APPLICATIONES MATHEMATICAE
31,3 (2004), pp. 303–312

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NONUNIQUENESS OF STEADY STATES IN ANNULAR DOMAINS FOR STREATER EQUATIONS

Abstract. Models introduced by R. F. Streater describe the evolution of the density and temperature of a cloud of self-gravitating particles. We study nonuniqueness of steady states in annular domains in $\mathbb{R}^d$, $d \geq 2$.

1. History of the problem and physical motivation. The system we will deal with is of the form

$$\begin{aligned} u_t &= -\nabla \cdot \mathbf{j}, \\ (u\theta)_t &= \nabla \cdot (\lambda \nabla \theta) - \nabla \cdot (\theta \mathbf{j}) - \nabla(\phi + \phi_0) \cdot \mathbf{j}, \\ \Delta \phi &= u, \end{aligned} \tag{1}$$

where $\mathbf{j} = -\kappa\big(\nabla u + \frac{u}{\theta}(\nabla\phi + \nabla\phi_0)\big)$. We will supplement this system with the following boundary conditions for u and θ:

$$\frac{\partial u}{\partial \nu} + \frac{u}{\theta}(\partial_\nu \phi + \partial_\nu \phi_0) = 0, \tag{2}$$

$$\frac{\partial \theta}{\partial \nu} = 0, \tag{3}$$

and we will consider two kinds of boundary conditions for ϕ:

$$\text{either the Dirichlet condition:} \quad \phi = 0 \quad \text{on } \partial\Omega, \tag{4}$$

$$\text{or the “free” condition:} \quad \phi = E_d * u_\Omega, \tag{5}$$

where $u_\Omega(x) = u(x)$ in Ω and vanishes outside Ω, and $\big(f * g\big)(x)$ means the convolution $\int_\Omega f(x-y)g(y)\,dy$ (E_d denotes the fundamental solution of the Laplacian in $\mathbb{R}^d$). We call (1) with (2)–(4) the Dirichlet problem, and (2), (3), (5) the “free” problem.

These systems describe the evolution of the density $u(x,t) \geq 0$ and the temperature $\theta(x,t) > 0$ in a cloud of self-gravitating particles under the

2000 *Mathematics Subject Classification*: 35Q, 35J60, 82C21.
Key words and phrases: Streater's models, existence and uniqueness of stationary solutions, Poisson–Boltzmann–Emden equation, phase plane method.

external potential $\phi_0(x)$ with self-induced potential ϕ. The coefficients κ and λ are nonnegative functions of x, u, θ and ϕ. Furthermore, they can vanish only when $\theta = 0$.

In [9] this system has been derived by Streater from the Smoluchowski equation (see [8]). This was generalized in [2] to the case of gravitational and electric self-interactions.

Some results on existence and uniqueness of solutions for (1) can be found e.g. in [3] and in the references therein.

2. Steady states equation. Following [1] we begin by recalling basic properties of solutions to (1)–(5).

PROPOSITION 2.1. *For sufficiently smooth solutions of the Dirichlet and the "free" problem we have*:

(i) *the mass* $M = \int_\Omega u\,dx$ *and the energy* $E = \int_\Omega u\big(\theta + \phi_0 + \frac{1}{2}\phi\big)\,dx$ *of the system are constant in time*,

(ii) *the entropy*

$$W = \int_\Omega u \log\left(\frac{u}{\theta}\right) dx$$

does not increase in time.

Proof. The first assertion follows easily from the boundary conditions, the Gauss–Green theorem and (1). For the entropy, a direct calculation shows that

$$\begin{aligned}
\frac{dW}{dt} &= \frac{d}{dt}\int_\Omega u \log\frac{u}{\theta}\,dx = \int_\Omega u_t \log\frac{u}{\theta}\,dx + \int_\Omega \left(\frac{u}{\theta}\right)_t \cdot \frac{\theta}{u}\,dx \\
&= -\int_\Omega \frac{1}{\theta}\nabla\cdot(\lambda\nabla\theta)\,dx - \int_\Omega \left(\frac{\nabla u}{u} + \frac{1}{\theta}(\nabla\phi + \nabla\phi_0)\cdot\mathbf{j}\right) dx \\
&= -\int_\Omega \lambda\,\frac{|\nabla\theta|^2}{\theta^2}\,dx - \int_\Omega \kappa u \left|\frac{\nabla u}{u} + \frac{1}{\theta}(\nabla\phi + \nabla\phi_0)\right|^2 dx \le 0,
\end{aligned} \tag{6}$$

which completes the proof. ∎

For a steady state $(u_\infty, \theta_\infty)$ (i.e. a solution independent of time), the entropy of the system is constant. From this trivial fact and from (6) we deduce that

$$\nabla u_\infty + \frac{u_\infty}{\theta_\infty}\nabla(\phi_\infty + \phi_0)$$

vanishes a.e. in Ω, and the temperature is constant: $\theta_\infty = \text{const}$. Multiplying by the nonzero factor $e^{(\phi_\infty+\phi_0)/\theta_\infty}$ and integrating, we obtain the equation for steady states

$$u_\infty = \frac{Me^{-(\phi_\infty+\phi_0)/\theta_\infty}}{\int_\Omega e^{-(\phi_\infty+\phi_0)/\theta_\infty}\,dx}. \tag{7}$$

For simplicity of notation we write $\phi_\infty := \phi$, $\theta_\infty := \theta$, $u_\infty := u$, and we will consider only the case $\phi_0 \equiv 0$.

As shown in [1], if we put $\phi = \theta\psi$ in (7), the energy relation becomes

$$E = \int_\Omega u\left(\theta + \frac{1}{2}\phi\right)dx = \theta\int_\Omega u\,dx + \frac{\theta^2}{2}\int_\Omega u\psi\,dx = \theta M + \frac{\theta^2}{2}\int_\Omega \psi\Delta\psi\,dx.$$

Then we can replace the problem of determination of steady states with prescribed mass $M > 0$ and energy E by the problem

$$\left(\frac{E}{M^2}\right)m^2 = m - \frac{1}{2}\int_\Omega \psi\Delta\psi\,dx = \mathcal{E}(m,\psi),$$

where $m = M/\theta$ and ψ solves the Poisson–Boltzmann–Emden equation

$$\begin{aligned} \Delta\psi &= m\,\frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx} && \text{in } \Omega, \\ \psi &= 0 && \text{on } \partial\Omega \end{aligned} \tag{8}$$

for the Dirichlet problem, and

$$\psi = \frac{m}{\int_\Omega e^{-\psi}\,dx}\,E_d * e^{-\psi} \tag{9}$$

for the "free" problem.

3. k-symmetric solutions for the Dirichlet problem. In this section we will deal with the two-dimensional Dirichlet problem (8) in an annular domain. Recall that the existence of radial solutions for all values of $m > 0$ was proved in [4]. Here we show the existence of nonradial solutions. Moreover, we show nonuniqueness of solutions and we study their exact shape.

To do this we take advantage of a reasoning from [7], concerning the existence, for $\lambda \geq 0$, of solutions of the problem

$$\begin{aligned} \Delta v + \lambda e^v &= 0 && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{10}$$

Without loss of generality, we set $\Omega = \{x \in \mathbb{R}^2 : 0 < a < |x| < 1\}$. Let $T_k x$ be the rotation of point x about the origin through $2\pi/k$, where $k = 1, 2, 3, \ldots$, and let

$$\begin{aligned} V_k &= \{v \in H_0^1(\Omega) : v(T_k x) = v(x) \text{ a.e. in } \Omega\}, \\ V_\infty &= \{v \in H_0^1(\Omega) : v \text{ is radial}\}. \end{aligned}$$

Then V_k and V_∞ are closed subspaces of $H_0^1(\Omega)$.

It is evident that for $v \in V_k$ $(k = \infty, 1, 2, 3, \ldots)$ and $\lambda \in \mathbb{R}$, if w solves

$$\begin{aligned} \Delta w + \lambda e^{v} &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

then w also belongs to V_k.

Next we introduce two functionals

$$\Phi(v) = \int_{\Omega} e^{v(x)}\, dx, \tag{11}$$

$$J(v) = \frac{1}{2} \int_{\Omega} |\nabla v(x)|^2\, dx, \tag{12}$$

and define for every $k \in \mathbb{N}$ a subset of V_k by

$$K_{k,\mu} = \{v \in V_k : \Phi(v) = \mu\} \quad \text{for } \mu \in \mathbb{R}.$$

Let $j_k[\mu] = \inf_{v \in K_{k,\mu}} J(v)$ whenever $K_{k,\mu} \neq \emptyset$. We will say that v is k-*symmetric* when $v \in V_k$ and $v \notin V_l$ for any $l > k$.

Our goal is to prove

THEOREM 3.1. *For every $k \in \mathbb{N}$ there exists a k-symmetric solution of the Dirichlet problem* (8) *for some $m > 0$.*

In [7] the following is proved:

THEOREM 3.2. *For every $k = \infty, 1, 2, 3, \ldots$ such that $K_{k,\mu} \subset V_k$ is not empty there exists $v_\mu \in K_{k,\mu}$ which minimizes* (12) (*i.e.* $j_k[\mu] = J(v_\mu)$). *Moreover*, *the minimizer v_μ solves*

$$\begin{aligned} \Delta v_\mu + \lambda_\mu e^{v_\mu} &= 0 \quad \text{in } \Omega, \\ v_\mu &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

for some parameter $\lambda_\mu \in \mathbb{R}$. ■

We also need two technical lemmas (for details see [7, Lemmas 3.3 and 3.4]).

LEMMA 3.3. *For every $k \in \mathbb{N}$ and sufficiently large μ we have*

$$j_k[\mu] < j_\infty[\mu].$$

LEMMA 3.4. *For every $k \in \mathbb{N}$ we have*

$$j_1[\mu] \le j_2[\mu] \le \cdots \le j_k[\mu] \le \cdots \le j_\infty[\mu].$$

Moreover, $j_k[\mu] < j_\infty[\mu]$ *implies*

$$j_1[\mu] < j_2[\mu] < \cdots < j_k[\mu].$$

Notice that for $\mu > |\Omega|$ we have $\lambda > 0$ and the solution $v \in K_{k,\mu}$ of the minimization problem also solves (10). Indeed, suppose, on the contrary, that $\lambda < 0$. From the maximum principle for elliptic problems we have $v \leq 0$ in Ω. Then

$$\mu = \int_{\Omega} e^{v}\, dx \leq \int_{\Omega} 1\, dx = |\Omega|,$$

which contradicts the assumption $\mu > |\Omega|$.

Now we recall the main theorem from [7, Th. 3.5] with its proof.

THEOREM 3.5. *For all $k \in \mathbb{N}$ there exists μ_k such that for all $\mu > \mu_k$ the problem* (10) *has a k-symmetric solution. This solution satisfies*

$$\int_{\Omega} e^{v}\, dx = \mu.$$

Proof. By Lemma 3.3 and Theorem 3.2 there exists $v = v_{k,\mu} \in V_k$ such that $J(v) = j_k[\mu] < j_\infty[\mu]$ for some μ sufficiently large. It suffices to show that v is k-symmetric.

Suppose, contrary to our claim, that there exists a natural number $l > k$ such that $v \in V_l$. Then

$$j_l[\mu] \leq J(v) = j_k[\mu] < j_\infty[\mu],$$

which contradicts Lemma 3.4. ■

Now let us put $v \mapsto -v$ in (10). Then we obtain

$$\begin{aligned} \Delta v &= \lambda e^{-v} && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$\Phi(-v) = \Phi_1(v) = \int_{\Omega} e^{-v}\, dx$, and (8) takes the form

$$\begin{aligned} \Delta v &= \frac{m}{\Phi_1(v)} e^{-v} && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Proof of Theorem 3.1. From Theorem 3.5 we have a k-symmetric solution to the problem (10) for sufficiently large values of $\mu = \Phi_1(v)$. This solution solves (8) with $m = \lambda\mu$. ■

Since (8) has radial solutions for all $m > 0$ (see [4]), the following corollary is obvious:

COROLLARY 3.6. *There is nonuniqueness of solutions for the problem* (8) *in a two-dimensional annulus for some values of $m > 0$.* ■

4. Radial solutions for the "free" problem. Now we will analyze (9) using dynamical systems methods. An example of use of this method is in [3].

Existence results for solutions to the problem (9) can be found in [3] and the references therein.

Our main goal in this section is to prove

THEOREM 4.1. *For any a and some values of A and m, the "free" problem* (9) *has at least two radial solutions in $\Omega_a^A = \{x \in \mathbb{R}^d : 0 < a < |x| < A < \infty\}$ for $3 \le d \le 9$.*

Assume that ψ is radial. Abusing the notation, we set $\psi(x) = \psi(|x|) = \psi(r)$. Then ψ solves

$$\psi = \frac{m}{\int_\Omega e^{-\psi}\,dx}\, E_d * e^{-\psi} \tag{13}$$

in the annulus Ω_a^A.

It is obvious that ψ also solves

$$\Delta\psi = m\,\frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx} \quad \text{in } \Omega.$$

We set $Q(r) = \int_{\Omega_a^r} \Delta\psi(x)\,dx$. Since ψ is radial,

$$\begin{aligned} Q(r) &= \int_{\Omega_a^r} \Delta\psi(x)\,dx = \int_{\Omega_a^r} \Big(\psi''(|x|) - \frac{d-1}{|x|}\,\psi'(|x|)\Big)\,dx \\ &= \int_a^r \int_{S(s)} \Big(\psi''(s) + \frac{d-1}{s}\psi'(s)\Big)\,dS\,ds \\ &= \sigma_d r^{d-1}\psi'(r) - \sigma_d a^{d-1}\psi'(a). \end{aligned}$$

But $\psi'(a) = 0$, and furthermore, from the right hand side of (13),

$$Q(r) = \frac{m}{\int_{\Omega_a^r} e^{-\psi(x)}\,dx} \int_a^r \int_{S(s)} e^{-\psi(s)}\,dS\,ds = \lambda \int_a^r \sigma_d s^{d-1} e^{-\psi(s)}\,ds,$$

which gives

$$Q'(r) = \frac{d}{dr}\Big(\lambda \int_a^r \sigma_d s^{d-1} e^{-\psi(s)}\,ds\Big) = \lambda\sigma_d r^{d-1} e^{-\psi(r)}. \tag{14}$$

Multiplying both sides of (14) by r^{1-d} and differentiating with respect to r we obtain

$$(1-d)r^{-d}Q'(r) + r^{d-1}Q''(r) = -\lambda\psi'(r)\sigma_d e^{-\psi(r)}. \tag{15}$$

Using $\sigma_d r^{d-1}\psi'(r) = Q(r)$ and multiplying (15) by r^d, we deduce from (14) and (15) the equation

$$Q''(r) + (1-d)\,\frac{1}{r}\,Q'(r) + \frac{1}{\sigma_d}\,r^{1-d}Q'(r)Q(r) = 0. \tag{16}$$

Moreover, this equation is supplemented by the boundary conditions

$$(17)\qquad \begin{aligned} Q(a) &= \int\limits_{\Omega_a^a} \Delta\Psi(x)\,dx = 0,\\ Q(A) &= \int\limits_{\Omega_a^A} \Delta\Psi(x)\,dx = \frac{m}{\int_{\Omega_a^A} e^{-\psi(x)}\,dx} \int\limits_{\Omega_a^A} e^{-\psi(x)}\,dx = m. \end{aligned}$$

Notice that this procedure reduces our nonlocal problem to a local one.

Now, we can set

$$\begin{aligned} s &= \log r,\\ v(s) &= \sigma_d^{-1} r^{3-d} Q'(r),\\ w(s) &= \sigma_d^{-1} r^{2-d} Q(r). \end{aligned}$$

Then (16) will take the form of the dynamical system

$$(18)\qquad \begin{cases} v'(s) = (2-w)v,\\ w'(s) = (2-d)w + v, \end{cases}$$

with the initial conditions

$$(19)\qquad \begin{aligned} w(\log a) &= \sigma_d^{-1} a^{2-d} Q(a) = 0,\\ w(\log A) &= \sigma_d^{-1} A^{2-d} Q(A) = \sigma_d^{-1} A^{2-d} m, \end{aligned}$$

resulting from (17).

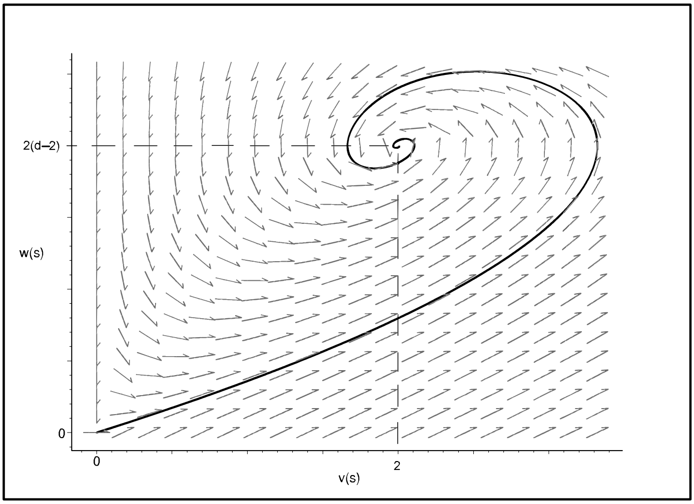

Fig. 1

Figure 1 shows the phase portrait of (18) in the first quadrant for $d = 3$. In the general case ($3 \le d \le 9$) the image is similar. We have two stationary points $P_1 = \binom{0}{0}$ i $P_2 = \binom{2(d-2)}{2}$. The first one is of source type, and the second one is a focus. These and other properties of this system can be found for example in [5].

Let us state two simple lemmas:

LEMMA 4.2. *The vector field generated by the system* (18) *is continuous, and transforms every interval* $\{(x,0) : 0 < \delta < x\}$ *to some curve homeomorphic to this interval.* ∎

LEMMA 4.3. *For every number* $q_0 < l < q_1$ *and every point* $(w_0, 0)$ *there exists* Δt *such that the trajectory of the system* (18), *starting at time* t_0 *from* $(w_0, 0)$, *intersects the line* $w = l$ *at least* k *times for every fixed* $k \in \mathbb{N}$ *in the time interval* $[t_0, t_0 + \Delta t]$. ∎

Lemma 4.3 may be easily deduced from Figure 1 and the Lyapunov function for the system (18),

$$L = \frac{1}{2}(w-2)^2 + (v - 2(d-2)) - 2(d-2)\log\left(\frac{v}{2(d-2)}\right).$$

Proof of Theorem 4.1. Since P_1 is a stationary point and the vector field is continuous, for any t_0, $\Delta t > 0$ and $\varepsilon_1 > 0$ we can find $0 < \varepsilon_2 \ll 1$ such that the trajectory starting from the point $(\varepsilon_2, 0)$ at time t_0 will remain in the ball $B((0,0), \varepsilon_1)$ in the time interval $[t_0, t_0 + \Delta t]$.

For simplicity, let us fix the inner radius a of our annulus to be 1. Then from (19) we have $t_0 = 0$. Now, we choose a point $(x_0, 0)$ on the v-axis and some positive number l_0 such that the line $w = l_0$ crosses the separatrix in at least two points. Next, we set the outer radius A of our annulus such that trajectory starting from $(x_0, 0)$ for $t_0 = 0$ intersects the line $w = l_0$ twice in the time interval $[0, \Delta t]$, where $\Delta t = \log A$. Finally, we determine the parameter m_0 such that $\sigma_d^{-1} A^{2-d} m_0 = l_0$.

As mentioned before we can now choose $(\varepsilon, 0)$ such that the trajectory starting from $(\varepsilon, 0)$ at time t_0 will not cross the line $w = l_0$ before time $t_0 + \Delta t$. Now by Lemma 4.2 the interval $I = \{(x,0) : \varepsilon < x < x_0\}$ is transformed by the dynamical system (18) to a curve Γ, homeomorphic to this interval. This means that there exist at least two points $\mathcal{K}_1 = (x_1^k, l_0)$, $\mathcal{K}_2 = (x_2^k, l_0)$ such that Γ crosses the line $w = l_0$ at $\mathcal{K}_1$, $\mathcal{K}_2$. Hence we can find two points $\mathcal{N}_1 = (x_1^n, 0)$, $\mathcal{N}_2 = (x_2^n, 0)$ in I such that the trajectories starting from $\mathcal{N}_1$, $\mathcal{N}_2$ will, after time $t_0 + \Delta t$, stop at line $w = l_0$. These are the trajectories we are looking for. They represent two different solutions of (9) on the annulus Ω_a^A with parameter $m = m_0$. ∎

COROLLARY 4.4. *Taking* a, A, m *in the proof of Theorem* 4.1 *more carefully, we can construct exactly* k *different radial solutions for the problem* (9) *in* Ω_a^A *and any fixed* $k \in \mathbb{N}$.

REMARK 4.5. By applying this approach, for a fixed m, it is not difficult to show uniqueness of radial solutions when $A/a > 0$ is sufficiently small and nonexistence of such solutions for $A/a \gg 1$.

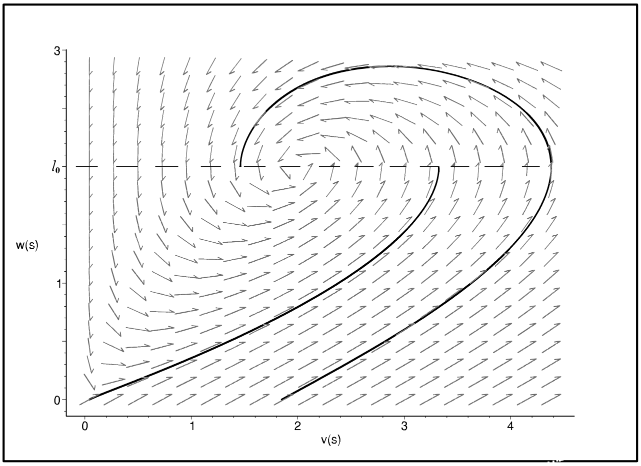

Fig. 2

Examples of such solutions can be found below. We put $r = 1$, $R = e^3$, $m = 504.8424623$. After a suitable change of variables we find two trajectories such that for $t_0 = 0$, $t_1 = 3$ we have $w(t_0) = 0$ and $w(t_1) = 2.00015$ (we used the Fehlberg fourth-fifth order Runge–Kutta method). In Figure 2 we have both trajectories (w, v). The left one starts from $(0.04026, 0)$, and the right one from $(1.77, 0)$. Figure 3 shows the section of the graph of the density function along some ray starting from $(0, 0)$.

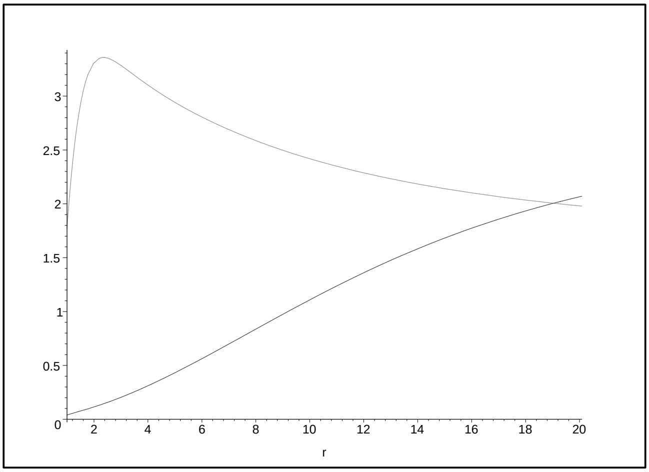

Fig. 3

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Received on 13.4.2004 (1738)