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WEIERSTRASS ELLIPTIC SOLUTIONS TO A ZAKHAROV EQUATION IN PLASMAS WITH POWER LAW NONLINEARITY

Abstract. In this paper, travelling wave solutions for the Zakharov equation in plasmas with power law nonlinearity are studied by using the Weierstrass elliptic function method. As a result, some previously known solutions are recovered, and at the same time some new ones are also given.

1. Introduction. Finding the exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics, optical fiber, and geochemistry. Thus, it is important to investigate the exact explicit solutions of NLEEs. In recent years, various powerful methods have been presented for finding exact solutions of NLEEs in mathematical physics, such as the Bäcklund transformation method [22, 35], Hirotas direct method [12], tanh-sech method [20], extended tanh method [2, 9], the exp-function method [10, 11], sine-cosine method [32, 38], Jacobi elliptic function expansion method [14], F-expansion method [6], Weierstrass elliptic function method [26]. Among those, one of the most effectively straightforward methods for constructing exact solutions is the Weierstrass elliptic function method [24, 27].

The generalized Zakharov equation (GZE) is a realistic model for plasma, which can be written as follows [21]:

$$iq_t + q_{xx} - 2\beta|q|q + 2qr = 0,$$
$$r_{tt} - r_{xx} + \alpha(|q|)_{xx} = 0, \tag{1}$$

where q is the envelope of the high-frequency electric field, and r is the

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plasma density measured from its equilibrium value. When $\beta = 0$, this system reduces to the classical Zakharov equation of plasma physics [13, 36].

Recently, some authors have investigated the travelling wave solutions of this equation and of its generalized forms by using various methods such as the exp-function method [4], the extended F-expansion method [31], He's variational iteration method [16], a new rational auxiliary equation method [18], He's semi-inverse method [37], the bifurcation method [29] and others.

The purpose of this paper is to apply the Weierstrass elliptic function method to the following Zakharov equation in plasmas with power law nonlinearity:

$$\begin{aligned} &iq_t + q_{xx} - 2\lambda|q|^{2n}q + 2qr = 0, \\ &r_{tt} - r_{xx} + (|q|^{2n})_{xx} = 0. \end{aligned} \tag{2}$$

Equations (2) are the ZE with power law nonlinearity and the parameter n is the power law nonlinearity parameter. The dependent variable $q(x,t)$ is a complex valued function while $r(x,t)$ is a real valued function. The coefficient λ is a real constant. Equations (2) have already been studied by He's variational principle [3], the ansatz method [30] and the bifurcation method [28].

The aim of this work is to investigate the travelling wave solutions of (2) systematically, by applying the Weierstrass elliptic function method.

The rest of this paper is organized as follows. In Section 2, we outline the Weierstrass elliptic function method. In Section 3, we give some particular travelling wave solutions of (2) by using the Weierstrass elliptic function method. Finally, some conclusions are given in Section 4.

2. Weierstrass elliptic functions. Let us consider the following nonlinear differential equation:

$$\left(\frac{d\phi(z)}{dz}\right)^2 = a_0\phi^4 + 4a_1\phi^3 + 6a_2\phi^2 + 4a_3\phi + a_4 \equiv f(\phi). \tag{3}$$

As is well-known [33, 34], the solutions ϕ of (3) can be expressed in terms of elliptic functions $\wp$:

$$\phi(z) = \phi_0 + \frac{f'(\phi_0)}{4\left[\wp(z, g_2, g_3) - \frac{1}{24}f''(\phi_0)\right]}, \tag{4}$$

where the primes denote differentiation with respect to ϕ, and ϕ_0 is a simple root of $f(\phi)$. The invariants g_2, g_3 of elliptic functions $\wp(z, g_2, g_3)$ are related to the coefficients of $f(\phi)$ by [5]

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \tag{5}$$

$$g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4, \tag{6}$$

When g_2 and g_3 are real, we have different behavior of $\wp(z)$ depending on whether the discriminant

$$\Delta = g_2^3 - 27g_3^2 \tag{7}$$

is positive, negative or zero.The conditions [26]

$$\Delta \neq 0 \quad \text{or} \quad \Delta = 0,\ g_2 > 0,\ g_3 > 0 \tag{8}$$

lead to periodic solutions, whereas the conditions

$$\Delta = 0, \qquad g_2 \geq 0, \qquad g_3 \leq 0 \tag{9}$$

lead to solitary wave solutions.

If $\Delta = 0$, then $\wp(z, g_2, g_3)$ degenerate into trigonometric or hyperbolic functions [1]. Thus, periodic solutions according to (11) are determined by

$$\phi(z) = \phi_0 + \frac{f'(\phi_0)}{4\left[-\frac{e_1}{2} - \frac{f''(\phi_0)}{24} + \frac{3}{2}e_1 \csc^2\left(\sqrt{\frac{3}{2}e_1}\, t\right)\right]}, \qquad \Delta = 0,\ g_3 > 0, \tag{10}$$

and solitary wave solutions by

$$\phi(z) = \phi_0 + \frac{f'(\phi_0)}{4\left[e_1 - \frac{f''(\phi_0)}{24} + 3e_1 \operatorname{csch}^2(\sqrt{3e_1}t)\right]}, \qquad \Delta = 0,\ g_3 < 0, \tag{11}$$

where $e_1 = \sqrt[3]{|g_3|}$ in (10) and $e_1 = \frac{1}{2}\sqrt[3]{|g_3|}$ in (11).

3. The Zakharov equation in plasmas with power law nonlinearity. We assume that the travelling wave solution of (2) is of the form

$$\begin{aligned} q(x,t) &= \exp(i\theta)g(\xi), \qquad & r(x,t) &= h(\xi), \\ \xi &= k(x - \nu t), & \theta &= -px + \omega t, \end{aligned} \tag{12}$$

where ν represents the velocity of the wave, while g and h are the wave profiles. In the phase component, p is the frequency and ω is the wave number. Substituting (12) into (2) and decomposing into real and imaginary parts gives

$$\nu = 2p, \tag{13}$$

and

$$\begin{aligned} k^2 g'' + 2gh - (\omega + p^2)g - 2\lambda g^{2n+1} &= 0, \\ k^2(4p^2 - 1)h'' + k^2(g^{2n})'' &= 0. \end{aligned} \tag{14}$$

Integrating the second equation of (14) twice and letting the first integral constant be zero, we have

$$h = \frac{g^{2n}}{1 - 4p^2} + c, \qquad p \neq 1/2, \tag{15}$$

where c is the second integral constant.

Substituting (15) into the first equation of (14), we have

$$g'' + \beta g - \alpha g^{2n+1} = 0 \tag{16}$$

where

$$\alpha = \frac{2}{k^2}\left(\lambda - \frac{1}{1-4p^2}\right), \qquad \beta = \frac{2c - p^2 - q}{k^2}.$$

By multiplying each side of (16) by g', and integrating once again, we deduce that

$$(g')^2 = R_1 - \beta g^2 + \frac{\alpha}{n+1} g^{2n+2}, \tag{17}$$

where R_1 is the integration constant. Making the transformation $g = \phi^p$, $p \neq 0, 1$, yields

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{1}{p^2}\left[R_1 \phi^{2-2p} - \beta\phi^2 + \frac{\alpha}{n+1}\phi^{2np+2}\right]. \tag{18}$$

If we want to guarantee the integrability of (18) in terms of elliptic functions, the powers of ϕ have to be integer numbers between 0 and 4 [15], and therefore, we have the following possible cases:

If $R_1 = 0$, then $p \in \{-1/n, -1/(2n), 1/(2n), 1/n\}$.

If $R_1 \neq 0$, then $n = 2$ and $p = \pm 1/2$.

Next, by using the results obtained in the preceding sections, we will construct the corresponding solutions of (18) in the above cases.

3.1. Case 1.

We consider several subcases.

Case (1)(i): $R_1 = 0$, $p = 1/(2n)$. In this case, (18) takes the form

$$\left(\frac{d\phi}{d\xi}\right)^2 = 4n^2\left[-\beta\phi^2 + \frac{\alpha}{n+1}\phi^3\right] = f(\phi). \tag{19}$$

The polynomial $f(\phi)$ has two roots: $\phi_0 = 0$ and $\phi_0 = \beta(n+1)/\alpha$. From (4), the solutions of (19) can be found to be

$$\phi = \frac{3\phi_0\wp(\xi, g_2, g_3) - 5n^2\beta\phi_0 + \frac{6n^2\alpha}{n+1}\phi_0^2}{3\wp(\xi, g_2, g_3) + n^2\beta - \frac{3n^2\alpha}{n+1}\phi_0} \tag{20}$$

where the invariants are

$$g_2 = 48n^4\beta^2, \qquad g_3 = 64n^6\beta^3. \tag{21}$$

Then $\Delta = 0$. The root $\phi_0 = 0$ gives the trivial solution $\phi = 0$, and the nonzero solution of (19) can be found by taking $\phi_0 = \beta(n+1)/\alpha$. Hence, from (10), we have the periodic wave solution to (19),

$$\phi = \frac{\beta(n+1)}{\alpha}\sec^2(n\sqrt{\beta}\,\xi) \tag{22}$$

for $\beta > 0$. From (11), we have the solitary wave solution

$$\phi = \frac{\beta(n+1)}{\alpha}\operatorname{sech}^2(n\sqrt{-\beta}\,\xi) \tag{23}$$

for $\beta < 0$.

Therefore, when $\beta > 0$, (2) has the periodic wave solution

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\sqrt{\frac{\beta(n+1)}{\alpha}}\sec(n\sqrt{\beta}\,\xi)\right)^{1/n}, \\ r(x,t) &= \frac{\beta(n+1)}{\alpha(1-4p^2)}(\sec(n\sqrt{\beta}\,\xi))^2 + c. \end{aligned} \tag{24}$$

When $\beta < 0$, it has the solitary wave solution

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\sqrt{\frac{\beta(n+1)}{\alpha}}\operatorname{sech}(n\sqrt{-\beta}\,\xi)\right)^{1/n}, \\ r(x,t) &= \frac{\beta(n+1)}{\alpha(1-4p^2)}(\operatorname{sech}(n\sqrt{-\beta}\,\xi))^2 + c. \end{aligned} \tag{25}$$

Case 1(ii): $R_1 = 0$, $p = -1/(2n)$. In this case, (18) takes the form

$$\left(\frac{d\phi}{d\xi}\right)^2 = 4n^2\left[-\beta\phi^2 + \frac{\alpha}{n+1}\phi\right] = f(\phi). \tag{26}$$

The second order polynomial $f(\phi)$ has two roots: $\phi_0 = 0$ and $\phi_0 = \alpha/(\beta(n+1))$. From (4), the solutions of (26) can be found to be

$$\phi = \frac{3\phi_0\wp(\xi, g_2, g_3) - 5n^2\beta\phi_0 + \frac{3n^2\alpha}{n+1}}{3\wp(\xi, g_2, g_3) + n^2\beta} \tag{27}$$

where the invariants are given by (21). Substituting the root $\phi_0 = 0$ into (27), we get

$$\phi = \frac{3n^2\alpha}{(n+1)(3\wp(\xi, g_2, g_3) + n^2\beta)}. \tag{28}$$

Since $\Delta = 0$, from (10) we have the periodic wave solution to (26),

$$\phi = \frac{\alpha}{\beta(n+1)}\sin^2(n\sqrt{\beta}\,\xi) \tag{29}$$

for $\beta > 0$. From (11), we have the solitary wave solution

$$\phi = -\frac{\alpha}{\beta(n+1)}\sinh^2(n\sqrt{-\beta}\,\xi) \tag{30}$$

for $\beta < 0$. Therefore, when $\beta > 0$, (2) has the periodic wave solution

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\sqrt{\frac{\beta(n+1)}{\alpha}}\csc(n\sqrt{\beta}\,\xi)\right)^{1/n}, \\ r(x,t) &= \frac{\beta(n+1)}{\alpha(1-4p^2)}(\csc(n\sqrt{\beta}\,\xi))^2 + c. \end{aligned} \tag{31}$$

When $\beta < 0$, it has the solitary wave solution

$$(32)\quad \begin{aligned} q(x,t) &= \exp(i\theta)\left(\sqrt{-\frac{\beta(n+1)}{\alpha}}\operatorname{csch}(n\sqrt{-\beta}\,\xi)\right)^{1/n}, \\ r(x,t) &= \frac{\beta(n+1)}{\alpha(1-4p^2)}(\operatorname{csch}(n\sqrt{-\beta}\,\xi))^2 + c. \end{aligned}$$

After substituting the second root $\phi_0 = \alpha/(\beta(n+1))$ into (27), we obtain

$$(33)\quad \phi = \frac{\alpha}{\beta(n+1)}\,\frac{3\wp(\xi, g_2, g_3) - 2n^2\beta}{3\wp(\xi, g_2, g_3) + n^2\beta}.$$

So from (10), we have the periodic wave solution to (26),

$$(34)\quad \phi = \frac{\alpha}{\beta(n+1)}\cos^2(n\sqrt{\beta}\,\xi)$$

for $\beta > 0$.

From (11), we have the solitary wave solution

$$(35)\quad \phi = \frac{\alpha}{\beta(n+1)}\cosh^2(n\sqrt{-\beta}\,\xi)$$

for $\beta < 0$.

Thus, we get the same solutions of (2) as those given by (24) and (25).

Case (1)(iii): $R_1 = 0$, $p = 1/n$. In this case, (18) takes the form

$$(36)\quad \left(\frac{d\phi}{d\xi}\right)^2 = 4n^2\left[-\beta\phi^2 + \frac{\alpha}{n+1}\phi^4\right] = f(\phi).$$

The fourth order polynomial $f(\phi)$ has two roots: $\phi_0 = 0$ (double) and $\phi_0 = \pm\sqrt{\beta(n+1)/\alpha}$. From (4), the solution of (26) in terms of ϕ_0 is

$$(37)\quad \phi = \frac{3\phi_0\wp(\xi, g_2, g_3) - 5n^2\beta\phi_0 + \frac{6n^2\alpha}{n+1}\phi_0^3}{3\wp(\xi, g_2, g_3) + n^2\beta - \frac{6n^2\alpha}{n+1}\phi_0^2}$$

where the invariants are

$$(38)\quad g_2 = \frac{4}{3}n^4\beta^2, \qquad g_3 = \frac{8}{27}n^6\beta^3,$$

Taking the root $\phi_0 = 0$ in (37), we get $\phi = 0$. However, when we take the root $\phi_0 = \pm\sqrt{\beta(n+1)/\alpha}$ in (37), we have

$$(39)\quad \phi = \pm\sqrt{\frac{\beta(n+1)}{\alpha}}\,\frac{3\wp(\xi, g_2, g_3) + n^2\beta}{3\wp(\xi, g_2, g_3) - 5n^2\beta}.$$

Since $\Delta = 0$, it is easy to see from (10), (11) that the above solutions (39) will generate the same periodic and solitary wave solutions to (2) as (24), (25).

Case 1(iv): $R_1 = 0$, $p = -1/n$. In this case, (18) takes the form

$$(40)\quad \left(\frac{d\phi}{d\xi}\right)^2 = n^2\left[-\beta\phi^2 + \frac{\alpha}{n+1}\right] = f(\phi).$$

The polynomial $f(\phi)$ hase two roots: $\phi_0 = \pm\sqrt{\frac{\alpha}{\beta(n+1)}}$. Using the same arguments as above, we can deduce that this case gives exactly the same solutions of (2) as Case 1(ii).

Exact travelling wave solutions of the Zakharov equation in plasmas with power law nonlinearity can be obtained by using the above results. We describe them in the following theorem.

THEOREM 1. *The Zakharov equation in plasmas with power law nonlinearity has solutions described as follows:*

(i) *When $\beta > 0$ and $\alpha(1-4p^2) \neq 0$, there exist the following explicit periodic wave solutions:*

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\sqrt{\frac{\beta(n+1)}{\alpha}}\sec(n\sqrt{\beta}\,\xi)\right)^{1/n}, \\ r(x,t) &= \frac{\beta(n+1)}{\alpha(1-4p^2)}(\sec(n\sqrt{\beta}\,\xi))^2 + c \end{aligned} \tag{41}$$

and

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\sqrt{\frac{\beta(n+1)}{\alpha}}\csc(n\sqrt{\beta}\,\xi)\right)^{1/n}, \\ r(x,t) &= \frac{\beta(n+1)}{\alpha(1-4p^2)}(\csc(n\sqrt{\beta}\,\xi))^2 + c. \end{aligned} \tag{42}$$

(ii) *When $\beta < 0$ and $\alpha(1-4p^2) \neq 0$, there exist the following explicit solitary wave solutions:*

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\sqrt{\frac{\beta(n+1)}{\alpha}}\operatorname{sech}(n\sqrt{-\beta}\,\xi)\right)^{1/n}, \\ r(x,t) &= \frac{\beta(n+1)}{\alpha(1-4p^2)}(\operatorname{sech}(n\sqrt{-\beta}\,\xi))^2 + c \end{aligned} \tag{43}$$

and

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\sqrt{-\frac{\beta(n+1)}{\alpha}}\operatorname{csch}(n\sqrt{-\beta}\,\xi)\right)^{1/n}, \\ r(x,t) &= \frac{\beta(n+1)}{\alpha(1-4p^2)}(\operatorname{csch}(n\sqrt{-\beta}\,\xi))^2 + c. \end{aligned} \tag{44}$$

REMARK 3.1. When $\beta > 0$, the solutions (41)–(42) that we obtained coincide with those obtained by the bifurcation method by M. Song [28, (3.4)]. When $\beta < 0$, the solutions (43)–(44) also coincide with those obtained by M. Song [28, (3.8)].

3.2. Case 2. Again, we consider several subspaces.

CASE 2(i): $R_1 \neq 0$, $n = 2$, $p = 1/2$. In this case, (18) takes the form

$$\left(\frac{d\phi}{d\xi}\right)^2 = 4\left[-\beta\phi^2 + \frac{\alpha}{3}\phi^4 + R_1\phi\right] = f(\phi). \tag{45}$$

According to (4), the solutions of (45) read

$$\phi = \frac{3\phi_0\wp(\xi, g_2, g_3) - 5\beta\phi_0 + 2\alpha\phi_0^3 + 3R_1}{3\wp(\xi, g_2, g_3) + \beta - 2\alpha\phi_0^2} \tag{46}$$

where the invariants are given by

$$g_2 = 4\beta^2, \qquad g_3 = \frac{4(\beta^3 - 9\alpha R_1^2)}{27}. \tag{47}$$

So we can obtain the general expressions for the solutions to (2):

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\frac{3\phi_0\wp(\xi, g_2, g_3) - 5\beta\phi_0 + 2\alpha\phi_0^3 + 3R_1}{3\wp(\xi, g_2, g_3) + \beta - 2\alpha\phi_0^2}\right)^{1/2}, \\ r(x,t) &= \frac{1}{1-4p^2}\left(\frac{3\phi_0\wp(\xi, g_2, g_3) - 5\beta\phi_0 + 2\alpha\phi_0^3 + 3R_1}{3\wp(\xi, g_2, g_3) + \beta - 2\alpha\phi_0^2}\right)^2 + c. \end{aligned} \tag{48}$$

For example, substituting the simplest root $\phi_0 = 0$ of $f(\phi)$ into (48), we get

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\frac{3R_1}{3\wp(\xi, g_2, g_3) + \beta}\right)^{1/2}, \\ r(x,t) &= \frac{1}{1-4p^2}\left(\frac{3R_1}{3\wp(\xi, g_2, g_3) + \beta}\right)^2 + c. \end{aligned} \tag{49}$$

CASE 2(ii): $R_1 \neq 0$, $n = 2$, $p = -1/2$. In this case, (18) takes the form

$$\left(\frac{d\phi}{d\xi}\right)^2 = 4\left[-\beta\phi^2 + \frac{\alpha}{3} + R_1\phi^3\right] = f(\phi). \tag{50}$$

Using similar arguments to those in Case 2(i), we get the following general expression for the solutions to (2):

$$\begin{aligned} q(x,t) &= \exp(i\theta)\left(\frac{-3\phi_0\wp(\xi, g_2, g_3) - 5\beta\phi_0 + 6R_1\phi_0^2}{3\wp(\xi, g_2, g_3) + \beta - 3R_1\phi_0}\right)^{-1/2}, \\ r(x,t) &= \frac{1}{1-4p^2}\left(\frac{-3\phi_0\wp(\xi, g_2, g_3) - 5\beta\phi_0 + 6R_1\phi_0^2}{3\wp(\xi, g_2, g_3) + \beta - 3R_1\phi_0}\right)^{-2} + c. \end{aligned} \tag{51}$$

4. Conclusion. From the above discussion, we find the travelling wave solutions of the Zakharov equation in plasmas with power law nonlinearity, expressed by hyperbolic functions and trigonometric functions, without the aid of mathematical software. The results show that the Weierstrass function method is a powerful mathematical tool to search for exact solutions to nonlinear differential equations, especially solitary ones. It may be advantageous that this quite general method can lead to free parameters as shown in the solution. We believe that this approach can also be used to solve other nonlinear equations.

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