langle y_2, y_2\rangle = 0$. Hence $(T_{\rm c})^*(Y) \subset Z$, so that T is adjointable on Y. ∎

Also, $\mathbb{B}_A(Y)$ is closed in $B_B(Y)$ as in the complex case. So $D = \mathbb{B}_A(Y)$ is a real unital C^*-algebra, and we may view it as a real C^*-subalgebra of $E = \mathbb{B}_{A_{\rm c}}(Y_{\rm c})$. For $y, x, w \in Y$ we have

$$(|y\rangle\langle y|)x + i(|y\rangle\langle y|)w = y\langle y, x\rangle + iy\langle y, w\rangle.$$

Also within $\mathbb{B}_{A_{\rm c}}(Y_{\rm c})$ we have $|y\rangle\langle y|(x + iw) = y(\langle y, x\rangle + i\langle y, w\rangle)$. Thus $(|y\rangle\langle y|)_{\rm c} = |y\rangle\langle y|$ in $\mathbb{B}_{A_{\rm c}}(Y_{\rm c})$, which is a positive operator. Now $D_{\rm c}$ may be viewed as a complex C^*-subalgebra of $E_{\rm c}$, and $d \in D$ is in D_+ iff $d \in (D_{\rm c})_+$ iff $d \in (E_{\rm c})_+$ iff $d \in E_+$, where the latter are the positive operators in E, with E considered as a real C^*-algebra. However, by [35, Proposition 5.2.2] this is equivalent to d being positive in E in the usual sense. Thus $|y\rangle\langle y| \geq 0$ in $\mathbb{B}_A(Y)$. It follows that Y is a real left C^*-module over $\mathbb{B}_A(Y)$, and over $\mathbb{K}_A(Y)$, where the latter is the closure of the span of the $|y\rangle\langle z|$ for $y, z \in Y$.

Since every real C^*-algebra has an increasing contractive approximate identity (see the proof of [35, Proposition 5.2.4]), 8.1.3 and 8.1.4(1) in [7] hold. Since Cohen's factorization theorem works for real Banach modules over real C^*-algebras, as we said at the start of our discussion of Chapter 2 in [7], 8.1.4(2) holds and so does the rest of 8.1.4. Lemma 8.1.5 may be proved by complexification. Proposition 8.1.6 holds with the same proof, but the argument after that proof fails. This is the assertion that the inner product on a C^*-module Y is completely determined by, and may be recovered from, the Banach module structure. To see this, simply complexify and use the complex case of this remark.

The definitions and arguments in 8.1.7 hold except for the point involving the polarization identity. However, this point is proved in [29]: A surjective isometric A-module map between real right C^*-modules is an (adjointable) unitary A-module map. Indeed, we have the real version of 8.1.8; and 8.1.9, 8.1.11, 8.1.14 and 8.1.15 hold with the same arguments (in some of these using again that Cohen's factorization theorem holds in the real case). We do not have the results in 8.1.12 and 8.1.13, of course. It is easy to see that the real variant of [7, Proposition 8.1.16(1)–(3)] holds. Item (4) there fails (take $Y = B$ a real unital C^*-algebra whose selfadjoint elements do not span B). The theory of the linking C^*-algebra in 8.1.17–8.1.19 is unchanged. We have:

THEOREM 9.2. *For real right C^*-modules Y, Z over A we have*

$$\begin{aligned} \mathbb{K}_{A_c}(Y_c, Z_c) &\cong \mathbb{K}_A(Y, Z)_c, \\ \mathbb{B}_{A_c}(Y_c, Z_c) &\cong \mathbb{B}_A(Y, Z)_c, \\ (B_A(Y, Z))_c &\cong B_{A_c}(Y_c, Z_c) \end{aligned}$$

completely isometrically. If $Y = Z$ then the first two of these isomorphisms are $*$-isomorphisms, and the last is also an algebra isomorphism. For the linking algebra, $\mathcal{L}(Y)_c \cong \mathcal{L}(Y_c)$ $*$-isomorphically.

Proof. By replacing the spaces by the C^*-module sum $Y \oplus Z$ we may assume that $Y = Z$. Then, since Y_c is an equivalence $\mathbb{K}_{A_c}(Y_c)$-A_c-bimodule and also an equivalence $\mathbb{K}_A(Y)_c$-A_c-bimodule, it follows that $\mathbb{K}_{A_c}(Y_c) = \mathbb{K}_A(Y)_c$. The action of $\mathbb{K}_A(Y)_c$ on Y_c is as

$$\begin{aligned} (R + iS)(x + iy) &= Rx - Sy + i(Ry + Sx) = R_c(x + iy) + iS_c(x + iy) \\ &= (R_c + iS_c)(x + iy). \end{aligned}$$

If $R_c = iS_c$ then $Rx + iRy = -Sy + iSx$ for all $x, y \in Y$ and $R = S = 0$. Thus $\mathbb{K}_{A_c}(Y_c)$ may be identified as a C^*-algebra with $\{R_c + iS_c \in \mathbb{B}_{A_c}(Y_c) : R, S \in \mathbb{K}_A(Y)\}$. We have

$$(B_A(Y))_c \cong (LM(\mathbb{K}_A(Y)))_c \cong LM((\mathbb{K}_A(Y))_c) \cong LM(\mathbb{K}_{A_c}(Y_c)) \cong B_{A_c}(Y_c),$$

and similarly $(\mathbb{B}_A(Y))_c \cong \mathbb{B}_{A_c}(Y_c)$. Finally,

$$\mathcal{L}(Y)_c \cong \mathbb{K}_A(Y \oplus A)_c \cong \mathbb{K}_{A_c}((Y \oplus A)_c) \cong \mathbb{K}_{A_c}(Y_c \oplus A_c) \cong \mathcal{L}(Y_c),$$

and $\mathcal{L}(Y)$ is clearly a $*$-subalgebra of $\mathcal{L}(Y_c)$. ∎

By taking $Y = C_n(A)$ and considering the natural $*$-homomorphism $M_n(A) \to \mathbb{K}_A(C_n(A))$ we see that the norm of a matrix in $M_n(A)$ is given by the formulae in [7, Corollary 8.1.13]. However, the equivalence with (ii) in that result is not valid even if $A = \mathbb{R}$ and a is selfadjoint and $n = 2$.

Corollary 8.1.20 is true, but needs a change at a couple of points of the proof. One first replaces the appeal to Proposition 8.1.16(4) to an appeal to a fact about the diagonal mentioned after Theorem 6.1. This shows that the contractive unital homomorphism $\pi : Z(M(B)) \to CB_B(Y)$, which may be viewed as mapping into $LM(\mathbb{K}_B(Y))$, actually maps into $\Delta(LM(\mathbb{K}_B(Y))) = M(\mathbb{K}_B(Y)) = \mathbb{B}_B(Y)$.

One may avoid the problematic parts of the proof of Corollary 8.1.21 in the real case by instead proving the result by complexification. Indeed, in that proof P_{c} will be adjointable on Y_{c} by the complex version of this result, so that P is adjointable on Y by Lemma 9.1. In the proof of 8.1.22 we may apply the argument for contractivity of R to $R_{\mathrm{c}} = \sum_i (Q_i)_{\mathrm{c}}$, to see that R_{c}, and hence R, is contractive. The rest of that proof works in the real case. Items 8.1.23–8.1.25 are true with the same proof, and 8.1.26 follows by complexification. The proof of Theorem 8.1.27(1) uses the polar decomposition of maps on complex C^*-modules from the source cited there. Inspecting that source shows that the argument there works in the real case.

Item 8.2.1 is valid in the real case, and is important (some of this we have stated already). If $T : Y \to Y$ is a contractive A-module map, then by Theorem 9.2 we see that T_{c} is contractive, hence completely contractive by the complex case. Thus [7, Proposition 8.2.2] holds. We also deduce that $\|T\| = \|T_{\mathrm{c}}\| = \|T_{\mathrm{c}}\|_{\mathrm{cb}}$, by a property of the operator space complexification. Also, the paragraph after that proposition, and the items in 8.2.3 and 8.2.4, hold in the real case. Items 8.2.5–8.2.7 are valid in the real case, although one needs to check that the facts cited in 8.2.5 from other sources hold in the real case. For this, it is useful to note that if $a \in A$ is a strictly positive element in a real C^*-algebra A (so that $a = a^* \geq 0$ and $\varphi(a) > 0$ for all real states φ on A), then a is strictly positive in A_{c}. This is because if ψ is a state on A_{c} then $\psi(a) \geq 0$ and $\mathrm{Re}\, \psi_A$ is a state on A so that $\psi(a) > 0$. If a is a strictly positive element then $a^{1/n}$ will be a countable cai, as usual.

Essentially all of the rest of Section 8.2 is true in the real case with the same arguments. For example, item 8.2.24 is true in the real case, and we note that lattices (1)–(4) there are also in correspondence to the analogous lattices in the complexified spaces. Similar statements hold for the real variant of 8.2.25. Some of the real versions of basic facts about TROs in [7, Section 8.3] are discussed at the end of the introduction to [5]. The rest of the items in Section 8.3 are also valid with the same proofs.

In Section 8.4 a word needs to be said about formula (8.17) since the proof of this there uses the span of Hermitian elements. However, (8.17) may be seen directly: let C be the algebra on the right side of (8.16). Then $\Delta(C)$ is a $*$-subalgebra of $\Delta(CB_{\mathcal{F}}(\mathcal{T}(X))) = \mathbb{B}_{\mathcal{F}}(\mathcal{T}(X))$. Thus $\Delta(C)$ is a $*$-subalgebra of the algebra on the right side of (8.17). Conversely, the latter algebra is clearly contained in $\Delta(C)$.

Similarly, the proof that (iii) implies (i) in Theorem 8.4.4 needs to be altered in the real case: if T and R are as in (iii), note that T_{c} and R_{c} satisfy the analogous relation for the complexifications. Thus $T_{\mathrm{c}} \in \mathcal{A}_\ell(X_{\mathrm{c}}) \cong \mathcal{A}_\ell(X)_{\mathrm{c}}$ and hence $T \in \mathcal{A}_\ell(X)$. (Alternatively, T_{c} is left multiplication by an element in $I_{11}(X_{\mathrm{c}})$, so can be identified with left multiplication by $a + ib$

for $a, b \in \mathcal{M}_\ell(X) \subset I_{11}(X)$. Since $(a^* + ib^*)j_c(X_c) \subset j_c(X_c)$ we see that $a \in \mathcal{A}_\ell(X)$.) Items 8.5.1–8.5.21 are as in the complex case, although in 8.5.20, the M-ideal case seems difficult and only recently proved in [12] (see the "Added in proof" below). In the complete M-ideal case the results follow by working in the complexification.

Reaching fatigue, we will say nothing about the real case of the remaining results 8.5.22–8.5.40, except that most of these items seemed unproblematic on our first pass through them, but we have not checked them carefully. This would be a nice project since many of these results are extremely important. Most of these items are not operator space results, but rather are von Neumann algebraic, so one might expect that a few of these results will be in the literature.

Finally, the real versions of most results in the first part of Section 8.6 are essentially already noted by Ruan at the end of his two papers on real operator spaces, although he leaves the details to the reader. Indeed, he points out these operator space properties hold on X if and only if they hold on X_c, so that the real cases of results in 8.6.1–8.6.3 follow from their complex versions.

Added in proof. Much of the present paper concerns real versions of the theory of operator spaces and algebras represented in [7] or from around the time of that text. Recently we have extended some more current aspects of this theory, such as e.g. real positivity, and some aspects of complex Jordan operator algebras, to the real setting. Also, the questions we raised on M-ideals in real operator algebras and real TRO's were solved after the acceptance of this paper (for the latter see [12]).

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