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Włodzimierz Szwarc (Milwaukee, WI)

ON CYCLING IN THE SIMPLEX METHOD OF THE TRANSPORTATION PROBLEM

Abstract. This paper shows that cycling of the simplex method for the $m \times n$ transportation problem where $k-1$ zero basic variables are leaving and reentering the basis does not occur once it does not occur in the $k \times k$ assignment problem. A method to disprove cycling for a particular k is applied for $k = 2, 3, 4, 5$ and 6.

1. Introduction. It is well known ([1], [2], [5], [6]) that cycling does occur in the simplex method of linear programming when this method generates a segment of non-optimal basic solutions $X, X_1, \dots, X_t$ where $X = X_1 = \dots = X_t$ and X and X_t share the same basis. Then the optimal solution can never be reached since this segment is replicated infinitely many times. To prevent cycling a special slightly modified linear programming problem is solved instead.

B. J. Gassner [4] presented two 4×4 and 5×5 assignment examples where cycling does occur. However, she did not apply the classical rule where the cell with the largest negative reduced cost enters the basis at each iteration.

G. B. Dantzig [3] in his 2003 book co-authored with M. N. Thapa posed the following open question while discussing Gassner's [4] result:

Does cycling occur for the simplex method in the transportation problem if the classical rule is applied to the original problem? In other words, does the simplex method of the transportation problem guarantee the optimality of the resulting solution? The paper addresses this particular issue.

We take advantage of the classical rule to establish the following result (Sections 3 and 4): To disprove cycling for the transportation problem, it is sufficient to show that it does not occur for a special $k \times k$ assignment

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problem that deals with basic solutions $X = \{x_{ij}\}$ where

(a) the only positive elements x_{ij} are on the main diagonal,
(b) the reduced cost is non-negative for at least one cell of each off-diagonal pair (i, j), (j, i). The reduced costs of the basic cells are zero.

This result was established by using a surrogate simplex transportation method of [7] and [8] outlined in Section 2.

This paper presents a method of disproving cycling in a $k \times k$ assignment problem for a *particular* k. Consider all pairs (i, j), (j, i) of off-diagonal cells. Mark a cell with a non-negative reduced cost by "+" and the other one by "−". The $k-1$ off-diagonal basic cells are marked by "+". Each basic solution X is then described by a k-node weighted directed tree T. The trees turn out to be a convenient tool to generate all cycles for a given k. Cycling does not occur in the assignment problem if the trees form no cycle $T_0, T_1, \ldots, T_t$, $T_0 = T_t$, or if for each cycle the respective solution sequence $X, X_1, X_2, \ldots$ terminates with an optimal basic solution X_u, $u < t$. The Appendix uses this method to disprove cycling for $k = 2, 3, 4, 5$ and 6.

2. The surrogate simplex method for the transportation problem. The transportation problem deals with minimizing

$$z(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m, \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n, \tag{3}$$

$$x_{ij} \geq 0, \qquad i = 1, \ldots, m,\ j = 1, \ldots, n, \tag{4}$$

where $a_i > 0$, $b_j > 0$, and c_{ij} are given numbers with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

A *loop* L is a set of $2k$ cells (i, j), $2 \leq k \leq \min(m, n)$, arranged in a sequence $(i_1, j_1), (i_1, j_2), (i_2, j_2), \ldots, (i_k, j_{k-1}), (i_k, j_1)$. Let B be a set of $m+n-1$ cells (i, j). B is a *basis* if none of its subsets forms a loop. A matrix $X(B) = \{x_{ij}(B)\}$ is a *basic feasible solution* and B a *feasible basis* if $X(B)$ satisfies (2)–(4) and $x_{ij}(B) = 0$ for each $(i, j) \notin B$. It is known that for each basis B there exists a unique matrix X that satisfies (2) and (3). The simplex method generates basic feasible solutions

$$X(B),\ X(B_1),\ X(B_2),\ \ldots, \tag{5}$$

leading to an optimal solution. Notice that replacing the c_{ij} of (1) with $c_{ij} + u_i + v_j$ where the u_i and v_j are arbitrary numbers results in an equivalent transportation problem since $z(X)$ changes by a constant $\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$ for every X. The *surrogate* simplex method of [7] and [8] generates for each basis of (5) matrices $\{c_{ij} + u_i + v_j\}$ where $c_{ij} + u_i + v_j = 0$ for each $(i,j) \in B$. Such matrices are symbolized by $C(B) = \{c_{ij}(B)\}$. Then the $-u_i$ and $-v_j$ are dual variables associated with B and the original c_{ij}. $X(B)$ is optimal if all $c_{ij}(B) \geq 0$ since $z'[X(B)] = \sum_i \sum_j c_{ij}(B) x_{ij}(B) = 0 \leq z'(X)$ for every X. Once $\min_{i,j \notin B} c_{ij}(B) = -c_1 < 0$ a new feasible solution $X(B_1)$ is generated. A matrix $C(B_1)$ is created by adding c_1 or $-c_1$ to certain rows and columns of $C(B)$. To illustrate this procedure consider Figure 1.

$$C(B) = \begin{array}{|c|c|c|c|} \hline 0 & -2 & 0 & 4 \\ \hline 0 & 0 & -1 & 0 \\ \hline 2 & -3 & 0 & -5 \\ \hline \end{array}$$

Fig. 1

Here $B = \{(1,1),(1,3),(2,1),(2,2),(2,4),(3,3)\}$, $\min c_{ij}(B) = -5$, and $B_1 = B + (3,4) - (1,1)$. To get $C(B_1)$ add 5 to columns 1, 2 and 4, and -5 to row 2 of $C(B)$. This can also be accomplished by adding 5 to rows 1 and 3, and -5 to column 3 of $C(B)$.

3. Problem reduction. Recall that cycling occurs in the transportation problem if the simplex method generates a sequence of feasible bases

$$B,\ B_1,\ B_2,\ \ldots,\ B_t,\ B_{t+1},\ \ldots, \tag{6}$$

where $B = B_t$ and $\min_{(i,j) \notin B_{s-1}} c_{ij}(B_{s-1}) = -c_s < 0$ for all $s \geq 1$ ($B_0 = B$). Notice that c_s is the maximum implicit cost of $C = \{c_{ij}\}$ for B_{s-1} and $z[X(B_r)]$ is constant for each $X(B_r)$ of (6).

A cell $(i,j) \in B$, where $x_{ij}(B) = 0$, is called *white* if it leaves and reenters B, and *black* if it stays in every basis of (6).

THEOREM 1. *Each cycling in an $m \times n$ transportation problem where $k-1$ cells reenter $B = B_t$ of (6) can be replicated in some $k \times k$ transportation problem where all k diagonal cells stay in the initial basis B' while $k-1$ off-diagonal cells leave and reenter B'.*

Proof. Let W and $B - W$ be sets of white and black cells respectively. $B - W$ is composed of k subsets $V_1, \ldots, V_k$ where two black cells are assigned to the same V_s if they share the same row or column. Notice that for each $(i,j) \in W$, $x_{uj}(B) > 0$ and $x_{iv}(B) > 0$ for some u and v. Rearrange for convenience the rows and columns so that the set V_1 is situated in rows $I_1 =$

$\{1,\ldots,m_1\}$ and columns $J_1=\{1,\ldots,n_1\}$, V_2 in rows $I_2=\{m_1+1,\ldots,m_2\}$ and columns $J_2=\{n_1+1,\ldots,n_2\}$, and so on.

Let $A_{11},A_{22},\ldots,A_{kk}$, where $A_{ss}=I_s\times J_s$, be areas occupied by $V_1,\ldots,V_k$ respectively. The white cells lie outside the A_{ss} areas. Figure 2 presents an 8×12 matrix $C(B)$ where $W=\{(2,7),(1,9),(4,11)\}$ and $I_1=\{1,2,3\}$, $J_1=\{1,2,3\}$, $I_2=\{4,5\}$, $J_2=\{4,5,6,7\}$, $I_3=\{6,7\}$, $J_3=\{8,9,10\}$, $I_4=\{8\}$, $J_4=\{11,12\}$.

$$C(B)=\begin{array}{|c|c|c||c|c|c|c||c|c|c||c|c|}
\hline
0 & & & & & & & & 0 & & & \\
\hline
 & 0 & 0 & & & & 0 & & & & & -3 \\
\hline
0 & 0 & & & & & & & & & & \\
\hline\hline
 & & 3 & 0 & 0 & 0 & & & & & 0 & \\
\hline
 & & & & & 0 & 0 & & -2 & & & \\
\hline\hline
 & & & & & 2 & & 0 & & & & -6 \\
\hline
 & -4 & & 2 & & & & 0 & 0 & 0 & & \\
\hline\hline
1 & & & & -5 & & & 7 & & & 0 & 0 \\
\hline
\end{array}$$

Fig. 2

Figure 2 presents only the basic entries $c_{ij}(B)=0$ and $\min c_{ij}(B)$ for each $A_{uv}=I_u\times J_v$ that does not contain the cells of B. A white cell *links* V_r and V_s if it shares a row with a black cell of V_r and a column with a black cell of V_s. Notice that two cells of W cannot link the same pair V_r and V_s since B would contain a loop. Since the black cells stay in every basis of (6), $c_{ij}(B)=c_{ij}(B_s)=0$ for each $(i,j)\in A_{uu}$ and each u and k. This can only happen if $C(B_s)$ is a result of adding to $c_{ij}(B)$: $-d_1$ to each row of I_1 and d_1 to each column of I_1, $-d_2$ to each row of I_2 and d_2 to each column of J_2, and so on. Then for each $(i,j)\in A_{uv}$, $c_{ij}(B_s)=c_{ij}(B)+d_u-d_v$. This implies that only the cells with $\min_{(i,j)\in A_{uv}} c_{ij}(B_s)$ can enter B_{s+1} as white cells. This also includes the white cells which left B and entered some basis of (6). Notice that for each A_{rs} area with $r\neq s$ only cells with the smallest $c_{ij}(B)$ may leave and reenter B when cycling occurs. We squeeze each A_{rs} area into a single cell and define a $k\times k$ matrix $C'(B')$ by placing $\min_{(i,j)\in A_{rs}} c_{ij}(B)$ in cells (r,s) and 0 in cells (r,r). Recall that for a white cell of A_{rs} this minimum is zero. B' consists of k black cells (r,r) and $k-1$ white cells. Figure 3 presents $C'(B')$ derived from the 8×12 matrix $C(B)$ of Figure 2.

Here $B'=\{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(2,4)\}$.

Consider $B_1=B-(i_1,j_1)+(i_2,j_2)$ where (i_1,j_1) links V_r and V_s and (i_2,j_2) links V_p and V_q. Since $-c_1=c_{i_2j_2}(B)=\min c_{ij}(B)=c'_{pq}(B')$ cell (p,q) enters B'_1. Next we examine the loops L and L' contained in $B+(i_2,j_2)$ and $B'+(p,q)$ respectively. Let $(u_1,v_1),S,(u_2,v_2)$ be a segment of L where (u_1,v_1) and (u_2,v_2) are white cells while S is a sequence of black cells of V_h.

$$C'(B') = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & -3 \\ \hline 3 & 0 & -2 & 0 \\ \hline -4 & 2 & 0 & -6 \\ \hline 1 & -5 & 7 & 0 \\ \hline \end{array}$$

Fig. 3

The following four cases are possible: 1) $u_1, u_2 \in I_h$, 2) $v_1, v_2 \in J_h$, 3) $v_1 \in J_h$, $u_2 \in I_h$, 4) $u_1 \in I_h$, $v_2 \in J_h$. It is easy to see that the number of cells in S is even for cases 1 and 2, and odd for cases 3 and 4. Suppose that (u_1, v_1) and (u_2, v_2) link: V_h with V_{k_1} and V_h with V_{k_2} (case 1); V_{k_1} with V_h and V_{k_2} with V_h (case 2); V_{k_1} with V_h and V_h with V_{k_2} (case 3); V_h with V_{k_1} and V_{k_2} with V_h (case 4). Then in L', $(u_1, v_1), S, (u_2, v_2)$ corresponds to a segment: $(h, k_1), (h, k_2)$ (case 1); $(k_1, h), (k_2, h)$ (case 2); $(k_1, h), (h, h), (h, k_2)$ (case 3) and $(h, k_1), (h, h), (k_2, h)$ (case 4). Notice that (h, h) is a black cell. Thus if the number of cells of $(u_1, v_1), S, (u_2, v_2)$ is even (odd) then it remains so in the corresponding segment of L'. Since (i_1, j_1) leaves B, there are an even number of cells in both $(i_2, j_2), \ldots, (i_1, j_1)$ segments of L. This number remains even for both $(p, q), \ldots, (r, s)$ segments of L'. Thus cell (r, s) that corresponds to (i_1, j_1) can also be deleted from B'.

The $c'_{rs}(B'_1)$ coincide with the $\min_{A_{rs}} c_{ij}(B_1)$, $r \neq k$, for a simple reason. If c_1 is added to column s of $C'(B'_1)$ then it is also added to each column of J_s of the matrix $C(B_1)$. Then $-c_1$ is added to each row s of $C'(B'_1)$ and each row of I_s of $C(B_1)$. Thus cycling in sequence (6) can be replicated by the sequence $B', B'_1, \ldots, B'_t, \ldots$, where $B' = B'_1$. ∎

To illustrate the second part of the proof, consider Figure 2, where $B_1 = B-(2,7)+(6,12)$. $B+(6,12)$ contains a 16-cell loop L that passes the A_{rs} areas in order $A_{34}, A_{33}, A_{13}, A_{11}, A_{12}, A_{22}, A_{24}, A_{44}$. In addition to $(6,12)$ it involves three white cells, four cells of V_1, three cells of V_2, three cells of V_3, and two cells of V_4. According to cases 1–4, $L' = (3,4), (3,3), (1,3), (1,2), (2,2), (2,4)$ (see Figure 3). Since $(2,7)$ links V_1 and V_2 and $(6,12)$ links V_2 and V_4, we have $B'_1 = B'_1 - (1,2) + (2,4)$. To get $C'(B'_1)$ we add 6 to columns 2 and 4, and -6 to rows 2 and 4 of $C'(B')$. The same numbers are added to each column of J_2 and J_4 and row I_2 and I_4 of $C(B)$ in order to get $C(B_1)$.

4. Properties of $k \times k$ matrices $C(B_r)$. From now on we assume that cycling occurs in a $k \times k$ transportation problem where each basis of (6) consists of k black cells (i, i), $i = 1, \ldots, k$, and $k - 1$ white cells which leave B and enter some subsequent bases. Consider the formula $c_{ij}(B_r) = c_{ij}(B) + u_i + v_j$. Since all $c_{ii}(B_r) = 0$ we have $u_i = -v_i$. Then

$$c_{ij}(B_r) = c_{ij}(B) + v_j - v_i. \tag{7}$$

Suppose a basic off-diagonal cell (p,q) leaves B_{r-1}. Define a subset A of $B_{r-1}-(p,q)$ as follows: 1) each (i,q) of the set $B_{r-1}-(p,q)$ belongs to A, 2) if (i,j) belongs to A then so does each cell (u,j) and (i,v). Let J_{r-1} be the set of columns of A.

Notice that if (i,j) replaces (p,q) in B_r then both q and j belong to J_{r-1}, while i and p do not. To get $C(B_r)$ we add c_r and $-c_r$ to the columns and rows of J_{r-1} of $C(B_{r-1})$.

Notice that the v_j satisfy the condition

$$v_j=\sum_{s\in\alpha_j}c_s, \tag{8}$$

where α_j is a subset of $\{1,\ldots,r\}$.

Theorem 2. *Suppose that for some r all white cells left B (some may have reentered later). Then for each (p,q) with $p\neq q$ and some $s\geq r$ either $c_{pq}(B_r)=c_{pq}(B_{s-1})+c_s$ or $c_{qp}(B_r)=c_{qp}(B_{s-1})+c_s$.*

Proof. Suppose (p,q), the basic cell of B_r, entered B_s and stayed basic in $B_{s+1},B_{s+2},\ldots,B_r$. Then $c_{pq}(B_{s-1})=-c_s$ and $c_{pq}(B_r)=c_{pq}(B_s)=c_{pq}(B_{s-1})+c_s$. Next assume that $(p,q)\notin B_r$. Then $B_r+(p,q)$ contains a loop $L=(p,q),(p,j_2),(i_2,j_2),\ldots,(i_u,j_{u-1}),(i_u,q)$. Assign all odd cells of L to L_1 and all even cells to L_2. Then for any u_i and v_j,

$$\sum_{L_1}c_{ij}-\sum_{L_2}c_{ij}=\sum_{L_1}(c_{ij}+u_i+v_j)-\sum_{L_2}(c_{ij}+u_i+v_j). \tag{9}$$

Notice that $c_{ij}(B_r)=0$ for each $(i,j)\in L-(p,q)$. Hence $c_{pq}(B_r)=\sum_{L_1}c_{ij}(B_r)-\sum_{L_2}c_{ij}(B_r)$.

Suppose all cells of $L-(p,q)$ belong to $B_s,B_{s+1},\ldots,B_r$ and (k,l) is one of its cells that entered B_s. Then $c_{kl}(B_{s-1})=-c_s<0$ and $c_{ij}(B_{s-1})=0$ for each $(i,j)\in L-(p,q)-(k,l)$. Due to (9),

$$c_{pq}(B_r)=\sum_{L_1}c_{ij}(B_{s-1})-\sum_{L_2}c_{ij}(B_{s-1}).$$

Then

$$c_{pq}(B_r)=\begin{cases}c_{pq}(B_{s-1})+c_s & \text{if } (k,l)\in L_2,\\ c_{pq}(B_{s-1})-c_s & \text{if } (k,l)\in L_1.\end{cases} \tag{10}$$

If $c_{pq}(B_r)=c_{pq}(B_{s-1})-c_r$, then $c_{qp}(B_r)=c_{qp}(B_{s-1})+c_r$. ∎

We call (p,q) a "+" (resp. "−") *cell* if it satisfies the first (resp. second) condition of (10). There are $\frac{1}{2}k(k-1)$ cells in each category. Notice that $c_{pq}(B_r)\geq 0$ for each "+" cell. Only "−" cells can replace the "+" cells in a basis.

Corollary 1. *Under the assumption of Theorem 2, $\alpha_p\neq\alpha_q$ for each $p\neq q$, $p,q=1,\ldots,k$.*

Proof. Suppose $\alpha_p = \alpha_q$. Then $u_p + v_q = -v_q + v_q = 0$ and $c_{pq}(B_r) = c_{pq}(B_u)$ for each $u \le r$, contrary to the fact that (p, q) is either a "+" or "−" cell. ∎

Since cycling occurs we can expand sequence (6) to $B_{-t}, B_{-t+1}, \ldots, B_{-1}, B, B_1, \ldots, B_t, \ldots$, where the initial basis is $B_{-t} = B = B_t$. This is why we can assume that Theorem 2 holds for $C(B_u)$ of any B_u of (6).

We say that column j *dominates* column i in $C(B_u)$ if $v_j - v_i = c_r + L(c_r)$ for some $r \le u$, where $L(c_r) = a_1c_1 + a_2c_2 + \cdots + a_{r-1}c_{r-1}$ and a_i are 0, 1 and -1.

Due to Corollary 1, for each $C(B_u)$ either column j dominates i or i dominates j. Notice that (7) implies that (i, j) of $C(B_u)$ is a "+" cell if column j dominates column i. Otherwise (i, j) is a "−" cell. Then for each $C(B_u)$ one can list the columns such that the number of "+" cells is $0, 1, 2, \ldots, k-1$ respectively. Use the following rule. Assign weight $w_i = r$ to column i if there are $r-1$ "+" cells in this column. Then (i, j) is a "+" cell of $C(B_u)$ if $w_i < w_j$ and a "−" cell if $w_i > w_j$. We can assume that all "+" cells of $C(B) = C(B_t)$ are above the main diagonal since this can be done by rearranging the rows and columns of $C(B)$. Then $w_i = i$ for each column i. Notice that all (i, j) of $C(B)$ below the main diagonal are "−" cells.

Consider a basis B^* where each white basic cell appears in a separate row and column. We prove the following:

PROPERTY 1. *B^* does not belong to sequence* (6).

Proof. Suppose $B^* = B_u$ for some B_u of (6). We can rearrange the rows and columns of $C(B^*)$ so that all cells above (resp. below) the main diagonal are "+" (resp. "−") cells. Let (i, j) satisfy $c_{ij}(B_u) = -c_{u+1}$. Then $B_{u+1} = B_u + (i, j) - (p, q)$ for some cell (p, q). Then (see proof of Theorem 2), $B_u + (i, j)$ contains a loop $L = L_1 + L_2$ where (i, j) belongs to the set L_1 of odd elements of L and (p, q) belongs to L_2, the set of even elements of L. Since all elements of L_2 are black diagonal cells, $z(B_{u+1}) = z(B_u) - c_{u+1}$. Hence B^* cannot appear in (6) where $z(B_u)$ is constant for each $u = 0, 1, \ldots, t$, where $B_0 = B$.

5. Directed weighted trees. Since the columns of the matrices $C(B_u)$ have assigned distinct weights we are in a position to define a corresponding k-node weighted directed tree T_u as follows:

Each column i of $C(B_u)$ is a node of T_u while each white basic cell (i, j) is a directed link $i \to j$ of T_u where $w_i < w_j$ since (i, j) is a "+" cell of $C(B_u)$. Link $i \to j$ will also be described as $[w_i]i \to [w_j]j$. Now, instead of $C(B), C(B_1), \ldots, C(B_t)$ where $C(B) = C(B_t)$ we deal with a sequence, called *cycle*:

(11) $$T_0, T_1, \ldots, T_t, \quad \text{where } T_0 = T_t,$$

where each link of T_0 leaves and reenters T_0.

For convenience assume that $w_i = i$ in T_0. Then all cells (i, j), $i < j$, of $C(B)$ are "+" cells. Let T^* be the tree corresponding to $C(B^*)$ of Property 1. Then $T^* = i_1 \to i_2 \to i_3 \to \cdots \to i_{k-1} \to i_k$ where the weights of nodes $i_1, \ldots, i_k$ are $1, \ldots, k$. Due to Property 1, T^* is of the type $(1,2), (2,3), \ldots, (k-1, k)$ and cannot appear in cycle (11). Next consider any tree T other than T^*. To construct a successor T' of T apply the following procedure:

1. Remove a link $p \to q$ from T thus splitting it into two non-empty subtrees.
2. Let A and $\bar{A}$ be sets of nodes of the two subtrees.
3. Assign p to A and q to $\bar{A}$. Assume $|A| = m$. Obviously $m \geq 1$.
4. Select nodes i of A and j of $\bar{A}$ where $w_i > w_j$.
5. Define T' as follows: Replace $p \to q$ with $i \to j$. Distribute new weights w_i' in T' by assigning weights $1, 2, \ldots, m$ to the nodes of A and $m+1$, $\ldots, k$ to the nodes of $\bar{A}$ so that for each pair (r, s) belonging to the same subset A or $\bar{A}$, $w_r' < w_s'$ if $w_r < w_s$.

Due to 4, T^* has no successors.

Let T_0 correspond to $C(B)$. Then (p, q) is a "+" basic white cell while (i, j) is a "−" cell of $C(B)$. To disprove cycling consider each of the k^{k-2} trees T_0 where $w_i = i$. For each T_0 we construct all possible sequences $T_0, T_1, T_2, \ldots$ where T_{u+1} is a successor of T_u.

Definition. Trees T_s and T_r of (11) are *equivalent* if to every link $[w_i]i \to [w_j]j$ of T_s there corresponds a link $[w_i]f(i) \to [w_j]f(j)$ of T_r where $f(1), \ldots, f(k)$ is a permutation of numbers $1, \ldots, k$. The set of pairs (w_i, w_j) describes their identical type.

Next we focus on those T_0 for which there exists a sequence

(12) $$T_0, T_1, T_2, \ldots, T_r,$$

where T_r is the earliest tree equivalent to T_0.

Notice that T_u and T_{r+u}, $u \geq 1$, are equivalent if for each u the weights of nodes i, j, p, and q in T_u and T_{r+u} are identical. Then (11) can also be written as

$$T_0, \ldots, T_r, \ldots, T_{2r}, \ldots, T_{hr} = T_t = T_0,$$

where $n \geq 1$ and $T_0, T_r, T_{2r}, \ldots,$ are equivalent trees.

Each of the h sequences $T_0, \ldots, T_r$; $T_r, \ldots, T_{2r}$; $\ldots$; $T_{(h-1)r}, \ldots, T_{hr}$ is called a *subcycle*.

To create a subcycle consider the tree

$$T_0 = [1]1 \to [3]3,\ [2]2 \to [3]3,\ [2]2 \to [5]5,\ [4]4 \to [5]5,$$

which in terms of (w_i, w_j) is of the type $(1,3),(2,3),(2,5),(4,5)$. Remove from T_0 the link with weights 2, 3, i.e. link $2 \to 3$. Then $A = \{1,3\}$ and $\bar{A} = \{2,4,5\}$. Select nodes with weights 4 and 3, i.e. nodes 4 and 3, and replace it with link $4 \to 3$ to create a successor tree $T_1 = [1]2 \to [3]5, [2]4 \to [3]5, [2]4 \to [5]3, [4]1 \to [5]3$.

Note that the tree T_0 has seven other successors.

T_1 is equivalent to T_0 since its (w_i, w_j) type is also $(1,3),(2,3),(2,5),(4,5)$ where $f(1), f(2), f(3), f(4), f(5) = 2,4,5,1,3$.

The shape of T_0 and T_1 can be represented as follows:

$$[1] \to [3] \leftarrow [2] \to [5] \leftarrow [4],$$

where the numbers indicate the weights of the nodes. Repeating the same procedure for T_1, where $p,q = 4,5$ and $i,j = 1,5$, $A = \{2,5\}$ and $\bar{A} = \{1,3,4\}$ we get

$$T_2 = [1]4 \to [3]3,\ [2]1 \to [3]3,\ [2]1 \to [5]5,\ [4]2 \to [5]5.$$

Continuing in the same fashion we get a cycle $T_0, T_1, \ldots, T_6$ where $T_0 = T_6$.

Notice that cycling does not occur if no subcycle exists for any T_0. To study the other properties of subcycles it is sufficient to consider subcycle (12). Let $i \to j$ symbolize the link that enters T_{u+1} by replacing some link of T_u. Link $i \to j$ varies with u. Let $w_{r,u-1}$ and w_{ru} be the weights of node r in T_{u-1} and T_u. The following properties eliminate certain T_0 trees.

PROPERTY 2. *Link $1 \to k$ does not appear in $T_0 = T_t$.*

Proof. Suppose $1 \to k$ belongs to T_0. Then $w_{01} = w_{lt} = 1$ and $w_{k1} = w_{kt} = k$ and for some u, link $1 \to k$ reenters T_{u+1} by replacing some link $p \to q$ of T_u and stays in all subsequent trees $T_{u+1}, \ldots, T_t$. Suppose $w_{l,u+1} = 1$ and $w_{k,u+1} = k$. Then $w_{pu} > w_{lu} > w_{ku} > w_{qu}$, which contradicts $w_{pu} < w_{qu}$. This contradiction also holds when $p = 1$ or $q = k$. Hence $w_{k,u+1} - w_{l,u+1} < k - 1$. But $w_{k,u+1} - w_{l,u+1} \geq w_{k,u+2} - w_{l,u+2} \geq \cdots \geq w_{kt} - w_{lt} = k - 1$, which contradicts the last inequality. ∎

PROPERTY 3. *Links $1 \to (k-1)$ and $2 \to k$ cannot simultaneously belong to T_0.*

Proof. It is sufficient to show that each of those links can only reenter $T_t = T_0$.

Notice that during the transition from T_{t-1} to T_t we have $A = \{1, \ldots, m\}$ and $\bar{A} = \{m+1, m+2, \ldots, k\}$ for some $1 \leq m < k$. Suppose $1 \to (k-1)$ is in T_{t-1}. Then $A = \{1, \ldots, k-1\}$, $\bar{A} = \{k\}$ and $1 < w_{k,t-1} < k$. Thus $w_{l,t-1} = 1$ and $w_{k-l,t-1} = k$, contradicting Property 2. If $2 \to k$ is in T_{t-1}

then $A = \{1\}$, $\bar{A} = \{2, \ldots, k\}$ and $1 < w_{l,t-1} < k$. Then $w_{2,t-1} = 1$ and $w_{k,t-1} = k$, again contradicting Property 2. ■

Consider the sequence $C(B), C(B_1), \ldots, C(B_u)$ where $-c_{u+1} = c_{rs}(B_u) = \min c_{ij}(B_u)$. Let A_m and $\bar{A}_m$ be sets used in the transition from the tree T_{m-1} to T_m. Then

$$-c_{u+1} = c_{rs}(B) + a_1 c_1 + a_2 c_2 + \cdots + a_u c_u, \tag{13}$$

where

$$a_m = \begin{cases} 0 & \text{if } r \text{ and } s \text{ both belong to either } A_m \text{ or } \bar{A}_m, \\ -1 & \text{if } r \text{ belongs to } A_m, \\ 1 & \text{if } r \text{ belongs to } \bar{A}_m. \end{cases}$$

Notice that if $-c_{u+1} = c_{rs}(B_{u-1}) < 0$ then $c_u \geq c_{u+1}$.

If $-c_{u+1} = c_{rs}(B_{u-1}) - c_u$, then $c_u \geq c_{u+1}$ if $c_{rs}(B_{u-1}) \geq 0$, and $c_u < c_{u+1}$ if $c_{rs}(B_{u-1}) < 0$.

To disprove cycling for a particular k, consider the set of k^{k-2} trees T_0 where $w_i = i$. From this set eliminate first those T_0 that satisfy Properties 2 and 3. For each remaining tree create a list of possible successors that are still on the list. Next eliminate trees whose only successor is T^* and update the list of successors for the remaining trees.

Repeating this elimination process, we end up with a list of trees T_0 which participate in subcycle (12) which is a segment of cycle (11). If no T_0 remains on the list then each tree sequence $T_0, T_1, \ldots,$ terminates with T^* which corresponds to $X(B^*)$. Then, due to Property 1, cycling does not occur in the assignment problem. Otherwise generate all possible cycles for each T_0.

Consider $-c_{u+1}$ defined by (13). We state the following.

CONJECTURE. Cycling does not occur in the assignment problem if for each tree cycle (11) there exists a $u < t$ such that $-c_{u+1} \geq 0$.

If for each cycle (11), $-c_{u+1} = \min c_{ij}(B_u) \geq 0$ for some $u < t$, then sequence (5) terminates with $X(B_u)$ which is an optimal basic solution. Hence cycling does not occur.

We prove in the Appendix that the Conjecture holds for $k = 2, 3, 4, 5$, and 6.

6. Concluding remarks. This paper demonstrates that to disprove cycling for the transportation problem it is sufficient to disprove it for a very special $k \times k$ assignment problem with an arbitrary k. It also makes a Conjecture (Section 5) which guarantees that cycling does not occur. It presents a method of disproving cycling for a particular k and shows that the Conjecture holds for $k \leq 6$. A challenging question remains how to prove this conjecture for an arbitrary k. So far only a few necessary conditions have been established that reduce the types of trees in a subcycle. The sufficient conditions might help to identify additional criteria when $-c_{u+1} \geq 0$ for $u < t$ in a cycle. According to the Appendix all 21 cycles for $k = 5$ were handled by Criterion I while all but two out of 7083 cycles for $k = 6$ were handled by Criteria I and II.

One should mention another property of trees. Let $\operatorname{sym} x = k+1-x$ where x is an integer $1, \ldots, k$. Define $\operatorname{sym} T$ as a symmetrical tree of T by replacing every link $[w_i]i \to [w_j]j$ of T by the link $[\operatorname{sym} w_i] \operatorname{sym} i \leftarrow [\operatorname{sym} w_j] \operatorname{sym} j$. Notice that $w_i < w_j$ implies that $\operatorname{sym} w_i > \operatorname{sym} w_j$.

The following property holds: $T_0, T_1, \ldots, T_r$ is a subcycle if and only if $\operatorname{sym} T_0, \operatorname{sym} T_1, \ldots, \operatorname{sym} T_r$ is a subcycle. This result reduces considerably the number of T_0 types to identify subcycles.

Appendix: Proof that no cycling occurs for $k = 2, 3, 4, 5$ and 6. To disprove cycling for a particular k, we identify the types of T_0 for which there exists a subcycle (12). Hence type $(1,2), (2,3), \ldots, (k-1,k)$ that has no successors is not considered. We assume that $w_i = i$ in T_0.

Case $k = 2$. The set of types is empty. ∎

Case $k = 3$. Due to Properties 2 and 3, the set is empty. ∎

Case $k = 4$. Properties 2 and 3 reduce the set of 16 types of T_0 to the following four types: $(1,2), (1,3), (3,4)$; $(1,2), (2,3), (2,4)$; $(1,2), (2,4), (3,4)$; and $(1,3), (2,3), (3,4)$. Each of the four types converges to the type $(1,2)$, $(2,3), (3,4)$ after at most two iterations. Thus no subcycle exists for $k = 4$. ∎

Case $k = 5$. We identify 14 types of T_0 for which there exists at least one subcycle (12). Recall that $w_i = i$ in T_0. The types form four groups: 1) 24, 2) 74, 3) 14, 38, 42, 100, and 4) 21, 25, 39, 45, 49, 72, 73, 99. Here is the (w_i, w_j) description of the 14 types.

24 :	$(1,2), (1,4), (3,4), (3,5)$	25 :	$(1,2), (1,4), (3,4), (4,5)$
74 :	$(1,3), (2,3), (2,5), (4,5)$	39 :	$(1,2), (2,3), (2,5), (4,5)$
14 :	$(1,2), (1,3), (3,4), (4,5)$	45 :	$(1,2), (2,4), (3,4), (3,5)$
38 :	$(1,2), (2,3), (3,4), (2,5)$	49 :	$(1,2), (2,5), (3,4), (4,5)$
42 :	$(1,2), (2,3), (3,5), (4,5)$	72 :	$(1,3), (2,3), (2,4), (4,5)$
100 :	$(1,4), (2,3), (3,4), (4,5)$	73 :	$(1,3), (2,3), (3,4), (2,5)$
21 :	$(1,2), (1,4), (2,3), (4,5)$	99 :	$(1,4), (2,3), (3,4), (3,5)$

Tables 1–4 present w_i, w_j, w_p, and w_q during the transition from T_u to T_{u+1} for each of the four groups. The upper entry is w_i-w_j (in type 24 it is 3-2), the lower entry is w_q-w_p (i.e. 3-4 in 24).

Table 1

	24
24	3-2 3-4

Table 2

	74
74	4-3 2-3

Table 1 means that if T_u is of type 24 and link of weights $3 \to 2$ replaces link of weights $3 \to 4$, then T_{u+1} is also of type 24. The same goes for type 74. If T_0 is of type 24 then $T_0 = 2 \leftarrow 1 \to 4 \leftarrow 3 \to 5$ where $w_i = i$. Since $3 \to 2$ replaces $3 \to 4$, we have $A = \{3, 5\}$ and $\bar{A} = \{1, 2, 4\}$. Hence the weights in T_1 are $1, 2$ for nodes $3, 5$ of A and $3, 4, 5$ for nodes $1, 2, 4$ of $\bar{A}$.

Tables 3 and 4 handle the other two types:

Table 3

	14	38	42	100
14		3-2 1-2		
38			5-4 2-3	
42				4-3 4-5
100	2-1 3-4			

Table 4

	21	25	39	45	49	72	73	99
21			4-3 1-2					
25						3-2 3-4	3-2 1-2	
39				4-3 2-3				4-3 4-5
45		3-2 3-4			5-4 3-4			
49		3-2 4-5						
72	2-1 2-3		4-3 2-3					
73					5-4 2-3	5-4 3-4		
99	2-1 3-4			2-1 2-3				

Based on Tables 1, 2, 3 and 4 we prove the following:

PROPERTY 4. *If cycling occurs for $k = 5$ then $c_u \geq c_{u+1}$ for each u.*

Proof. Consider a cycle (11) where $c_u < c_{u+1}$ for some u. Let $w_{i,u-1}$, $w_{j,u-1}$ and w_{iu}, w_{ju} be the weights of nodes i and j in the trees T_{u-1} and T_u of (11). By assumption (i, j) is a "$-$" cell both in $C(B_{u-1})$ and $C(B_u)$.

Hence both differences $w_{i,u-1} - w_{j,u-1}$ and $w_{iu} - w_{ju}$ are positive. According to Tables 1–4, for each s, $w_{is} - w_{js} = 1$ when link $i \to j$ enters T_{s+1}. Since $c_u < c_{u+1}$, i and j belong to different sets A and $\bar{A}$, $0 < w_{i,u-1} - w_{j,u-1} < w_{iu} - w_{ju} = 1$, which is impossible. Hence $c_u \geq c_{u+1}$. ∎

Using Tables 1–4 we generate the following 21 subcycles:

1. $24, 24$
2. $74, 74$
3. $14, 38, 42, 100, 14$
4. $21, 39, 45, 25, 73, 72, 21$
5. $21, 39, 45, 25, 72, 21$
6. $21, 39, 45, 49, 25, 73, 72, 21$
7. $21, 39, 45, 49, 25, 72, 21$
8. $21, 39, 99, 21$
9. $21, 39, 99, 45, 25, 73, 72, 21$
10. $21, 39, 99, 45, 25, 72, 21$
11. $21, 39, 99, 45, 49, 25, 73, 72, 21$
12. $21, 39, 99, 45, 49, 25, 72, 21$
13. $25, 73, 72, 39, 45, 25$
14. $25, 73, 72, 39, 45, 49, 25$
15. $25, 73, 72, 39, 99, 45, 25$
16. $25, 73, 72, 39, 99, 45, 49, 25$
17. $25, 73, 49, 25$
18. $25, 72, 39, 45, 25$
19. $25, 72, 39, 45, 49, 25$
20. $25, 72, 39, 99, 45, 25$
21. $25, 72, 39, 99, 45, 49, 25$

The number of subcycles in a cycle varies from 2 to 4. The largest 33-tree cycle is composed of four subcycles 11 (see the list above). The smallest 7-tree cycle is composed of seven identical type trees (of types 24, 74). To disprove cycling we have to show that for each tree cycle (11) there exists a $u < t$ where $-c_u = \min c_{ij}(B_{u-1}) \geq 0$. This means that $X(B_{u-1})$ is an optimal solution.

Consider the cycle $T_0, T_1, \dots, T_6$ where each tree is of type 74, handled in Section 5. Using formula (13) we get $-c_1 = c_{43}(B) < 0$, $A_1 = \{1, 3\}$ and $-c_2 = c_{15}(B) - c_1$, $A_2 = \{2, 5\}$. Consider T_2. To create T_3 replace link p, q with weights $2, 3$ by a link with weights $4, 3$; i.e. $i, j = 2, 3$, $-c_3 = c_{23}(B) + c_1 - c_2 \geq 0$ since $(2, 3)$ is a "+" cell of $C(B)$ and $c_1 \geq c_2$ (Property 4).

Cycling is disproved for this tree cycle since $X(B_2)$ is an optimal solution.

Next consider the first four trees T_0, T_1, T_2, T_3 of the largest 33-tree cycle composed of subcycles 11. Here $-c_1 = c_{43}(B)$, $A_1 = \{2, 3\}$, $-c_2 = c_{25}(B) - c_1$, $A_2 = \{1, 3, 4, 5\}$, $-c_3 = c_{12}(B) + c_1 - c_2 \geq 0$. Again $X(B_2)$ is an optimal solution for this cycle tree.

For nine subcycles $-c_4 = c_{15}(B) + c_2 - c_3 \geq 0$ while for the remaining eleven subcycles $-c_3 = c_{ij}(B) + c_1 - c_2 \geq 0$ where $(i, j) = (1, 2), (2, 3)$ or $(3, 4)$. ∎

Case $k = 6$. There are 420 types of T_0 participating in at least one of the 7083 subcycles. The computer program used two criteria to identify cycles (composed of subcycles) where $-c_{u+1} \geq 0$ for some $u < t$.

Criterion I: There exist numbers u and r with $r < u-1$ such that

$$-c_{u+1} = c_{ij}(B_r) + c_{u-1} - c_u, \quad c_{ij}(B_r) \geq 0 \quad \text{and} \quad c_{u-1} \geq c_u. \tag{14}$$

Hence $-c_{u+1} \geq 0$.

Criterion II: There exist numbers u, r and s with $s+1 < r < u$ such that

$$-c_{u+1} = c_{ij}(B_r) + c_s + c_{u-1} - c_u, \quad c_{ij}(B_r) \geq 0, \tag{15}$$

where

$$c_s \geq c_{s+1} \geq \cdots \geq c_{u-1}. \tag{16}$$

To prove that $-c_{u+1} \geq 0$ it is sufficient to show that $d = c_s + c_{u-1} \geq 0$. Notice that $d \geq 0$ if $c_{u-1} \geq c_u$.

Consider the case $c_{u-1} < c_u$. Suppose cell (p,q) enters the basis B_u. The assumption $c_{u-1} < c_u$ implies that $-c_u = c_{pq}(B_{u-1}) = c_{pq}(B_{u-2}) - c_{u-1}$ ($c_{pq}(B_{u-1}) = c_{pq}(B_{u-2})$ implies that $c_{u-1} \geq c_u$). Thus

$$c_{u-1} - c_u = c_{pq}(B_{u-2}) \geq \min c_{ij}(B_{u-2}) = -c_{u-1} \geq -c_s.$$

Thus $d \geq 0$. The computer program disproved cycling for 6894 subcycles by Criterion I and for 183 subcycles by Criterion II.

For the remaining two subcycles one or both of the conditions (16) and $c_{u-1} \geq c_u$ may not hold. There $-c_8 = c_{56}(B) + c_2 + c_6 - c_7$ and $-c_8 = c_{12}(B) + c_2 + c_6 - c_7$. Here $c_{56}(B) \geq 0$ and $c_{12}(B) \geq 0$; however, the inequality $c_2 \geq c_3 \geq \cdots \geq c_6 \geq c_7$ does not hold. The respective $-c_7$ for those subcycles happen to be identical: $-c_7 = c_{25}(B) - c_1 - c_6 = c_{25}(B_1) - c_6$. Due to $c_{25}(B_1) \geq -c_2$ we get $-c_7 \geq -c_2 - c_6$. Hence $d \geq 0$ for both subcycles. This concludes the proof of Case $k = 6$. ∎

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School of Business Administration
University of Wisconsin-Milwaukee
Milwaukee, WI 53201, U.S.A.
E-mail: wlodek.1@comcast.net

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