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GEBELEIN INEQUALITY IN A HILBERT SPACE

Abstract. We present the Gebelein inequality in a separable real Hilbert space. As an application we prove the Strong Law of Large Numbers for Gaussian functionals with values in a separable real Banach space.

1. Introduction. Let μ be a standard Gaussian measure on the real line $\mathbb{R}$ and $|\rho| \le 1$. We use $L^2(\mu)$ for $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of subsets of $\mathbb{R}$. In $L^2(\mu)$ we have the inner product

$$\langle f, g\rangle_\mu = \int_{\mathbb{R}} f(x)g(x)\,d\mu(x), \qquad f, g \in L^2(\mu),$$

and the norm

$$\|f\|_2 = \Big(\int_{\mathbb{R}} f^2(x)\,d\mu(x)\Big)^{1/2}, \qquad f \in L^2(\mu).$$

We recall the Hermite polynomials

$$H_0 \equiv 1, \qquad H_n(x) = (-1)^n \exp(x^2/2)\frac{d^n}{dx^n}(\exp(-x^2/2)), \qquad x \in \mathbb{R},\ n \ge 1,$$

and their generating function

$$w(t,x) := \exp(tx - t^2/2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x), \qquad t, x \in \mathbb{R}. \tag{1.1}$$

We put $h_n := H_n/\sqrt{n!}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It is known that the collection $\{h_n\}_{n\in\mathbb{N}_0}$ forms an orthonormal basis in $L^2(\mu)$.

The Ornstein–Uhlenbeck operator

$$P_\rho : L^2(\mu) \to L^2(\mu)$$

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is defined by

$$(P_\rho f)(y) = \int_{\mathbb{R}} f(\rho y + \sqrt{1-\rho^2}\, z)\, d\mu(z), \qquad y \in \mathbb{R},\ f \in L^2(\mu).$$

The Ornstein–Uhlenbeck operator has the following probabilistic interpretation. Let random variables X and Y have the Gaussian distribution μ and let $\operatorname{cov}(X,Y) = E(XY) = \rho$. Then for $f, g \in L^2(\mu)$, we have

$$E[f(X)g(Y)] = E[E(f(X)\,|\,Y)g(Y)] = E[h(Y)g(Y)], \tag{1.2}$$

where $h(y) = E(f(X)\,|\,Y = y)$, $y \in \mathbb{R}$. On the other hand, if Z is a Gaussian random variable with the standard distribution and independent of Y, then

$$\mathcal{L}(U, Y) = \mathcal{L}(X, Y),$$

where $U = \rho Y + \sqrt{1-\rho^2}\, Z$ and $\mathcal{L}(X,Y)$ denotes the distribution of the random vector (X,Y). Thus

$$\begin{aligned} E[f(X)g(Y)] &= E[f(U)g(Y)] = E[E(f(U)\,|\,Y)g(Y)] \\ &= E[E(f(\rho Y + \sqrt{1-\rho^2}\, Z)\,|\,Y)g(Y)] = E[P_\rho f(Y)g(Y)]. \end{aligned} \tag{1.3}$$

By comparing (1.2) and (1.3) we see that $(P_\rho f)(Y)$ is a version of the conditional expectation $E[f(X)\,|\,Y]$. It is easy to see that P_ρ is symmetric ($\langle P_\rho f, g\rangle_\mu = \langle f, P_\rho g\rangle_\mu$, $f, g \in L^2(\mu)$) and a linear contraction in $L^2(\mu)$. It is clear that P_ρ is an isometric isomorphism in $L^2(\mu)$ when $|\rho| = 1$. Moreover, the Hermite polynomials H_n, $n \in \mathbb{N}_0$, are its eigenvectors, that is,

$$P_\rho H_n = \rho^n H_n, \qquad n \in \mathbb{N}_0,$$

and P_ρ has the following expansion in the Hermite basis:

$$P_\rho f = \sum_{n=0}^{\infty} \rho^n \langle f, h_n\rangle_\mu h_n, \qquad f \in L^2(\mu).$$

We recall the Gebelein inequality.

THEOREM 1.1 ([G], [DK], [B]). *If $f \in L^2(\mu)$, $\langle f, 1\rangle_\mu = 0$ and $|\rho| \le 1$, then*

$$\|P_\rho f\|_2 \le |\rho|\, \|f\|_2,$$

with equality if and only if f is a linear function. ∎

Using the Gebelein inequality, one can prove the Strong Law of Large Numbers for Gaussian functionals.

THEOREM 1.2 ([BC]). *Let $\{X_i\}_{i\ge1}$ be a Gaussian sequence of standard random variables such that*

$$\sup_{i\ge1} \sum_{j=1}^{\infty} |\rho_{ij}| < \infty,$$

where $\rho_{ij} = E(X_iX_j)$, $i,j \geq 1$. *Then for* $f \in L^1(\mu)$ *we have*

$$\frac{f(X_1) + \cdots + f(X_n)}{n} \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} f\, d\mu \quad a.s. \ \blacksquare$$

Note that for a centered Gaussian vector $\mathbf{V} = (X, Y)$ with covariance matrix

$$\operatorname{cov}(\mathbf{V}) = \begin{bmatrix} \sigma^2 & \sigma^2\rho \\ \sigma^2\rho & \sigma^2 \end{bmatrix},$$

where $\sigma^2 = EX^2 = EY^2$ and ρ is a correlation coefficient of $\mathbf{V}$, we can also define the Ornstein–Uhlenbeck operator. Namely, for $f \in L^2(\mu_\sigma)$, where μ_σ is the distribution of X, we put

$$(P_\rho f)(y) = \int_{\mathbb{R}} f(\rho y + \sqrt{1-\rho^2}\, z)\, d\mu_\sigma(z), \qquad y \in \mathbb{R}.$$

Now, the orthogonal Hermite polynomials have the form

$$H_{\sigma,n}(x) := H_n(x/\sigma), \qquad x \in \mathbb{R},$$

and if normalized in $L^2(\mu_\sigma)$,

$$h_{\sigma,n}(x) := H_{\sigma,n}(x)/\sqrt{n!}, \qquad x \in \mathbb{R}.$$

The orthonormal system $\{h_{\sigma,n}\}_{n\geq 0}$ is a basis in $L^2(\mu_\sigma)$ and

$$P_\rho h_{\sigma,n} = \rho^n h_{\sigma,n}, \qquad n \geq 0.$$

Moreover, it is easy to check that in this case the Gebelein inequality has the same form as in Theorem 1.1. This observation concerning the random vector $\mathbf{V}$ shows that we can extend our considerations about the Gebelein inequality and the Ornstein–Uhlenbeck operator to the case of a Hilbert space.

2. Gaussian measures on the Cartesian product of Hilbert spaces. Let H be a fixed (infinite-dimensional) real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote by $L(H) := L(H,H)$ the Banach algebra of all continuous linear operators from H into H. It is well known that the Cartesian product $H \times H$ is also a real separable Hilbert space with inner product

$$\langle x, y\rangle_{H\times H} := \langle x_1, y_1\rangle + \langle x_2, y_2\rangle,$$

where $x = (x_1, x_2) \in H \times H$ and $y = (y_1, y_2) \in H \times H$. Thus the norm of $H \times H$ is equal to

$$\|x\|_{H\times H} := \sqrt{\|x_1\|^2 + \|x_2\|^2}, \qquad x = (x_1, x_2) \in H \times H.$$

It is known that if a system $\{e_n\}_{n\geq 1}$ is an orthonormal basis in H, then the system $\{(e_n, 0)\}_{n\geq 1} \cup \{(0, e_n)\}_{n\geq 1}$ is an orthonormal basis in $H \times H$.

Let $B \in L(H \times H)$. Then for $x, y \in H$ we have

$$B(x, y) = (B_1(x, y), B_2(x, y)),$$

where $B_1, B_2 \in L(H \times H, H)$. Note that for $x, y \in H$,

$$B_1(x, y) = B_1(x, 0) + B_1(0, y),$$
$$B_2(x, y) = B_2(x, 0) + B_2(0, y).$$

Hence we can introduce operators $B_{ij} \in L(H)$, $i, j = 1, 2$, as follows:

$$B_{11}(x) = B_1(x, 0), \quad B_{12}(y) = B_1(0, y), \quad x, y \in H,$$
$$B_{21}(x) = B_2(x, 0), \quad B_{22}(y) = B_2(0, y), \quad x, y \in H.$$

and we have

$$B_1(x, y) = B_{11}(x) + B_{12}(y), \quad x, y \in H,$$
$$B_2(x, y) = B_{21}(x) + B_{22}(y), \quad x, y \in H.$$

Therefore, we can represent the operator B in matrix form:

$$B(x, y) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (x, y) \in H \times H,$$

and briefly

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

We denote by $\mathcal{B}(H)$ the Borel σ-algebra of H and by μ_Q a fixed centered Gaussian measure on $(H, \mathcal{B}(H))$ with covariance operator Q such that $\operatorname{Ker} Q = \{0\}$ (then $\operatorname{supp}(\mu_Q) = H$). It is well known that there exists a complete orthonormal basis $\{e_n\}_{n \geq 1}$ on H and a sequence $\{\lambda_n\}_{n \geq 1}$ of positive numbers such that

$$Q(e_n) = \lambda_n e_n, \quad n \in \mathbb{N}, \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Without loss of generality we may and will assume that

$$\lambda_1 = \cdots = \lambda_{d_1} > \lambda_{d_1+1} = \cdots = \lambda_{d_2} > \lambda_{d_2+1} = \cdots.$$

Then for each $i \geq 1$ we have

$$d_i - d_{i-1} = \dim[\operatorname{Ker}(\lambda_{d_i} I - Q)], \quad d_0 := 0.$$

We recall that the *Cameron–Martin space* $Q^{1/2}(H) \subset H$ can be defined by

$$Q^{1/2}(H) = \Big\{ y \in H : \sum_{n=1}^{\infty} \langle y, e_n \rangle^2 / \lambda_n < \infty \Big\}.$$

Let us consider the mapping

$$W : Q^{1/2}(H) \to L^2(H, \mu_Q), \quad Q^{1/2}(H) \ni y \mapsto W_y \in L^2(H, \mu_Q),$$

where $W_y(x) = \langle x, Q^{-1/2}y\rangle$ for $x \in H$. If $y_1, y_2 \in Q^{1/2}(H)$ then

$$\int_H W_{y_1} W_{y_2}\, d\mu_Q = \langle QQ^{-1/2}y_1, Q^{-1/2}y_2\rangle = \langle y_1, y_2\rangle.$$

Hence W is an isometry. Since $Q^{1/2}(H)$ is dense in H, the mapping W can be uniquely extended to H. The operator W is called the *white noise mapping.* Note that for fixed $y \in H$ the random variable W_y is a centered Gaussian random variable on the probability space $(H, \mathcal{B}(H), \mu_Q)$ with variance $\|y\|^2$. Moreover, if $y = \sum_{n=1}^{\infty} \langle y, e_n\rangle e_n$, then

$$W_y = \sum_{n=1}^{\infty} \langle y, e_n\rangle W_{e_n} = \sum_{n=1}^{\infty} \langle y, e_n\rangle \frac{\langle \cdot, e_n\rangle}{\sqrt{\lambda_n}} \tag{2.4}$$

and for $S \in L(H)$,

$$W_{Sy} = \sum_{n=1}^{\infty} \langle Sy, e_n\rangle W_{e_n} = \sum_{n=1}^{\infty} \langle y, e_n\rangle W_{Se_n} \quad \text{in } L^2(\mu_Q).$$

Let $B \in L(H \times H)$. We say that B is *positive* (written $B \ge 0$) if

$$\langle B(x,y), (x,y)\rangle_{H\times H} \ge 0 \quad \text{for all } x, y \in H.$$

Let Q be the covariance operator as above and let $R \in L(H)$. Assume that an operator $B \in L(H \times H)$ has the form

$$B = \begin{bmatrix} Q & QR \\ R^*Q & Q \end{bmatrix}, \tag{2.5}$$

where R^* is the adjoint operator of R. We see at once that B is symmetric (i.e. $B = B^*$).

LEMMA 2.1. *An operator $B \in L(H \times H)$ of the form* (2.5) *is positive if and only if*

$$\|Q^{1/2}RQ^{-1/2}\|_{Q^{1/2}(H)} \le 1, \tag{2.6}$$

where $\|\cdot\|_{Q^{1/2}(H)}$ denotes the norm of H restricted to the Cameron–Martin space $Q^{1/2}(H)$. Moreover, if $QR = RQ$ then B is positive if and only if $\|R\| \le 1$.

Proof. Let $(x,y) \in H \times H$. Then

$$\begin{aligned}
\langle B(x,y), (x,y)\rangle_{H\times H} &= \langle (Qx + QRy, R^*Qx + Qy), (x,y)\rangle_{H\times H} \\
&= \langle Qx, x\rangle + \langle QRy, x\rangle + \langle R^*Qx, y\rangle + \langle Qy, y\rangle \\
&= \langle Qx, x\rangle + 2\langle QRy, x\rangle + \langle Qy, y\rangle \\
&= \|Q^{1/2}x\|^2 + 2\langle Q^{1/2}RQ^{-1/2}Q^{1/2}y, Q^{1/2}x\rangle + \|Q^{1/2}y\|^2 \\
&= \|u\|^2 + 2\langle Q^{1/2}RQ^{-1/2}v, u\rangle + \|v\|^2,
\end{aligned}$$

where $u := Q^{1/2}x$ and $v := Q^{1/2}y$. Hence B is positive if and only if

$$\|u\|^2 + 2\langle Q^{1/2}RQ^{-1/2}v, u\rangle + \|v\|^2 \geq 0 \quad \text{for all } u, v \in Q^{1/2}(H). \tag{2.7}$$

Let us assume that (2.7) is fullfilled. Then

$$\|u\|^2 + \|v\|^2 \geq -2\langle Q^{1/2}RQ^{-1/2}v, u\rangle, \quad u, v \in Q^{1/2}(H). \tag{2.8}$$

Putting $-u$ instead of u in (2.8), we get

$$\|u\|^2 + \|v\|^2 \geq 2\langle Q^{1/2}RQ^{-1/2}v, u\rangle, \quad u, v \in Q^{1/2}(H). \tag{2.9}$$

From (2.8) and (2.9) we obtain

$$\|u\|^2 + \|v\|^2 \geq 2|\langle Q^{1/2}RQ^{-1/2}v, u\rangle|, \quad u, v \in Q^{1/2}(H).$$

Taking the sup over all $u, v \in Q^{1/2}(H)$ such that $\|u\| = \|v\| = 1$ we obtain (2.6). Conversely, assume that (2.6) is fullfilled. Then

$$|\langle Q^{1/2}RQ^{-1/2}v, u\rangle| \leq \|v\|\,\|u\|, \quad u, v \in Q^{1/2}(H).$$

Hence

$$\langle Q^{1/2}RQ^{-1/2}v, u\rangle \geq -\|v\|\,\|u\|, \quad u, v \in Q^{1/2}(H).$$

Therefore

$$\|u\|^2 + 2\langle Q^{1/2}RQ^{-1/2}v, u\rangle + \|v\|^2 \geq \|u\|^2 - 2\|u\|\,\|v\| + \|v\|^2 = (\|u\| - \|v\|)^2 \geq 0,$$

where $u, v \in Q^{1/2}(H)$. From (2.7) it follows that B is positive. The second part of the lemma follows from the first part and from the density of the Cameron–Martin $Q^{1/2}(H)$ space in H. ∎

An operator $T \in L(H)$ is said to be *nuclear* if there exist two sequences $\{h_i\}_{i\geq 1}, \{g_i\}_{i\geq 1} \subset H$ such that $\sum_{i=1}^{\infty} \|h_i\|\,\|g_i\| < \infty$ and T has the representation

$$Tx = \sum_{i=1}^{\infty} \langle x, h_i\rangle g_i, \quad x \in H.$$

For a nuclear operator T, we can define its *trace* by $\operatorname{tr}(T) = \sum_{i=1}^{\infty} \langle Tf_i, f_i\rangle$, where $\{f_i\}_{i\geq 1}$ is an othonormal basis of H. It is known that $\operatorname{tr}(T)$ is a well-defined number, independent of the choice of $\{f_i\}_{i\geq 1}$. Moreover, a symmetric positive operator $T \in L(H)$ is nuclear if and only if for some (or each) orthonormal basis $\{f_i\}_{i\geq 1}$ of H we have $\sum_{i=1}^{\infty} \langle Tf_i, f_i\rangle < \infty$.

Now, we are going to show that the operator $B \in L(H \times H)$ of the form (2.5) is under certain assumptions the covariance operator of some centered Gaussian measure on $(H \times H, \mathcal{B}(H \times H))$.

THEOREM 2.2. *Let Q be the covariance operator as above and $R \in L(H)$ be such that $\|R\| \leq 1$ and $RQ = QR$. Then the operator $B \in L(H \times H)$ of the form* (2.5) *is the covariance operator of some centered Gaussian measure on $(H \times H, \mathcal{B}(H \times H))$.*

Proof. The symmetry of B is obvious. Lemma 2.1 implies positivity. An easy computation shows that B has a finite trace. Then the conclusion follows from the Mourier theorem (see e.g. [VTC]). ∎

3. The Ornstein–Uhlenbeck operator on a Hilbert space. Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space and let $(X, Y) : \Omega \to H \times H$ be a centered Gaussian vector with covariance operator B of the form (2.5), where $QR = RQ$ and $\|R\| \le 1$. By definition of the covariance operator of (X, Y), we have

$$\begin{aligned} Q(x) &= \int_\Omega X\langle X, x\rangle\, dP = \int_\Omega Y\langle Y, x\rangle\, dP, \\ (QR)(x) &= \int_\Omega X\langle Y, x\rangle\, dP, \\ (R^*Q)(x) &= \int_\Omega Y\langle X, x\rangle\, dP, \end{aligned}$$

where $x \in H$ and the above integrals are in the Bochner sense. Let $Z : \Omega \to H$ be a centered Gaussian vector with covariance operator Q and independent of the random vector Y. Let us denote

$$U = RY + \sqrt{I - RR^*}Z,$$

where I is the identity operator on H. Note that $I - RR^*$ is a symmetric and positive operator.

Now, we determine the covariance operator of the random vector (U, Y). For $x \in H$ we have

$$\begin{aligned} \int_\Omega U\langle U, x\rangle\, dP &= \int_\Omega (RY + \sqrt{I - RR^*}\, Z)\langle RY + \sqrt{I - RR^*}\, Z, x\rangle\, dP \\ &= \int_\Omega RY\langle RY, x\rangle\, dP + \int_\Omega RY\langle \sqrt{I - RR^*}\, Z, x\rangle\, dP \\ &\quad + \int_\Omega \sqrt{I - RR^*}\, Z\langle RY, x\rangle\, dP + \int_\Omega \sqrt{I - RR^*}\, Z\langle \sqrt{I - RR^*}\, Z, x\rangle\, dP \\ &= \int_\Omega RY\langle RY, x\rangle\, dP + \int_\Omega \sqrt{I - RR^*}\, Z\langle \sqrt{I - RR^*}\, Z, x\rangle\, dP \\ &= RQR^*x + \sqrt{I - RR^*}\, Q\sqrt{I - RR^*}\, x = Q(x). \end{aligned}$$

Similarly,

$$\int_\Omega U\langle Y, x\rangle\, dP = \int_\Omega RY\langle Y, x\rangle\, dP = RQ(x) = QR(x)$$

and

$$\int_\Omega Y\langle U, x\rangle\, dP = \int_\Omega Y\langle RY, x\rangle\, dP = QR^*(x) = R^*Q(x).$$

Hence, we see that (X,Y) and (U,Y) have equal covariance operators. This implies that $\mathcal{L}(X,Y)=\mathcal{L}(U,Y)$. So, we can define the Ornstein–Uhlenbeck operator $P_R : L^2(\mu_Q)\to L^2(\mu_Q)$, where $L^2(\mu_Q)$ is shorthand for $L^2(H,\mathcal{B}(H),\mu_Q)$,

$$\begin{aligned}(P_R f)(y) &= E[f(X)\,|\,Y=y] = E[f(U)\,|\,Y=y]\\ &= \int\limits_H f(Ry+\sqrt{I-RR^*}\,z)\,d\mu_Q(z), \qquad y\in H.\end{aligned}$$

It is easy to see that P_R is a contraction on $L^2(\mu_Q)$. Let us point out that we can define P_R on $L^p(\mu_Q)$, $p\ge 1$, and in this case P_R is also a contraction. The operator P_R is symmetric if R is symmetric.

For a sequence $n=\{n_i\}_{i\ge 1}\subset \mathbb{N}_0$ we define

$$|n| := \sum_{i=1}^{\infty} n_i \quad \text{and} \quad n! := \prod_{i=1}^{\infty} n_i!.$$

Let us introduce the sets

$$\begin{aligned}\Lambda &:= \{n=\{n_i\}_{i\ge 1}\in \mathbb{N}_0^{\mathbb{N}} : |n|<\infty\},\\ \Lambda_r &:= \{n=\{n_i\}_{i\ge 1}\in \Lambda : n_i=0,\ i>r\}, \qquad r\in\mathbb{N}.\end{aligned}$$

For $n=\{n_i\}_{i\ge1}\in\Lambda$ we define Hermite polynomials on H by

$$H_n(x)=\prod_{i=1}^{\infty} H_{n_i}(W_{e_i}(x)), \qquad x\in H,$$

and

$$h_n(x)=H_n(x)/\sqrt{n!}, \qquad x\in H,$$

where $\{e_n\}_{n\ge1}$ is as above (i.e. $\{e_n\}_{n\ge1}$ is the basis of H composed of normalized eigenvectors of the operator Q) and W is the white noise mapping.

THEOREM 3.1 ([N]). *The system $\{h_n\}_{n\in\Lambda}$ is an orthonormal basis in $L^2(\mu_Q)$.* ∎

For any $n\ge 1$ we will denote by $\mathcal{H}_n$ the closed linear subspace of $L^2(\mu_Q)$ generated by the random variables $\{H_n(W_y): y\in H,\ \|y\|=1\}$, and $\mathcal{H}_0$ will be the set of constants. It is well known that the subspaces $\mathcal{H}_n$ and $\mathcal{H}_m$ are orthogonal whenever $n\ne m$. The subspace $\mathcal{H}_n$, $n\ge 0$, is called the *Wiener chaos of order* n, and the set $\{h_m : |m|=n,\ m\in\Lambda\}$ is an orthonormal basis in $\mathcal{H}_n$ (see e.g. [N]).

THEOREM 3.2 ([N]). *The space $L^2(\mu_Q)$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{H}_n$, $n\ge0$, i.e.*

$$L^2(\mu_Q)=\bigoplus_{n=0}^{\infty}\mathcal{H}_n. \ ∎$$

In the next assertion we determine a generating function of the Hermite polynomials $\{h_n\}_{n\in\Lambda}$.

LEMMA 3.3. *For $t \in H$ we have*

$$\exp(W_t - \|t\|^2/2) = \sum_{n\in\Lambda} \frac{t^n}{n!} H_n = \sum_{n\in\Lambda} \frac{t^n}{\sqrt{n!}} h_n, \tag{3.10}$$

where $t = \sum_{i\geq 1} t_i e_i$, $t_i = \langle t, e_i\rangle$, $i \geq 1$, and $t^n = \prod_{i\geq 1} t_i^{n_i}$ with the convention $0^0 := 1$. The convergence in (3.10) *is in the norm of $L^2(\mu_Q)$.*

Proof. We will show that the Fourier coefficients of $\omega_t := \exp(W_t - \|t\|^2/2)$ with respect to the basis $\{h_n\}_{n\geq 1}$ are equal to $t^n/\sqrt{n!}$, $n \in \Lambda$. Let $n = \{n_i\}_{i\geq 1} \in \Lambda$. Assume that $n = 0 := (0, 0, \ldots)$. Then

$$\langle \omega_t, h_0\rangle_{\mu_Q} = \langle \omega_t, 1\rangle_{\mu_Q} = \int_H \exp(W_t - \|t\|^2/2)\, d\mu_Q = 1.$$

Now, let $n \neq 0$. Then there is $m \in \mathbb{N}$ such that $n_i = 0$ for $i > m$. Using (2.4), independence of $\{W_{e_i}\}_{i\geq 1}$ and (1.1) we have

$$\begin{aligned}
\langle \omega_t, h_n\rangle_{\mu_Q} &= \int_H \exp(W_t - \|t\|^2/2) h_n\, d\mu_Q \\
&= \frac{1}{\sqrt{n!}} \exp(-\|t\|^2/2) \int_H \exp(W_t) H_n\, d\mu_Q \\
&= \frac{1}{\sqrt{n!}} \exp(-\|t\|^2/2) \int_H \exp\Big(\sum_{i=1}^{\infty} t_i W_{e_i}\Big) \prod_{i=1}^{m} H_{n_i}(W_{e_i})\, d\mu_Q \\
&= \frac{1}{\sqrt{n!}} \exp(-\|t\|^2/2) \int_H \exp\Big(\sum_{i=m+1}^{\infty} t_i W_{e_i}\Big)\, d\mu_Q \\
&\quad \times \int_H \exp\Big(\sum_{i=1}^{m} t_i W_{e_i}\Big) \prod_{i=1}^{m} H_{n_i}(W_{e_i})\, d\mu_Q \\
&= \frac{1}{\sqrt{n!}} \exp(-\|t\|^2/2) \exp\Big(\frac{1}{2} \sum_{i=m+1}^{\infty} \langle t, e_i\rangle^2\Big) \\
&\quad \times \prod_{i=1}^{m} \int_H \exp(t_i W_{e_i}) H_{n_i}(W_{e_i})\, d\mu_Q \\
&= \frac{1}{\sqrt{n!}} \exp(-\|t\|^2/2) \prod_{i=m+1}^{\infty} \exp(t_i^2/2) \\
&\quad \times \prod_{i=1}^{m} \int_H \exp(t_i^2/2) \sum_{j=0}^{\infty} \frac{t_i^j}{j!} H_j(W_{e_i}) H_{n_i}(W_{e_i})\, d\mu_Q \\
&= \frac{1}{\sqrt{n!}} \prod_{i=1}^{m} \frac{t_i^{n_i}}{n_i!} n_i! = \frac{1}{\sqrt{n!}} t^n. \quad \blacksquare
\end{aligned}$$

COROLLARY 3.4. *Assume that* $t = \sum_{i=1}^{r} t_i e_i \in H$, $r \in \mathbb{N}$. *Then equality* (3.10) *has the form*

$$\exp(W_t - \|t\|^2/2) = \sum_{n \in \Lambda_r} \frac{t^n}{n!} H_n = \sum_{n \in \Lambda_r} \frac{t^n}{\sqrt{n!}} h_n. \ \blacksquare$$

4. Main result. The set of all infinite matrices (with countable rows and columns) with elements from $\mathbb{R}$ (or $\mathbb{N}_0$) is denoted by $\mathcal{M}_\infty(\mathbb{R})$ (resp. $\mathcal{M}_\infty(\mathbb{N}_0)$). If $M \in \mathcal{M}_\infty(\mathbb{R})$, the jth column and ith row of M are denoted by M_j and M^i respectively. From time to time we shall use the shorthand $M = [M_j^i]$. As usual we identify rows and columns of M with vectors from $\mathbb{R}^\infty$. Let us introduce the set

$$\mathcal{M}_\Lambda(\mathbb{N}_0) = \{K \in \mathcal{M}_\infty(\mathbb{N}_0) : |K| \in \Lambda\},$$

where $|K| = (|K^1|, |K^2|, \ldots)$. If $K \in \mathcal{M}_\Lambda(\mathbb{N}_0)$, it is easy to see that K has a finite number of non-zero columns and rows. Moreover, for $K \in \mathcal{M}_\Lambda(\mathbb{N}_0)$ and $M \in \mathcal{M}_\infty(\mathbb{R})$, we denote

$$K! := \prod_{i=1}^{\infty} K^i! = \prod_{i,j=1}^{\infty} K_j^i! \quad \text{and} \quad M^K := \prod_{i=1}^{\infty} (M^i)^{K^i} = \prod_{i,j=1}^{\infty} (M_j^i)^{K_j^i},$$

with the convention $0^0 = 1$. From the above definitions we immediately get

COROLLARY 4.1. *Let* $K \in \mathcal{M}_\Lambda(\mathbb{N}_0)$ *and* $M \in \mathcal{M}_\infty(\mathbb{R})$. *Then*

(i) $K! = (K^T)!$ *and* $M^{K^T} = (M^T)^K$ (*here and hereafter*, T *stands for transposition*).
(ii) *Let* $K_j^i \neq 0$ *and* $M_j^i = 0$ *for some* $i, j \in \mathbb{N}$. *Then* $M^K = 0$.
(iii) *If* $|K| = n$ *and* $|K^T| = m$, *then* $|n| = |m|$. $\blacksquare$

Given $M \in \mathcal{M}_\infty(\mathbb{R})$ such that $M^i \in l_1$ for $i \geq 1$, $n \in \Lambda$ and $t \in l_\infty$. It is easy to check that

$$(Mt)^n = \sum_{\substack{K \in \mathcal{M}_\Lambda(\mathbb{N}_0) \\ |K| = n}} \frac{n!}{K!} M^K t^{|K^T|}. \tag{4.11}$$

Putting $t = (1, 1, \ldots)$ in (4.11) we obtain

$$|M|^n = \sum_{\substack{K \in \mathcal{M}_\Lambda(\mathbb{N}_0) \\ |K| = n}} \frac{n!}{K!} M^K, \tag{4.12}$$

where $|M| = (|M^1|, |M^2|, \ldots)$. We now turn to the Ornstein–Uhlenbeck operator P_R, where $R \in L(H)$, $\|R\| \leq 1$, $RQ = QR$, where Q is as above. The matrix of the operator R in the orthonormal basis $\{e_n\}_{n \geq 1}$ (we recall that $\{e_n\}_{n \geq 1}$ is the basis of H composed of the normalized eigenvectors of the operator Q) is denoted by

$$\mathbf{R} = [R^i_j]_{i,j\geq 1} \quad \text{where} \quad R^i_j = \langle Re_j, e_i\rangle,\ i,j\geq 1.$$

Since Q and R commute, the spaces $\mathrm{Ker}(\lambda_{d_i} I - Q)$, $i \in \mathbb{N}$, are invariant under R, i.e.

$$R(\mathrm{Ker}(\lambda_{d_i} I - Q)) \subset \mathrm{Ker}(\lambda_{d_i} I - Q), \qquad i \in \mathbb{N}.$$

This implies that $\mathbf{R}$ is a block diagonal matrix, where the dimensions of the blocks are $(d_i - d_{i-1}) \times (d_i - d_{i-1})$, $i \in \mathbb{N}$.

Lemma 4.2. *Let $K \in \mathcal{M}_\Lambda(\mathbb{N}_0)$ and $|K| = n$, $|K^T| = m$ (obviously $n = \{n_i\}_{i\geq 1}, m = \{m_i\}_{i\geq 1} \in \Lambda$) and $\mathbf{R}^K \neq 0$. Then for each $r \in \mathbb{N}$ we have $n \in \Lambda_{d_r}$ if and only if $m \in \Lambda_{d_r}$.*

Proof. ($\Rightarrow$) Assume that $n \in \Lambda_{d_r}$ for some fixed $r \in \mathbb{N}$. Then $|K^i| = 0$ for $i > d_r$. Assume that there exists $j_0 > d_r$ such that $m_{j_0} \neq 0$. It follows that there exists $1 \leq i_0 \leq d_r$ such that $K^{i_0}_{j_0} \neq 0$. Since $R^{i_0}_{j_0} = 0$ we get $\mathbf{R}^K = 0$. This contradicts our assumption.

($\Leftarrow$) The proof is similar. ∎

Note that if the matrix $\mathbf{R}$ satisfies the condition $\sup_{i\geq 1}\sum_{j\geq 1}|R^i_j| < \infty$, then it defines an operator (denoted by the same letter) $\mathbf{R} : l_\infty \to l_\infty$ with the norm

$$\|\mathbf{R}\|_\infty = \sup_{i\geq 1}\sum_{j\geq 1}|R^i_j|.$$

Lemma 4.3. *Let $\|\mathbf{R}\|_\infty \leq 1$ and $\|\mathbf{R}^T\|_\infty \leq 1$. Then $\|R\| \leq 1$ (here $\|R\|$ means the operator norm of R).*

Proof. This is immediate from the Frobenius theorem (see [HLP]). ∎

Theorem 4.4. *Let $m \in \Lambda$. Then*

$$P_R(H_m) = \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K^T|=m}} \frac{m!}{K!}\,\mathbf{R}^{K^T} H_{|K|}. \tag{4.13}$$

Proof. Let us point out that the number of terms in the above sum is finite. For any $t \in H$ we define

$$\omega_t(x) := \exp(-\|t\|^2/2 + W_t(x)), \qquad x \in H.$$

Let $t = \sum_{k\geq 1} t_k e_k$, where $t_k = \langle x, e_k\rangle$, $k \geq 1$. Hence and from (2.4) we obtain

$$\begin{aligned}(P_R\omega_t)(x) &= \int_H \exp(-\|t\|^2/2 + W_t(Rx + \sqrt{I-RR^*}\,y))\,d\mu_Q(y)\\ &= \exp(-\|t\|^2/2)\int_H \exp\Big(\sum_{k=1}^{\infty} t_k \frac{\langle Rx + \sqrt{I-RR^*}\,y, e_k\rangle}{\sqrt{\lambda_k}}\Big)\,d\mu_Q(y)\\ &= \exp(-\|t\|^2/2)\exp\Big(\sum_{k=1}^{\infty} t_k \frac{\langle Rx, e_k\rangle}{\sqrt{\lambda_k}}\Big)\\ &\quad\times \int_H \exp\Big(\sum_{k=1}^{\infty} t_k \frac{\langle \sqrt{I-RR^*}y, e_k\rangle}{\sqrt{\lambda_k}}\Big)\,d\mu_Q(y).\end{aligned}$$

Note that

$$\frac{\langle Rx, e_k\rangle}{\sqrt{\lambda_k}} = \frac{\langle x, R^*e_k\rangle}{\sqrt{\lambda_k}} = \frac{\langle x, Q^{-1/2}Q^{1/2}R^*e_k\rangle}{\sqrt{\lambda_k}} = \langle x, Q^{-1/2}R^*e_k\rangle$$

and similarly

$$\frac{\langle \sqrt{I-RR^*}\,y, e_k\rangle}{\sqrt{\lambda_k}} = \langle y, Q^{-1/2}\sqrt{I-RR^*}\,e_k\rangle.$$

It follows that

$$\begin{aligned}(P_R\omega_t)(x) &= \exp(-\|t\|^2/2)\exp\Big(\sum_{k=1}^{\infty} t_k\langle x, Q^{-1/2}R^*e_k\rangle\Big)\\ &\quad\times \int_H \exp\Big(\sum_{k=1}^{\infty} t_k\langle y, Q^{-1/2}\sqrt{I-RR^*}\,e_k\rangle\Big)\,d\mu_Q(y)\\ &= \exp(-\|t\|^2/2)\exp\Big(\sum_{k=1}^{\infty} t_k W_{R^*e_k}(x)\Big)\int_H \exp\Big(\sum_{k=1}^{\infty} t_k W_{\sqrt{I-RR^*}\,e_k}(y)\Big)\,d\mu_Q(y).\end{aligned}$$

From (2.4) we conclude that

$$W_{R^*t} = \sum_{k=1}^{\infty} t_k W_{R^*e_k} \quad \text{and} \quad W_{\sqrt{I-RR^*}\,t} = \sum_{k=1}^{\infty} t_k W_{\sqrt{I-RR^*}\,e_k} \quad \text{in } L^2(\mu_Q).$$

Therefore

$$\begin{aligned}P_R(\omega_t) &= \exp(-\|t\|^2/2)\exp(W_{R^*t})\int_H \exp[W_{\sqrt{I-RR^*}\,t}(y)]\,d\mu_Q(y)\\ &= \exp(-\|t\|^2/2)\exp(W_{R^*t})\exp(\|\sqrt{I-RR^*}\,t\|^2/2)\\ &= \exp[W_{R^*t} - \|R^*t\|^2/2] = \sum_{n\in\Lambda}\frac{(R^*t)^n}{n!}H_n = \sum_{n\in\Lambda}\frac{(\mathbf{R}^Tt)^n}{n!}H_n.\end{aligned}$$

Let us fix $m \in \Lambda$. There exist $s, j_0 \in \mathbb{N}$ such that $m \in \Lambda_s$ and $d_{j_0-1} < s \le d_{j_0}$. Let $r := d_{j_0}$. Then for any $t = \sum_{1\le i\le r} t_i e_i$ we have

$$P_R(\omega_t) = \sum_{n\in\Lambda_r} \frac{(\mathbf{R}^T t)^n}{n!} H_n = \lim_{l\to\infty} \sum_{\substack{n\in\Lambda_r\\ |n|\le l}} \frac{(\mathbf{R}^T t)^n}{n!} H_n \quad \text{in } L^2(\mu_Q).$$

From (4.11) it follows that

$$\begin{aligned} P_R(\omega_t) &= \lim_{l\to\infty} \sum_{\substack{n\in\Lambda_r\\ |n|\le l}} \frac{1}{n!} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{n!}{K!} \mathbf{R}^{K^T} t^{|K^T|} H_n \\ &= \lim_{l\to\infty} \sum_{\substack{n\in\Lambda_r\\ |n|\le l}} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{\mathbf{R}^{K^T}}{K!} t^{|K^T|} H_{|K|}. \end{aligned}$$

Note that the number of terms in the above two sums is finite. By Corollary 4.1(iii) and Lemma 4.2 we obtain

$$\begin{aligned} P_R(\omega_t) &= \lim_{l\to\infty} \sum_{\substack{n\in\Lambda_r\\ |n|\le l}} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K^T|=n}} \frac{\mathbf{R}^{K^T}}{K!} t^{|K^T|} H_{|K|} \\ &= \lim_{l\to\infty} \sum_{\substack{n\in\Lambda_r\\ |n|\le l}} \frac{t^n}{n!} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K^T|=n}} \frac{n!}{K!} \mathbf{R}^{K^T} H_{|K|} = \sum_{n\in\Lambda_r} \frac{t^n}{n!} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K^T|=n}} \frac{n!}{K!} \mathbf{R}^{K^T} H_{|K|}. \end{aligned}$$

On the other hand, from Corollary 3.4 and from the continuity of P_R in $L^2(\mu_Q)$, we get

$$P_R(\omega_t) = \sum_{n\in\Lambda_r} \frac{t^n}{n!} P_R(H_n).$$

By comparing this formula with the formula obtained above, we get (4.13), and the proof is complete. ∎

Corollary 4.5. *For each $n \in \mathbb{N}_0$,*

$$P_R(\mathcal{H}_n) \subset \mathcal{H}_n. \ ∎$$

For the Hermite polynomials orthonormal in $L^2(\mu_Q)$, formula (4.13) takes the form

$$P_R(h_m) = \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K^T|=m}} \frac{\sqrt{m!}\sqrt{|K|!}}{K!} \mathbf{R}^{K^T} h_{|K|}. \tag{4.14}$$

The next theorem is a generalization of the Gebelein inequality to Hilbert spaces.

THEOREM 4.6. *Let $Q \in L(H)$ be as above and let $R \in L(H)$ satisfy $QR = RQ$ and $\|\mathbf{R}\|_\infty \le 1$ and $\|\mathbf{R}^T\|_\infty \le 1$. Then for $f \in L^2(\mu_Q)$ such that $\langle f, 1\rangle_{\mu_Q} = 0$ we have*

$$\|P_R(f)\|_2 \le \sqrt{\|\mathbf{R}\|_\infty \|\mathbf{R}^T\|_\infty}\, \|f\|_2. \tag{4.15}$$

Proof. Let us first see that by Lemma 4.3 the operator P_R is properly defined. For $\|\mathbf{R}\|_\infty = 0$, inequality (4.15) holds trivially. Assume that $\|\mathbf{R}\|_\infty \ne 0$. Let us consider the linear operator $S_R : L^2(\mu_Q) \to L^2(\mu_Q)$ defined as

$$S_R(f) = \sum_{n\in\Lambda} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \langle f, h_{|K^T|}\rangle_{\mu_Q} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!}\, \mathbf{R}^{K^T} h_n, \qquad f \in L^2(\mu_Q).$$

We shall prove that S_R is continuous. Let $f \in L^2(\mu_Q)$. Then

$$\begin{aligned}
\|S_R(f)\|_2^2 &= \int_H \Big| \sum_{n\in\Lambda} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \langle f, h_{|K^T|}\rangle_{\mu_Q} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!}\, \mathbf{R}^{K^T} h_n \Big|^2 d\mu_Q \\
&= \sum_{n\in\Lambda} \Big(\sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!}\, \mathbf{R}^{K^T} \langle f, h_{|K^T|}\rangle_{\mu_Q} \Big)^2 \\
&\le \sum_{n\in\Lambda} \Big(\sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!}\, (\overline{\mathbf{R}}^T)^K |\langle f, h_{|K^T|}\rangle_{\mu_Q}| \Big)^2,
\end{aligned}$$

where $\overline{\mathbf{R}} = [|R^i_j|]_{i,j\ge 1}$ (here $|R^i_j|$ means the absolute value of R^i_j). From what has already been proved, from (4.12) and by the Jensen inequality we see that

$$\begin{aligned}
\|S_R(f)\|_2^2 &\le \sum_{n\in\Lambda} \frac{(|\overline{\mathbf{R}}^T|^n)^2}{n!} \\
&\quad \times \Big(\frac{1}{|\overline{\mathbf{R}}^T|^n} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{n!}{K!} (\overline{\mathbf{R}}^T)^K \sqrt{|K^T|!}\, |\langle f, h_{|K^T|}\rangle_{\mu_Q}| \Big)^2 \\
&\le \sum_{n\in\Lambda} \frac{(|\overline{\mathbf{R}}^T|^n)^2}{n!} \frac{1}{|\overline{\mathbf{R}}^T|^n} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{n!}{K!} (\overline{\mathbf{R}}^T)^K\, |K^T|!\, \langle f, h_{|K^T|}\rangle_{\mu_Q}^2 \\
&= \sum_{n\in\Lambda} |\overline{\mathbf{R}}^T|^n \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{|K^T|!}{K!} (\overline{\mathbf{R}}^T)^K \langle f, h_{|K^T|}\rangle_{\mu_Q}^2
\end{aligned}$$

$$\le \|\mathbf{R}^T\|_\infty \sum_{n\in\Lambda} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{|K^T|!}{K!} (\overline{\mathbf{R}}^T)^K \langle f, h_{|K^T|}\rangle_{\mu_Q}^2$$

$$= \|\mathbf{R}^T\|_\infty \sum_{n\in\Lambda} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K^T|=n}} \frac{|K^T|!}{K^T!} \overline{\mathbf{R}}^{K^T} \langle f, h_{|K^T|}\rangle_{\mu_Q}^2$$

$$= \|\mathbf{R}^T\|_\infty \sum_{n\in\Lambda} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \frac{|K|!}{K!} \overline{\mathbf{R}}^{K} \langle f, h_n\rangle_{\mu_Q}^2.$$

Now, from (4.12) we conclude that

$$\|S_R(f)\|_2^2 \le \|\mathbf{R}^T\|_\infty \sum_{n\in\Lambda} |\overline{\mathbf{R}}|^n \langle f, h_n\rangle_{\mu_Q}^2. \tag{4.16}$$

By assumption $\|\mathbf{R}^T\|_\infty \le 1$ and $|\overline{\mathbf{R}}|^n \le 1$ ($|\overline{\mathbf{R}}|^0 = 1$). Therefore

$$\|S_R(f)\|_2 \le \|f\|_2, \quad f \in L^2(\mu_Q),$$

i.e. S_R is a continuous linear operator on $L^2(\mu_Q)$. Moreover, $P_R(h_m) = S_R(h_m)$for $m \in \Lambda$: indeed,

$$S_R(h_m) = \sum_{n\in\Lambda} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n}} \langle h_m, h_{|K^T|}\rangle_{\mu_Q} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!} \mathbf{R}^{K^T} h_n$$

$$= \sum_{n\in\Lambda} \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K|=n\\ |K^T|=m}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!} \mathbf{R}^{K^T} h_{|K|}$$

$$= \sum_{\substack{K\in\mathcal{M}_\Lambda(\mathbb{N}_0)\\ |K^T|=m}} \frac{\sqrt{|K^T|!}\sqrt{|K|!}}{K!} \mathbf{R}^{K^T} h_{|K|} = P_R(h_m).$$

Hence and from the continuity of P_R and S_R we conclude that $S_R = P_R$. Finally, from (4.16) and by assumption $\langle f, 1\rangle_{\mu_Q} = 0$ we obtain

$$\|P_R(f)\|_2^2 = \|S_R(f)\|_2^2 \le \|\mathbf{R}^T\|_\infty \sum_{n\in\Lambda} |\overline{\mathbf{R}}|^n \langle f, h_n\rangle_{\mu_Q}^2 \le \|\mathbf{R}^T\|_\infty \|\mathbf{R}\|_\infty \|f\|_2^2,$$

where $f \in L^2(\mu_Q)$ and the proof of (4.15) is complete. ∎

Example. Assume that $\mathbf{R}$ is a diagonal matrix with main diagonal $\{\rho_i\}_{i\ge1}$ (e.g. if R is symmetric then we can find an orthonormal basis of H such that in this basis both operators Q and R have diagonal matrices). It is clear that $\mathbf{R}^T = \mathbf{R}$ and (by the assumption of Theorem 4.6)

$\|\rho\|_\infty = \|\mathbf{R}\|_\infty \le 1$, where $\rho = (\rho_1, \rho_2, \ldots)$. From (4.14) it follows that

$$P_R(h_n) = \sum_{\substack{K \in \mathcal{M}_\Lambda(\mathbb{N}_0) \\ |K^T| = n}} \frac{\sqrt{n!}\sqrt{|K|!}}{K!} \mathbf{R}^{K^T} h_{|K|}$$

$$= \sum_{\substack{K \in \mathcal{M}_\Lambda(\mathbb{N}_0) \\ |K| = |K^T| = n}} \frac{n!}{K!} \mathbf{R}^K h_n = \rho^n h_n, \qquad n \in \Lambda.$$

Thus

$$P_R(f) = \sum_{n \in \Lambda} \rho^n \langle f, h_n \rangle h_n, \qquad f \in L^2(\mu_Q), \tag{4.17}$$

and the Gebelein inequality has the form

$$\|P_R(f)\|_2 \le \|\rho\|_\infty \|f\|_2, \qquad f \in L^2(\mu_Q),\ \langle f, 1\rangle_{\mu_Q} = 0. \tag{4.18}$$

In order to examine the equality case in (4.18) we shall consider three cases, the proof of which is an immediate consequence of (4.17).

(i) If $|\rho_i| < \|\rho\|_\infty$, $i \in \mathbb{N}$, then we have equality in (4.18) if and only if $f = 0$.

(ii) If $|\rho_i| = \|\rho\|_\infty = 1$, $i \in I \subset \mathbb{N}$, then we have equality in (4.18) if and only if

$$f = \sum_{n \in \Lambda_I} t_n h_n, \qquad \sum_{n \in \Lambda_I} t_n^2 < \infty,$$

where $\Lambda_I = \{n = \{n_i\}_{i \ge 1} \in \Lambda : n_i \ne 0 \Rightarrow i \in I\}$.

(iii) If $|\rho_i| = \|\rho\|_\infty < 1$, $i \in I \subset \mathbb{N}$, then we have equality in (4.18) if and only if

$$f = \sum_{n \in \Lambda_{I_1}} t_n h_n, \qquad \sum_{n \in \Lambda_{I_1}} t_n^2 < \infty,$$

where $\Lambda_{I_1} = \{n = \{n_i\}_{i \ge 1} \in \Lambda_I : |n| = 1\}$.

Assume additionally that H is finite-dimensional, say $\dim(H) = d$. It is clear that in this case

$$P_R(h_n) = \rho^n h_n, \qquad n \in \mathbb{N}_0^d,\ \rho = (\rho_1, \ldots, \rho_d).$$

It follows that

$$\|P_R(f)\|_2 \le \|\rho\|_{\max} \|f\|_2, \qquad f \in L^2(\mu_Q),\ \langle f, 1\rangle_{\mu_Q} = 0,$$

where $\|\rho\|_{\max} = \max_{1 \le i \le d} |\rho_i|$. It is well known that for every $f \in L^2(\mu_Q)$ there exists a Borel function $g : \mathbb{R}^d \to \mathbb{R}$ such that

$$f(x) = g(\langle x, e_1\rangle, \ldots, \langle x, e_d\rangle), \qquad x \in H,$$

and

$$\int_H f^2(x)\, d\mu_Q(x) = \int_{\mathbb{R}^d} g^2(t_1, \ldots t_d)\, d\nu(t_1, \ldots, t_d),$$

where $\nu = \mu_{\lambda_1} \times \cdots \times \mu_{\lambda_d}$. If we replace the condition $\langle f, 1\rangle_{\mu_Q} = 0$, $f \in L^2(\mu_Q)$ with the stronger condition

$$\int_{\mathbb{R}} g(t_1, \ldots, t_d)\, d\mu_{\lambda_i}(t_i) = 0, \qquad i = 1, \ldots, d, \tag{4.19}$$

then the Gebelein inequality has the form

$$\|P_R(f)\|_2 \le |\rho_1|^{\varepsilon_1} \cdots |\rho_d|^{\varepsilon_d} \|f\|_2 = \overline{\rho}^{\varepsilon} \|f\|_2, \qquad \varepsilon_i = \begin{cases} 0 & \text{if } \rho_i = 0, \\ 1 & \text{if } \rho_i \neq 0, \end{cases} \tag{4.20}$$

for $i = 1, \ldots, d$ and $\overline{\rho} = (|\rho_1|, \ldots, |\rho_d|)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$. If H is infinite-dimensional, then condition (4.19) has the form

$$\int_{\mathbb{R}} g(t_1, t_2, \ldots)\, d\mu_{\lambda_i}(t_i) = 0, \qquad i = 1, 2, \ldots, \tag{4.21}$$

where $g : \mathbb{R}^\infty \to \mathbb{R}$ is a Borel function such that

$$f(x) = g(\langle x, e_1\rangle, \langle x, e_2\rangle, \ldots), \qquad x \in H,\ f \in L^2(\mu_Q).$$

It is easy to check that (4.21) implies $\langle f, h_n\rangle_{\mu_Q} = 0$, $n \in \Lambda$, i.e. $f = 0$. Therefore in the infinite-dimensional case inequality (4.20) under the assumption (4.21) has a trivial form.

5. Applications. Let $Q \in L(H)$ be as above and let $\{X_n\}_{n\ge1}$ be a centered Gaussian sequence of random vectors $X_n : \Omega \to H$, $n \ge 1$, with covariance operator Q and such that the covariance operator of (X_i, X_j), $i, j \ge 1$, has the form

$$\operatorname{cov}[(X_i, X_j)] = \begin{bmatrix} Q & QR_{ij} \\ R_{ij}Q & Q \end{bmatrix}, \qquad i, j \ge 1,$$

where for $i, j \ge 1$ the operators $R_{ij} \in L(H)$ are symmetric, $R_{ij}Q = QR_{ij}$ and $\|\mathbf{R}_{ij}\|_\infty \le 1$ (note that $R_{ii} = I$ is the identity operator). Assume additionally that

$$\sup_{i\ge1} \sum_{j=1}^{\infty} \|\mathbf{R}_{ij}\|_\infty < \infty.$$

Adopting now the methods from [BC] we obtain the following statement.

Theorem 5.1. *Let $\{X_n\}_{n\ge1}$ be a centered Gaussian sequence as above. Suppose that $f \in L^1(\mu_Q)$. Then*

$$\frac{f(X_1) + \cdots + f(X_n)}{n} \xrightarrow[n\to\infty]{} \int_H f\, d\mu_Q \qquad P\text{-a.s.} \ ■$$

Let E be a separable real Banach space with norm $\|\cdot\|_E$. We denote by $L^1(\mu_Q; E)$ the space of (equivalence classes of) Bochner measurable functions $g : H \to E$ such that $\int_H \|g\|\, d\mu_Q < \infty$. Now Theorem 5.1 and a slight change in the proof of Ranga Rao (see e.g. [DS]) of the Strong Law of Large Numbers for independent random vectors show that for a Gaussian sequence $\{X_n\}_{n\geq 1}$ (under the above assumptions) and for $g \in L^1(\mu_Q; E)$ we have

$$\frac{g(X_1) + \cdots + g(X_n)}{n} \xrightarrow[n\to\infty]{} \int_H g\, d\mu_Q \qquad P\text{-a.s.}$$

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