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## BLOW-UP VERSUS GLOBAL EXISTENCE OF SOLUTIONS TO AGGREGATION EQUATIONS

*Abstract.* A class of nonlinear viscous transport equations describing aggregation phenomena in biology is considered. General conditions on an interaction potential are obtained which lead either to the existence or to the nonexistence of global-in-time solutions.

**1. Introduction.** The following Cauchy problem for the heat equation corrected by the nonlocal and nonlinear transport term

$$(1.1) \quad u_t = \Delta u - \nabla \cdot (u(\nabla K * u)), \quad x \in \mathbb{R}^n, t > 0,$$

$$(1.2) \quad u(x, 0) = u_0(x)$$

has been used to describe a collective motion and aggregation phenomena in biology and mechanics of continuous media. Here, the unknown function  $u = u(x, t) \geq 0$  is either the population density of a species or the density of particles in a granular medium. From the mathematical point of view, equation (1.1) can be considered as either a viscous conservation law with a nonlocal (quadratic) nonlinearity or a viscous transport equation with nonlocal velocity, and its character depends strongly on the properties of the given kernel  $K$ . If this kernel is radially symmetric, the *nonincreasing* function  $K(r)$ ,  $r = |x|$ , corresponds to the attraction of particles while *nondecreasing* one is repulsive.

Let us first emphasize that problem (1.1)–(1.2) contains, as a particular case, the (simplified) Patlak–Keller–Segel system for chemotaxis describing the motion of cells, usually bacteria or amoebae, that are attracted by a chemical substance and are able to emit it (see e.g. [17] for a general intro-

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2010 *Mathematics Subject Classification*: 35K15, 35B40, 92C17.

*Key words and phrases*: nonlocal parabolic equations, blowup of solutions, chemotaxis, moment method.

duction to chemotaxis). This parabolic-elliptic system has the form

$$(1.3) \quad u_t = \nabla \cdot (\nabla u - u \nabla v), \quad x \in \mathbb{R}^n, t > 0,$$

$$(1.4) \quad 0 = \Delta v - \alpha v + u,$$

where  $\alpha > 0$  is a given constant. In this model, the function  $u = u(x, t)$  represents the cell density and  $v = v(x, t)$  is the concentration of the chemical attractant which induces a drift force. Computing  $v$  from equation (1.4) and substituting it into the transport term in equation (1.3), we immediately obtain equation (1.1) with the kernel  $K = K(x)$  given by the fundamental solution of the operator  $-\Delta + \alpha$  on  $\mathbb{R}^n$ . In this case, the function  $K$  is called the Bessel potential and it is singular at the origin if  $n \geq 2$ , more precisely, it satisfies  $|\nabla K(x)| \sim |x|^{-n+1}$  as  $|x| \rightarrow 0$  and it decays exponentially when  $|x| \rightarrow \infty$  (see [20, Ch. 5.3] for more details). Hence, when  $n \geq 2$ , we see that  $\nabla K \in L^{q'}(\mathbb{R}^n)$  for every  $q' < n/(n - 1)$  and  $\nabla K \notin L^p(\mathbb{R}^n)$  if  $p \geq n/(n - 1)$ . On the other hand, for  $n = 1$ , this fundamental solution is given explicitly by  $K(x) = \exp(-\sqrt{\alpha}|x|)$ , hence  $\nabla K \in L^{q'}(\mathbb{R})$  for all  $q' \in [1, \infty]$ . We refer the reader to the recent works [4, 7–10, 12, 14, 18] (this list is by no means exhaustive) and to the references therein for mathematical results on the Patlak–Keller–Segel system (1.3)–(1.4).

In this work, we are motivated by recent results on the local and global existence of solutions to the *inviscid* aggregation equation

$$(1.5) \quad u_t + \nabla \cdot (u(\nabla K * u)) = 0,$$

which has been thoroughly studied in [13] under some additional hypotheses on the kernel (see also [1, 3]). In particular, kernels that are smooth (not singular) at the origin  $x = 0$  lead to the global in time existence of solutions (see e.g. [3, 2, 13]). Nonsmooth kernels (and  $C^1$  off the origin, like  $K(x) = e^{-|x|}$ ) may lead to blowup of solutions either in finite or infinite time [1, 2, 3, 13, 14]. Singular kernels of potential type (arising in chemotaxis theory, cf. e.g. [5, 6] and the references therein)  $K(x) = c|x|^{\beta-d}$ , with  $1 < \beta < d$ , usually lead to finite time blowup of all nonnegative solutions (see e.g. [6]).

In particular, in the recent work by Bertozzi et al. [2] on the inviscid aggregation equation (1.5), the kernel  $K$  is assumed to be radially symmetric,  $K(x) = k(|x|)$  with the function  $k(r)$  increasing in  $r$ , smooth away from zero and bounded from below. The authors of [2] obtained natural conditions on  $K$  such that all solutions to equation (1.5), supplemented with bounded, nonnegative, and compactly supported initial data, either blow up in finite time or exist for all  $t > 0$ . More precisely, they introduce the quantity

$$(1.6) \quad \int_0^1 \frac{1}{k'(r)} dr$$

and show that if (1.6) is finite, the solution of (1.5) blows up in finite time.

On the other hand, if (1.6) is infinite, a global-in-time solution to (1.5) is constructed.

The purpose of this paper is to describe an analogous influence of singularities of the kernel  $\nabla K$  on the existence and nonexistence of global-in-time solutions of the “viscous” problem (1.1)–(1.2). Roughly speaking, our results can be summarized as follows. If  $\nabla K \in L^{q'}(\mathbb{R}^n)$  for some  $q' \in [1, \infty]$ , we can always construct local-in-time solutions to (1.1)–(1.2), however, some additional regularity assumptions on the initial conditions have to be imposed if  $\nabla K$  is too singular in the scale of  $L^p$ -spaces. Next, we show that the initial value problem (1.1)–(1.2) with a mildly singular interaction kernel, namely  $\nabla K \in L^{q'}(\mathbb{R}^n)$  for some  $q' \in (n, \infty]$ , has a global-in-time solution for any nonnegative and integrable initial datum (1.2). On the other hand, there are strongly singular kernels, such that some solutions of problem (1.1)–(1.2) blow up in finite time. In particular, we show that the behavior  $|\nabla K(x)| \sim |x|^{-1}$ , as  $|x| \rightarrow 0$ , appears to be critical for the existence and the nonexistence of global-in-time solutions to problem (1.1)–(1.2). In the next section, we state and discuss our results more precisely.

To conclude this introduction, we would like to mention that completely analogous results can be obtained for the aggregation equation with the fractional dissipation

$$(1.7) \quad u_t + \nabla \cdot (u(\nabla K * u)) = -\nu(-\Delta)^{\gamma/2}u,$$

where  $\nu > 0$  and  $\gamma \in (1, 2]$ . Some results in this direction, mainly for the kernel  $K$  either of the form  $K(x) = e^{-|x|}$  or given by the Bessel potential, were published in [5, 6, 14].

**Notation.** Throughout this paper, we denote the norm of the usual Lebesgue space  $L^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , by  $\|\cdot\|_{L^p}$ . The constants (always independent of  $x$ ,  $t$ , and  $u$ ) will be denoted by the same letter  $C$ , even if they may vary from line to line. Sometimes, we write, e.g.,  $C = C(T)$  when we want to emphasize the dependence of  $C$  on a parameter  $T$ . We write  $f(x) \sim g(x)$  if there is a constant  $C > 0$  such that  $C^{-1}g(x) \leq f(x) \leq Cg(x)$ .

**2. Results and comments.** We begin by introducing the terminology systematically used in this work.

DEFINITION 2.1. The interaction kernel  $K : \mathbb{R}^n \rightarrow \mathbb{R}$  is called

- *mildly singular* if  $\nabla K \in L^{q'}(\mathbb{R}^n)$  for some  $q' \in (n, \infty]$ ;
- *strongly singular* if  $\nabla K \in L^{q'}(\mathbb{R}^n)$  for some  $q' \in [1, n]$  and  $\nabla K \notin L^p(\mathbb{R}^n)$  for every  $p > n$ .

Notice that any function  $\nabla K$  satisfying  $|\nabla K(x)| \sim |x|^{-a}$  as  $|x| \rightarrow 0$  and rapidly decreasing if  $|x| \rightarrow \infty$  is mildly singular in the sense stated above if  $a < 1$  and strongly singular for  $a \geq 1$ . Hence, the Bessel potential  $K$

(appearing in the case of the chemotaxis system) is strongly singular when  $n \geq 2$  and mildly singular for  $n = 1$ .

In order to describe the influence of singularities of the function  $\nabla K$  on the existence/nonexistence of solutions to the initial value problem (1.1)–(1.2), we discuss separately conditions leading to the local-in-time existence of solutions, their global-in-time existence, as well as the blowup of solutions in finite time.

**Local existence of solutions.** First, we show that the critical exponent  $q' = n$  from Definition 2.1 appears already in the construction of local-in-time solutions to (1.1)–(1.2) with kernels satisfying  $\nabla K \in L^{q'}(\mathbb{R}^n)$ . Notice that for strongly singular kernels we have to consider more regular (in the sense of  $L^p$ -spaces) initial conditions. In the following two theorems, the quantity  $n/(n - 1)$  stands for  $\infty$  if  $n = 1$ .

**THEOREM 2.2** (Mildly singular kernels). *Assume that  $\nabla K \in L^{q'}(\mathbb{R}^n)$  with  $q' \in (n, \infty]$ . Let  $q \in [1, n/(n - 1))$  satisfy  $1/q + 1/q' = 1$ . For every  $u_0 \in L^1(\mathbb{R}^n)$  there exists  $T = T(\|u_0\|_{L^1}, \|\nabla K\|_{L^{q'}}) > 0$  and a unique mild solution of problem (1.1)–(1.2) in the space*

$$\mathcal{X}_T = C([0, T], L^1(\mathbb{R}^n)) \cap C((0, T], L^q(\mathbb{R}^n))$$

*equipped with the norm  $\|u\|_{\mathcal{X}_T} \equiv \sup_{0 \leq t \leq T} \|u\|_{L^1} + \sup_{0 \leq t \leq T} t^{\frac{n}{2}(1-1/q)} \|u\|_{L^q}$ .*

**THEOREM 2.3** (Strongly singular kernels). *Assume that  $\nabla K \in L^{q'}(\mathbb{R}^n)$  with  $q' \in [1, n]$ . Let  $q \in [n/(n - 1), \infty]$  satisfy  $1/q + 1/q' = 1$ . For every  $u_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ , there exists  $T = T(\|u_0\|_{L^1}, \|u_0\|_{L^q}, \|\nabla K\|_{L^{q'}}) > 0$  and a unique mild solution of problem (1.1)–(1.2) in the space*

$$\mathcal{Y}_T = C([0, T], L^1(\mathbb{R}^n)) \cap C([0, T], L^q(\mathbb{R}^n))$$

*equipped with the norm  $\|u\|_{\mathcal{Y}_T} \equiv \sup_{0 \leq t \leq T} \|u\|_{L^1} + \sup_{0 \leq t \leq T} \|u\|_{L^q}$ .*

Recall that, as usual, the function  $u = u(x, t)$  is called a *mild solution* of (1.1)–(1.2) if it satisfies the integral equation

$$(2.1) \quad u(t) = G(\cdot, t) * u_0 - \int_0^t \nabla G(\cdot, t - s) * (u(\nabla K * u))(s) ds$$

with the heat kernel denoted by  $G(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/(4t))$ . We construct solutions to the integral equation (2.1) using the Banach contraction principle and, in the proofs of Theorems 2.2 and 2.3, we emphasize that different estimates are necessary according to the singularity of  $\nabla K$ .

**REMARK 2.4.** In this work, we skip completely questions of regularity of mild solutions to (1.1)–(1.2) because there are standard and well-known results (see e.g. the monograph by Pazy [19] for more details). In particular, by a bootstrap argument, one can show that any mild solution

$u \in C([0, T], L^q(\mathbb{R}^n))$  of equation (2.1) satisfies  $u \in C^1((0, T], L^q(\mathbb{R}^n)) \cap C((0, T], W^{1,q}(\mathbb{R}^n))$  and  $u(t) \in W^{2,q}(\mathbb{R}^n)$  for every  $t \in (0, T]$ . Moreover, if the initial condition is nonnegative, so is the corresponding solution.

**Global existence of solutions.** For mildly singular kernels, nonnegative solutions to problem (1.1)–(1.2) are global in time.

**THEOREM 2.5** (Mildly singular kernels). *Let  $q, q' \in [1, \infty]$  satisfy  $1/q + 1/q' = 1$ . Assume that  $\nabla K \in L^{q'}(\mathbb{R}^n)$  with  $q' \in (n, \infty]$ . For every  $u_0 \in L^1(\mathbb{R}^n)$  such that  $u_0 \geq 0$ , there exists a unique global-in-time solution  $u$  of problem (1.1)–(1.2) satisfying*

$$u \in C([0, \infty), L^1(\mathbb{R}^n)) \cap C((0, \infty), W^{1,q}(\mathbb{R}^n)) \cap C^1((0, \infty), L^q(\mathbb{R}^n)).$$

On the other hand, problem (1.1)–(1.2) with strongly singular kernels has a global-in-time solution under suitable smallness assumptions imposed on initial conditions. To formulate this result, it is more convenient to extend the class of kernels considered and to assume that  $\nabla K \in L^{q',\infty}(\mathbb{R}^n)$ , where  $L^{q',\infty}(\mathbb{R}^n)$  is the weak  $L^{q'}$ -space defined as the space of all measurable functions  $f$  such that  $\sup_{\lambda>0} \lambda |\{x : |f(x)| > \lambda\}|^{1/q'} < \infty$ .

Here, let us recall the well-known embedding  $L^{q'}(\mathbb{R}^n) \subset L^{q',\infty}(\mathbb{R}^n)$  for all  $q' \in [1, \infty)$ . However, it follows immediately from the definition of the  $L^{q',\infty}$ -space that

$$(2.2) \quad |\cdot|^{-n/q'} \in L^{q',\infty}(\mathbb{R}^n) \setminus L^{q'}(\mathbb{R}^n) \quad \text{for } 1 < q' \leq n.$$

In the following, we are going to use the weak Young inequality

$$(2.3) \quad \|\nabla K * f\|_{L^k} \leq C \|\nabla K\|_{L^{q',\infty}} \|f\|_{L^p}$$

with  $p, q', k \in (1, \infty)$  satisfying  $1/p + 1/q' = 1 + 1/k$ , a constant  $C = C(n, k, p, q') > 0$  and all  $f \in L^p(\mathbb{R}^n)$  (see e.g. [15, Sect. 4.3] for the proof of (2.3)).

**THEOREM 2.6** (Strongly singular kernels). *Let  $n \geq 2$ . Assume that  $\nabla K \in L^{q',\infty}(\mathbb{R}^n)$  with  $q' \in (1, n]$ . Denote*

$$(2.4) \quad q_* = \frac{n}{n + 1 - n/q'} \in [1, n).$$

*There is an  $\varepsilon > 0$  such that for every  $u_0 \in L^{q_*}(\mathbb{R}^n)$  with  $\|u_0\|_{L^{q_*}} < \varepsilon$ , there exists a global-in-time mild solution of problem (1.1)–(1.2) satisfying  $u \in C([0, \infty), L^{q_*}(\mathbb{R}^n))$ .*

**REMARK 2.7.** Notice that, if  $n \geq 2$ , the Bessel potential  $K = K(x)$  satisfies  $\nabla K \in L^{q',\infty}(\mathbb{R}^n)$  with  $q' = n/(n - 1)$ .

**REMARK 2.8.** In the one-dimensional case, nonnegative solutions of problem (1.1)–(1.2) are always global in time for every kernel  $\nabla K \in L^1(\mathbb{R})$  provided initial conditions are regular enough (see [11, Thm. 2.1] for more details).

**Blowup versus non-blowup of solutions.** Next, we state conditions on strongly singular kernels under which we can observe the blowup in finite time of solutions to the initial value problem (1.1)–(1.2).

**THEOREM 2.9** (Strongly singular kernels). *Assume that the kernel  $K : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the following conditions:*

- (i)  $K(x) = K(|x|)$  for all  $x \in \mathbb{R}^n$ ,
- (ii) there exist  $\delta > 0$ ,  $\gamma > 0$ , and  $C > 0$  such that

$$\sup_{0 < s \leq \delta} sK'(s) \leq -\gamma \quad \text{and} \quad |sK'(s)| \leq Cs^2 \quad \text{for all } s \geq \delta.$$

For the initial datum  $u_0 \in L^1(\mathbb{R}^n)$  satisfying  $u_0 \geq 0$  and  $|x|^2 u_0 \in L^1(\mathbb{R}^n)$ , denote

$$I(0) = \int_{\mathbb{R}^n} |x|^2 u_0(x) \, dx \quad \text{and} \quad M = \int_{\mathbb{R}^n} u_0(x) \, dx = \|u_0\|_{L^1}.$$

If

$$M > \frac{2n + 4(C + \gamma/\delta^2)I(0)}{\gamma},$$

then there is  $T = T(M, I(0), \delta, \gamma, C) > 0$  such that the corresponding non-negative local-in-time solution to the initial value problem (1.1)–(1.2) cannot be extended beyond  $[0, T]$ .

**REMARK 2.10.** Any interaction kernel  $K = K(x)$  satisfying the assumptions of Theorem 2.9 has to be strongly singular in the sense of Definition 2.1. Indeed, this follows immediately from the inequalities

$$\begin{aligned} \|\nabla K\|_p^p &= \int_{\mathbb{R}^n} \left| \frac{x}{|x|} K'(|x|) \right|^p dx = \int_{\mathbb{R}^n} |K'(|x|)|^p dx = C \int_0^\infty |K'(s)|^p s^{n-1} ds \\ &\geq C\gamma^p \int_0^\delta s^{-p+n-1} ds = \infty \end{aligned}$$

for every  $p \in [n, \infty)$ . Notice that, for  $n = 1$ , these calculations imply that every one-dimensional kernel  $K$  satisfying the assumptions of Theorem 2.9 satisfies also  $\nabla K \notin L^p(\mathbb{R})$  for each  $p \in [1, \infty]$ .

**REMARK 2.11.** If  $n \geq 2$ , one can prove, following the reasoning e.g. from [12, Lem. 3.1], that the Bessel potential satisfies the assumptions of Theorem 2.9.

We conclude the presentation of our results by a non-blowup criterion for the initial value problem (1.1)–(1.2) with suitable strongly singular kernels.

**THEOREM 2.12** (Strongly singular kernels). *Let  $n \geq 1$ . Assume that  $\nabla K \in L^{q'}(\mathbb{R}^n)$  with  $q' \in [1, n]$  and let  $q \in [n/(n - 1), \infty]$  satisfy  $1/q + 1/q' = 1$ . Suppose, moreover, that the kernel  $K$  can be decomposed into two parts,*

$K = K_1 + K_2$ , where  $\Delta K_1$  is nonnegative (as e.g. a tempered distribution) and  $\nabla K_2 \in L^\infty(\mathbb{R}^n)$ . Then, for every  $u_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$  with  $u_0 \geq 0$ , the local-in-time solution from Theorem 2.3 exists, in fact, for all  $t > 0$ .

REMARK 2.13. Let  $n = 2$ . The function  $K$  defined by its Fourier transform

$$\widehat{K}(\xi) = \frac{-1}{|\xi|^2 + 1}$$

is an example of a strongly singular kernel satisfying the assumptions of Theorem 2.12 (this is the Bessel potential, discussed just below (1.3)–(1.4), with the reverse sign). Indeed, using the decomposition

$$\widehat{K}(\xi) = \widehat{K}_1(\xi) + \widehat{K}_2(\xi) \equiv \frac{-1}{|\xi|^2} + \frac{1}{|\xi|^2(|\xi|^2 + 1)},$$

we see that  $\Delta K_1$  is the Dirac delta and  $\nabla K_2 \in L^\infty(\mathbb{R}^2)$ , because  $\widehat{\nabla K}_2 \in L^1(\mathbb{R}^2)$ . In other words, the two-dimensional initial value problem for the parabolic-elliptic system (1.3)–(1.4), where the sign “ $-$ ” in the first equation is replaced by “ $+$ ”, is globally wellposed for any nonnegative initial condition from  $L^1(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$  for some  $q \in (2, \infty]$ .

The result of Theorem 2.12 on the global-in-time existence of nonnegative solutions is far from being optimal. We have stated it here to emphasize the important role of the sign of a strongly singular kernel  $K$  in the blowup phenomenon described by Theorem 2.9.

**3. Construction of local-in-time solutions.** As usual, a solution to the initial value problem (1.1)–(1.2) is obtained as a fixed point of the integral equation (2.1). Here, it is convenient to apply the following abstract approach proposed by Meyer [16].

LEMMA 3.1. *Let  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$  be a Banach space,  $y \in \mathcal{X}$ , and  $B : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$  be a bilinear form satisfying  $\|B(x_1, x_2)\|_{\mathcal{X}} \leq C\|x_1\|_{\mathcal{X}}\|x_2\|_{\mathcal{X}}$  with a positive constant  $C$  and all  $x_1, x_2 \in \mathcal{X}$ . If  $4C\|y\|_{\mathcal{X}} < 1$ , the equation  $x = y + B(x, x)$  has a solution in  $\mathcal{X}$  satisfying  $\|x\|_{\mathcal{X}} \leq 2\|y\|_{\mathcal{X}}$ . Moreover, the solution is unique in the ball  $U(0, 1/(2C)) \subset \mathcal{X}$ .*

We skip the proof of Lemma 3.1 which is a direct consequence of the Banach fixed point theorem.

To prove Theorems 2.2 and 2.3, we are going to apply Lemma 3.1 to the “quadratic” equation (2.1), written in the form  $u(t) = G(\cdot, t) * u_0 + B(u, u)(t)$ , with the bilinear form

$$(3.1) \quad B(u, v)(t) = - \int_0^t \nabla G(\cdot, t-s) * (u(\nabla K * v))(s) ds$$

defined on a suitable Banach space. In our reasoning, we use the following well-known estimates of the heat kernel which are immediate consequences of the Young inequality for convolution:

$$(3.2) \quad \|G(\cdot, t) * f\|_{L^p} \leq C t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})} \|f\|_{L^q},$$

$$(3.3) \quad \|\nabla G(\cdot, t) * f\|_{L^p} \leq C t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}} \|f\|_{L^q}$$

for every  $1 \leq q \leq p \leq \infty$ , each  $f \in L^q(\mathbb{R}^n)$ , and  $C = C(p, q)$  independent of  $t, f$ . Notice that  $C = 1$  in inequality (3.2) for  $p = q$  because  $\|G(\cdot, t)\|_{L^1} = 1$  for all  $t > 0$ .

*Proof of Theorem 2.2.* First, we observe that for every  $q' \in (n, \infty]$ , the relation  $1/q + 1/q' = 1$  implies  $q \in [1, n/(n - 1))$ .

Here, we use Lemma 3.1 with  $\mathcal{X} \equiv \mathcal{X}_T = C([0, T], L^1(\mathbb{R}^n)) \cap C((0, T], L^q(\mathbb{R}^n))$ , which is a Banach space with the norm  $\|u\|_{\mathcal{X}_T} \equiv \sup_{0 \leq t \leq T} \|u\|_{L^1} + \sup_{0 < t \leq T} t^{\frac{n}{2}(1 - \frac{1}{q})} \|u\|_{L^q}$ .

By inequality (3.2), we immediately obtain

$$\|G(\cdot, t) * u_0\|_{L^1} \leq \|u_0\|_{L^1} \quad \text{and} \quad t^{\frac{n}{2}(1 - \frac{1}{q})} \|G(\cdot, t) * u_0\|_{L^q} \leq C(q, 1) \|u_0\|_{L^1}$$

for every  $u_0 \in L^1(\mathbb{R}^n)$ , hence,  $y \equiv G(\cdot, t) * u_0 \in \mathcal{X}_T$  with  $\|y\|_{\mathcal{X}_T} \leq (1 + C(q, 1)) \|u_0\|_{L^1}$ .

Next, we show that the bilinear operator defined in (3.1) satisfies  $B : \mathcal{X}_T \times \mathcal{X}_T \rightarrow \mathcal{X}_T$  and there exists a constant  $C_1 > 0$  such that for all  $T > 0$  and all  $u, v \in \mathcal{X}_T$  we have

$$(3.4) \quad \|B(u, v)\|_{\mathcal{X}_T} \leq C_1 T^{\frac{1}{2}(1 - n(1 - \frac{1}{q}))} \|\nabla K\|_{L^{q'}} \|u\|_{\mathcal{X}_T} \|v\|_{\mathcal{X}_T}.$$

Assume that  $u, v \in \mathcal{X}_T$ . First, we compute the  $L^1$ -norm of  $B(u, v)(t)$ . By inequalities (3.2) and (3.3) combined with the Hölder inequality and the Young inequality, we have

$$\begin{aligned} \|B(u, v)(t)\|_{L^1} &\leq \int_0^t \|\nabla G(\cdot, t - s) * (u(\nabla K * v))(s)\|_{L^1} ds \\ &\leq C \int_0^t (t - s)^{-1/2} \|u(\nabla K * v)(s)\|_{L^1} ds \\ &\leq C \int_0^t (t - s)^{-1/2} \|u(s)\|_{L^q} \|\nabla K\|_{L^{q'}} \|v(s)\|_{L^1} ds \end{aligned}$$

$$\begin{aligned} &\leq C\|\nabla K\|_{L^{q'}}\left(\sup_{0<s<T} s^{\frac{n}{2}(1-\frac{1}{q})}\|u(s)\|_{L^q}\right)\left(\sup_{0<s<T}\|v(s)\|_{L^1}\right) \\ &\quad \times \int_0^t (t-s)^{-1/2} s^{-\frac{n}{2}(1-\frac{1}{q})} ds \\ &\leq C\|\nabla K\|_{L^{q'}}\|u\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T} \int_0^t (t-s)^{-1/2} s^{-\frac{n}{2}(1-\frac{1}{q})} ds, \end{aligned}$$

where  $C$  is a positive constant. Here, notice that  $-\frac{n}{2}(1-1/q) > -1$  because  $q \in [1, n/(n-1))$ , consequently,

$$\int_0^t (t-s)^{-1/2} s^{-\frac{n}{2}(1-\frac{1}{q})} ds = t^{\frac{1}{2}(1-n(1-1/q))} \mathcal{B}\left(1 - \frac{n}{2}\left(1 - \frac{1}{q}\right), \frac{1}{2}\right),$$

where  $\mathcal{B}$  denotes the beta function. Therefore, we obtain

$$(3.5) \quad \sup_{0<t\leq T} \|B(u, v)\|_{L^1} \leq CT^{\frac{1}{2}(1-n(1-\frac{1}{q}))} \|\nabla K\|_{L^{q'}} \|u\|_{\mathcal{X}_T} \|v\|_{\mathcal{X}_T}.$$

where  $\frac{1}{2}(1-n(1-1/q)) > 0$ .

To deal with the  $L^q$ -norm of  $B(u, v)(t)$ , we proceed similarly:

$$\begin{aligned} &t^{\frac{n}{2}(1-\frac{1}{q})} \|B(u, v)(t)\|_{L^q} \\ &\leq Ct^{\frac{n}{2}(1-\frac{1}{q})} \int_0^t (t-s)^{-1/2} \|u(s)\|_{L^q} \|\nabla K\|_{L^{q'}} \|v(s)\|_{L^q} ds \\ &\leq Ct^{\frac{n}{2}(1-\frac{1}{q})} \|\nabla K\|_{L^{q'}} \left(\sup_{0\leq s<T} s^{\frac{n}{2}(1-\frac{1}{q})}\|u(s)\|_{L^q}\right) \left(\sup_{0\leq s<T} s^{\frac{n}{2}(1-\frac{1}{q})}\|v(s)\|_{L^q}\right) \\ &\quad \times \int_0^t (t-s)^{-1/2} s^{-n(1-\frac{1}{q})} ds \\ &\leq Ct^{\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \|\nabla K\|_{L^{q'}} \mathcal{B}\left(1 - n\left(1 - \frac{1}{q}\right), \frac{1}{2}\right) \|u\|_{\mathcal{X}_T} \|v\|_{\mathcal{X}_T}. \end{aligned}$$

Hence, we have

$$(3.6) \quad \sup_{0\leq t\leq T} t^{\frac{n}{2}(1-1/q)} \|B(u, v)(t)\|_{L^q} CT^{\frac{1}{2}(1-n(1-1/q))} \|\nabla K\|_{L^{q'}} \|u\|_{\mathcal{X}_T} \|v\|_{\mathcal{X}_T}.$$

Estimates (3.5) and (3.6) imply that the bilinear form  $B$  satisfies (3.4). Hence, it follows from Lemma 3.1 that if we choose  $T > 0$  so small that

$$(3.7) \quad 4C_1 T^{\frac{1}{2}(1-n(1-\frac{1}{q}))} \|\nabla K\|_{L^{q'}} \|u_0\|_{L^1} (1 + C(q, 1)) < 1,$$

then there is a solution in the space  $\mathcal{X}_T$  with  $\|u\|_{\mathcal{X}_T} \leq 2\|u_0\|_{L^1} (1 + C(q, 1))$ .

By Lemma 3.1, this is the unique solution in the ball  $U(0, 1/(2C))$  with the constant  $C = C_1 T^{\frac{1}{2}(1-n(1-1/q))} \|\nabla K\|_{L^{q'}}$ . However, using a standard argument based on the Gronwall lemma combined with the estimates leading

to (3.5) and (3.6), one can show that this is the unique solution in the whole space  $\mathcal{X}_T$ . This completes the proof. ■

*Proof of Theorem 2.3.* Now, we assume that  $q' \in [1, n]$  and we apply Lemma 3.1 in the space  $\mathcal{X} \equiv \mathcal{Y}_T = C([0, T], L^1(\mathbb{R}^n)) \cap C([0, T], L^q(\mathbb{R}^n))$  endowed with the norm  $\|u\|_{\mathcal{Y}_T} \equiv \sup_{0 \leq t \leq T} \|u\|_{L^1} + \sup_{0 \leq t \leq T} \|u\|_{L^q}$ .

By inequality (3.2), it is clear that

$$\|G(\cdot, t) * u_0\|_{L^1} \leq \|u_0\|_{L^1} \quad \text{and} \quad \|G(\cdot, t) * u_0\|_{L^q} \leq \|u_0\|_{L^q}$$

for all  $u_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ . These inequalities imply that  $y \equiv G(\cdot, t) * u_0 \in \mathcal{Y}_T$  and  $\|G(\cdot, t) * u_0\|_{\mathcal{Y}_T} \leq \|u_0\|_{L^1} + \|u_0\|_{L^q}$ .

Next, for  $u, v \in \mathcal{Y}_T$ , we see that

$$\begin{aligned} \|B(u, v)(t)\|_{L^1} &\leq C \int_0^t (t-s)^{-1/2} \|u(s)\|_{L^q} \|\nabla K\|_{L^{q'}} \|v(s)\|_{L^1} ds \\ &\leq CT^{1/2} \|\nabla K\|_{L^{q'}} \|u\|_{\mathcal{Y}_T} \|v\|_{\mathcal{Y}_T}, \end{aligned}$$

where  $C$  is a positive constant. In a similar way, we show the  $L^q$ -estimate

$$\begin{aligned} \|B(u, v)(t)\|_{L^q} &\leq C \int_0^t (t-s)^{-1/2} \|u(\nabla K * u)(s)\|_{L^q} ds \\ &\leq C \int_0^t (t-s)^{-1/2} \|u(s)\|_{L^q} \|\nabla K\|_{L^{q'}} \|v(s)\|_{L^q} ds \\ &\leq CT^{1/2} \|\nabla K\|_{L^{q'}} \|u\|_{\mathcal{Y}_T} \|v\|_{\mathcal{Y}_T}. \end{aligned}$$

Summing up these inequalities, we obtain

$$(3.8) \quad \|B(u, v)\|_{\mathcal{Y}_T} \leq C\sqrt{T} \|\nabla K\|_{L^{q'}} \|u\|_{\mathcal{Y}_T} \|v\|_{\mathcal{Y}_T}.$$

Therefore, by Lemma 3.1, if we choose  $T > 0$  such that

$$(3.9) \quad 4C\sqrt{T} \|\nabla K\|_{L^{q'}} (\|u_0\|_{L^1} + \|u_0\|_{L^q}) < 1,$$

then we obtain a local-time solution in the space  $\mathcal{Y}_T$  which is unique in the open ball  $U(0, (2C\sqrt{T} \|\nabla K\|_{L^{q'}})^{-1})$ . As in the proof of Theorem 2.2, using an argument involving the Gronwall lemma, we can show that this is the unique solution in the whole space  $\mathcal{Y}_T$ . ■

### 4. Construction of global-in-time solutions

*Proof of Theorem 2.5.* We are going to show that any nonnegative local-in-time mild solution  $u = u(x, t)$  constructed in Theorem 2.2 exists, in fact, on every time interval  $[0, T]$ .

First, we note that the condition  $u_0(x) \geq 0$  implies  $u(x, t) \geq 0$  for all  $x \in \mathbb{R}^n$  and  $t \geq 0$ . Next, integrating equation (2.1) with respect to  $x$ , using

the Fubini theorem, and the identities

$$\int_{\mathbb{R}^n} G(x, t) \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} \nabla G(x, t) \, dx = 0 \quad \text{for all } t > 0,$$

we obtain the conservation of the  $L^1$ -norm of nonnegative solutions:

$$(4.1) \quad \|u(t)\|_{L^1} = \int_{\mathbb{R}^n} u(x, t) \, dx = \int_{\mathbb{R}^n} u_0(x) \, dx = \|u_0\|_{L^1}.$$

For this reason, the local existence time  $T = T(\|u_0\|_{L^1}, \|\nabla K\|_{L^{q'}})$  from Theorem 2.3 does not change for all nonnegative  $u_0 \in L^1(\mathbb{R}^n)$  with the same  $L^1$ -norm. From now on, it suffices to follow a standard procedure which consists in applying repeatedly Theorem 2.2 to equation (1.1) supplemented with the initial datum  $u(x, kT)$  to obtain a unique solution on the interval  $[kT, (k + 1)T]$  for every  $k \in \mathbb{N}$ . This completes the proof of Theorem 2.5. ■

Next, we deal with strongly singular kernels from the space  $L^{q', \infty}(\mathbb{R}^n)$  with  $1 < q' \leq n$ . The following lemma plays an important role in the proof of Theorem 2.6.

LEMMA 4.1. *Assume that  $\nabla K \in L^{q', \infty}(\mathbb{R}^n)$  with  $1 < q' \leq n$ . For every  $r, p \in (1, \infty)$  satisfying*

$$(4.2) \quad \frac{1}{r} = \frac{2}{p} + \frac{1}{q'} - 1$$

*there is a positive number  $C = C(r, p, n, q', \|\nabla K\|_{L^{q', \infty}})$  such that for all  $u, v \in L^p(\mathbb{R}^n)$  we have*

$$(4.3) \quad \|u(\nabla K * v)\|_{L^r} \leq C \|u\|_{L^p} \|v\|_{L^p}.$$

*Proof.* First, we use the Hölder inequality to estimate

$$\|u(\nabla K * v)\|_{L^r} \leq C \|u\|_{L^p} \|\nabla K * v\|_{L^k} \quad \text{with} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{k}.$$

Next, we apply the weak Young inequality (2.3) which leads to  $\nabla K * v \in L^k(\mathbb{R}^n)$  with  $1/k = 1/p + 1/q' - 1$ . ■

*Proof of Theorem 2.6.* Recall that  $q_* = 1/(1 + 1/n - 1/q')$ . For  $p$  satisfying

$$(4.4) \quad 1 \leq \max \left\{ q_*, \frac{1}{1 - 1/(2q')} \right\} < p < \frac{1}{1 - 1/q' + 1/(2n)},$$

we define the Banach space

$$\begin{aligned} \mathcal{X} = & C([0, \infty), L^{q_*}(\mathbb{R}^n)) \\ & \cap \left\{ C([0, \infty), L^p(\mathbb{R}^n)) \mid \sup_{t>0} t^{\frac{n}{2}(\frac{1}{q_*} - \frac{1}{p})} \|u(t)\|_{L^p} < \infty \right\} \end{aligned}$$

with the norm  $\|u\|_{\mathcal{X}} \equiv \sup_{t>0} \|u(t)\|_{L^{q_*}} + \sup_{t>0} t^{\frac{n}{2}(\frac{1}{q_*} - \frac{1}{p})} \|u(t)\|_{L^p}$ .

For every  $u_0 \in L^{q^*}(\mathbb{R}^n)$ , it follows immediately from estimates (3.2) that

$$(4.5) \quad \|G(\cdot) * u_0\|_{\mathcal{X}} \leq C_3 \|u_0\|_{L^{q^*}}$$

for some constant  $C_3 > 0$ .

In the next step, we estimate the bilinear form  $B(u, v)$  defined in (3.1) for any  $u, v \in \mathcal{X}$ . By estimates (3.3) and (4.3), we have

$$(4.6) \quad \begin{aligned} \|B(u, v)(t)\|_{L^{q^*}} &\leq C \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q^*})-\frac{1}{2}} \|u(\nabla K * v)(s)\|_{L^r} ds \\ &\leq C \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q^*})-\frac{1}{2}} \|u(s)\|_{L^p} \|v(s)\|_{L^p} ds \\ &\leq C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q^*})-\frac{1}{2}} s^{-n(\frac{1}{q^*}-\frac{1}{p})} ds, \end{aligned}$$

where  $r$  is defined in (4.2). Inequalities in (4.6) make sense and involve convergent integrals because, by a direct calculation, it follows from (4.4) that

$$1 < r \leq q^*, \quad -\frac{n}{2} \left( \frac{1}{r} - \frac{1}{q^*} \right) - \frac{1}{2} > -1, \quad -n \left( \frac{1}{q^*} - \frac{1}{p} \right) > -1.$$

Therefore, after changing the variable on the right-hand side of (4.6), we see that

$$\begin{aligned} \|B(u, v)(t)\|_{L^{q^*}} &\leq C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q^*})-\frac{1}{2}-n(\frac{1}{q^*}-\frac{1}{p})+1} \\ &\quad \times \mathcal{B} \left( 1 - n \left( \frac{1}{q^*} - \frac{1}{p} \right), \frac{1}{2} \left( 1 - n \left( \frac{1}{r} - \frac{1}{q^*} \right) \right) \right), \end{aligned}$$

where  $\mathcal{B}$  denotes the beta function. However, for  $q^*$  defined by (2.4), it follows from (4.2) that

$$-\frac{n}{2} \left( \frac{1}{r} - \frac{1}{q^*} \right) - \frac{1}{2} - n \left( \frac{1}{q^*} - \frac{1}{p} \right) + 1 = 0,$$

hence, the  $L^{q^*}$ -norm of  $B(u, v)$  is estimated as

$$(4.7) \quad \sup_{t>0} \|B(u, v)(t)\|_{L^{q^*}} \leq C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}$$

with a positive constant  $C$ .

By similar arguments to those in the case of the  $L^{q^*}$ -estimate, we obtain

$$\begin{aligned} t^{\frac{n}{2}(\frac{1}{q^*}-\frac{1}{p})} \|B(u, v)(t)\|_{L^p} &\leq C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} t^{\frac{n}{2}(\frac{1}{q^*}-\frac{1}{p})} \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}} s^{-n(\frac{1}{q^*}-\frac{1}{p})} ds \\ &= C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}. \end{aligned}$$

Therefore, there is a constant  $C > 0$  independent of  $t$  such that

$$(4.8) \quad \sup_{t>0} t^{\frac{n}{2}(\frac{1}{q_*} - \frac{1}{p})} \|B(u, v)(t)\|_{L^p} \leq C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}.$$

Finally, it follows from (4.7) and (4.8) that

$$(4.9) \quad \|B(u, v)\|_{\mathcal{X}} \leq \eta \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}$$

for a positive number  $\eta$  independent of  $t, u,$  and  $v.$  Hence, we conclude by Lemma 3.1 that the equation  $u(t) = G(t) * u_0 + B(u, u)$  has a solution in  $\mathcal{X}$  if  $4\eta \|G(t) * u_0\|_{\mathcal{X}} < 1.$  However, by (4.5), it suffices to assume that  $\|u_0\|_{L^{q_*}} < 1/(4\eta C_3)$  to complete the proof of Theorem 2.6. ■

### 5. Nonexistence of global-in-time solutions

*Proof of Theorem 2.9.* Let us recall that we limit ourselves to nonnegative solutions to (1.1)–(1.2) satisfying

$$M = \int_{\mathbb{R}^n} u(x, t) dx = \int_{\mathbb{R}^n} u_0(x) dx \quad \text{for all } t \in [0, T].$$

As a standard practice, we study the evolution of the second moment of a solution to (1.1)–(1.2),

$$I(t) = \int_{\mathbb{R}^n} |x|^2 u(x, t) dx.$$

Here, we skip the well-known argument (see e.g. [12]) that the quantity  $I(t)$  is finite if  $u_0 \in L^1(\mathbb{R}^n, (1 + |x|^2) dx).$

Differentiating the function  $I(t)$  with respect to  $t,$  using equation (1.1), and integrating by parts, we obtain

$$(5.1) \quad \begin{aligned} \frac{d}{dt} I(t) &= \int_{\mathbb{R}^n} |x|^2 (\Delta u - \nabla \cdot (u(\nabla K * u))) dx \\ &= 2nM + 2 \int_{\mathbb{R}^n} x \cdot u(\nabla K * u) dx \\ &= 2nM + 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x, t) u(y, t) x \cdot \nabla K(x - y) dx dy. \end{aligned}$$

Symmetrizing in  $x$  and  $y$  the double integral on the right-hand side of (5.1), we obtain

$$(5.2) \quad \begin{aligned} \frac{d}{dt} I(t) &= 2nM \\ &+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x, t) u(y, t) (x \cdot \nabla K(x - y) + y \cdot \nabla K(y - x)) dx dy. \end{aligned}$$

Now, notice that the interaction kernel is assumed to be radial,  $K(x) =$

$K(|x|)$ , hence  $\nabla K(x) = (x/|x|)K'(r)$ , where  $r = |x|$ . Therefore,  

$$x \cdot \nabla K(x - y) + y \cdot \nabla K(y - x) = x \cdot \frac{x - y}{|x - y|} K'(|x - y|) + y \cdot \frac{y - x}{|y - x|} K'(|y - x|)$$

$$= |x - y| K'(|x - y|).$$

Now, we apply the assumption (ii) imposed on the kernel  $K$  as follows:

$$\begin{aligned} \frac{d}{dt} I(t) &= 2nM + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x, t) u(y, t) |x - y| K'(|x - y|) \, dx \, dy \\ &\leq 2nM - \gamma \iint_{|x - y| \leq \delta} u(x, t) u(y, t) \, dx \, dy \\ &\quad + C \iint_{|x - y| > \delta} u(x, t) u(y, t) |x - y|^2 \, dx \, dy \\ &\leq 2nM - \gamma M^2 + (C + \gamma/\delta^2) \iint_{|x - y| > \delta} u(x, t) u(y, t) |x - y|^2 \, dx \, dy. \end{aligned}$$

Hence, using the elementary inequality  $|x - y| \leq 2(|x|^2 + |y|^2)$  we obtain

$$\begin{aligned} \frac{d}{dt} I(t) &\leq 2nM - \gamma M^2 + 2(C + \gamma/\delta^2) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x, t) u(y, t) (|x|^2 + |y|^2) \, dx \, dy \\ &= M(2n - \gamma M + 4(C + \gamma/\delta^2) I(t)), \end{aligned}$$

which implies that

$$\frac{d}{dt} I(t) \leq M(2n - \gamma M + 4(C + \gamma/\delta^2) I(0)) < 0 \quad \text{for all } t > 0$$

provided  $\gamma M > 2n + 4(C + \gamma/\delta^2) I(0)$ . Consequently,  $I(T) = 0$  for some  $0 < T < \infty$ . This contradicts the global-in-time existence of regular nonnegative solutions of problem (1.1)–(1.2). ■

*Proof of Theorem 2.12.* In order to show that the local-in-time solution from Theorem 2.3 exists for all  $t \in [0, \infty)$ , it is sufficient to obtain its a priori  $L^q$ -estimate. Indeed, if  $\|u(t)\|_{L^q}$  does not blow up in finite time, we can apply a continuation argument as in the proof of Theorem 2.5.

Multiplying both sides of (1.1) by  $u^{q-1}$  (recall that  $u$  is nonnegative), integrating over  $\mathbb{R}^n$ , and using the decomposition of  $K$ , we have

$$\begin{aligned} (5.3) \quad \frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^n} u^q \, dx &= \int_{\mathbb{R}^n} u^{q-1} \Delta u \, dx - \int_{\mathbb{R}^n} u^{q-1} \nabla \cdot (u(\nabla K * u)) \, dx \\ &= -(q - 1) \int_{\mathbb{R}^n} u^{q-2} |\nabla u|^2 \, dx + (q - 1) \int_{\mathbb{R}^n} u^{q-1} \nabla u \cdot (\nabla K_1 * u) \, dx \\ &\quad + (q - 1) \int_{\mathbb{R}^n} u^{q-1} \nabla u \cdot (\nabla K_2 * u) \, dx. \end{aligned}$$

Notice that the second term of the right-hand side of (5.3) is nonpositive due to the assumptions of  $K_1$  in view of the following calculation:

$$\begin{aligned} (q-1) \int_{\mathbb{R}^n} u^{q-1} \nabla u \cdot (\nabla K_1 * u) \, dx &= \frac{q-1}{q} \int_{\mathbb{R}^n} \nabla u^q \cdot (\nabla K_1 * u) \, dx \\ &= -\frac{q-1}{q} \int_{\mathbb{R}^n} u^q \cdot (\Delta K_1 * u) \, dx \leq 0. \end{aligned}$$

Here, we have assumed  $K_1$  to be sufficiently regular and the more general case can be handled by a standard regularization procedure.

Next, by the  $\varepsilon$ -Young inequality, the third term of the right-hand side of (5.3) is estimated as follows:

$$\begin{aligned} (q-1) \int_{\mathbb{R}^n} u^{q-1} \nabla u \cdot (\nabla K_2 * u) \, dx &\leq (q-1) \left[ \varepsilon \int_{\mathbb{R}^n} u^{q-2} |\nabla u|^2 \, dx + C(\varepsilon) \int_{\mathbb{R}^n} u^q |\nabla K_2 * u|^2 \, dx \right] \\ &\leq \varepsilon (q-1) \int_{\mathbb{R}^n} u^{q-2} |\nabla u|^2 \, dx + C(\varepsilon) \|\nabla K_2\|_{L^\infty}^2 \|u_0\|_{L^1}^2 \int_{\mathbb{R}^n} u^q \, dx \end{aligned}$$

since, by (4.1), we have  $\|\nabla K_2 * u(t)\|_{L^\infty} \leq \|\nabla K_2\|_{L^\infty} \|u_0\|_{L^1}$ .

Therefore, coming back to (5.3), for  $\varepsilon \leq 1$ , we see that

$$\begin{aligned} \frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^n} u^q \, dx &\leq -(q-1)(1-\varepsilon) \int_{\mathbb{R}^n} u^{q-2} |\nabla u|^2 \, dx \\ &\quad + C(\varepsilon) \|\nabla K_2\|_{L^\infty}^2 \|u_0\|_{L^1}^2 \int_{\mathbb{R}^n} u^q \, dx \\ &\leq C(\varepsilon) \|\nabla K_2\|_{L^\infty}^2 \|u_0\|_{L^1}^2 \int_{\mathbb{R}^n} u^q \, dx. \end{aligned}$$

Hence, by the Gronwall lemma,  $\|u(t)\|_{L^q} \leq e^{Ct} \|u_0\|_{L^q}$ , where  $C = C(\varepsilon) \|\nabla K_2\|_{L^\infty}^2 \|u_0\|_{L^1}^2$ . This implies that  $\|u(t)\|_{L^q}$  does not blow up in finite time and the proof of Theorem 2.12 is complete. ■

**Acknowledgments.** This work was partially supported by the MNiSzW grant No. N N201 418839, MEXT the Grant-in-Aid for Young Scientists (B) 20740087, and the Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007-2013 funded by European Regional Development Fund (Ph.D. Programme: Mathematical Methods in Natural Sciences).

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