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EXISTENCE OF SOLUTIONS TO THE $(\mathrm{rot}, \mathrm{div})$-SYSTEM IN L_2-WEIGHTED SPACES

Abstract. The existence of solutions to the elliptic problem $\mathrm{rot}\, v = w$, $\mathrm{div}\, v = 0$ in $\Omega \subset \mathbb{R}^3$, $v \cdot \overline{n}|_S = 0$, $S = \partial\Omega$, in weighted Hilbert spaces is proved. It is assumed that Ω contains an axis L and the weight is a negative power of the distance to the axis. The main part of the proof is devoted to examining solutions in a neighbourhood of L. Their existence in Ω follows by regularization.

1. Introduction. We consider the elliptic problem

$$\begin{aligned} &\mathrm{rot}\, v = w && \text{in } \Omega,\\ &\mathrm{div}\, v = 0 && \text{in } \Omega,\\ &v \cdot \overline{n} = b && \text{on } S, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $S = \partial\Omega$, $\overline{n}$ is the unit outward vector normal to S and the dot denotes the scalar product in $\mathbb{R}^3$. Problem (1.1) was considered, e.g., in [6].

For solutions of problem (1.1) to exist, the following compatibility conditions have to be satisfied:

$$\int_S b(s)\, dS = 0, \tag{1.2}$$

$$\mathrm{div}\, w = 0. \tag{1.3}$$

Let an axis L pass through Ω. Then we introduce weighted Sobolev spaces

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$H^k_{-\mu}(\Omega)$ with the norm

$$\|u\|_{H^k_{-\mu}(\Omega)} = \Big(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha_x u(x)|^2 \varrho(x)^{2(-\mu-k+|\alpha|)}\,dx\Big)^{1/2} < \infty,$$

where $\varrho(x) = \operatorname{dist}\{x, L\}$. The aim of this paper is to prove the following result:

THEOREM 1.1. *Assume the compatibility conditions* (1.2), (1.3) *hold. Assume that* $w \in H^k_{-\mu}(\Omega)$, $b \in H^{k+1/2}_{-\mu}(S)$, $\mu \in \mathbb{R}_+$, $\mu \notin \mathbb{Z}$, $k \in \mathbb{N}$. *Then there exists a solution to problem* (1.1) *such that* $v \in H^{k+1}_{-\mu}(\Omega)$ *and*

$$\|v\|_{H^{k+1}_{-\mu}(\Omega)} \le c(\|w\|_{H^k_{-\mu}(\Omega)} + \|b\|_{H^{k+1/2}_{-\mu}(S)}). \tag{1.4}$$

We are looking for solutions to problem (1.1) in the form (see [6])

$$v = \nabla\varphi + u, \tag{1.5}$$

where φ is a solution to the problem

$$\Delta\varphi = 0, \quad \overline{n}\cdot\nabla\varphi|_S = b, \tag{1.6}$$

and u satisfies

$$\begin{aligned} &\operatorname{rot} u = w \quad \text{in } \Omega,\\ &\operatorname{div} u = 0 \quad \text{in } \Omega,\\ &u\cdot\overline{n}|_S = 0. \end{aligned} \tag{1.7}$$

By Lemma 1 in [1], $(1.7)_{2,3}$ imply the existence of a vector e such that

$$u = \operatorname{rot} e, \quad \operatorname{div} e = 0, \quad e\cdot\overline{\tau}|_S = 0, \tag{1.8}$$

where $\overline{\tau}$ is any tangent vector to S.

In view of (1.8), problem (1.7) takes the form

$$-\Delta e = w, \quad e\cdot\overline{\tau}|_S = 0, \quad \operatorname{div} e|_S = 0, \tag{1.9}$$

where we have taken into account that $\Delta \operatorname{div} e = 0$, $\operatorname{div} e|_S = 0$ imply $\operatorname{div} e = 0$.

In a curvilinear orthonormal system of coordinates (τ_1, τ_2, n) in a neighbourhood of S we express the vector e in the form $e = \sum_{\mu=1}^2 e_\mu\overline{\tau}_\mu + e_n\overline{n}$, where $e_\mu = e\cdot\overline{\tau}_\mu$, $e_n = e\cdot\overline{n}$.

Then problem (1.9) can be replaced by

$$-\Delta e = w, \quad e_\tau|_S = 0, \quad (\overline{n}\cdot\nabla e_n + e_n \operatorname{div}\overline{n})|_S = 0. \tag{1.10}$$

To prove Theorem 1.1 we have to show the existence of solutions to problems (1.6) and (1.10) in the same weighted spaces.

To prove the existence of solutions to problem (1.6) in weighted spaces we need

LEMMA 1.1 (see [3, Ch. 4]). *Assume that $b \in H^{1/2}(S)$, $S \in C^2$. Then there exists a solution to problem* (1.6) *such that $\nabla\varphi \in H^1(\Omega)$ and*

$$\|\nabla\varphi\|_{H^1(\Omega)} \le c\|b\|_{H^{1/2}(S)}. \tag{1.11}$$

Similarly, for solutions to (1.10) we have

LEMMA 1.2 (see [3, Ch. 4]). *Assume that $w \in L_2(\Omega)$ and $S \in C^2$. Then there exists a solution to problem* (1.10) *such that $e \in H^2(\Omega)$ and*

$$\|e\|_{H^2(\Omega)} \le c\|w\|_{L_2(\Omega)}. \tag{1.12}$$

Now we formulate the following main results:

THEOREM 1.2. *Let $b \in H_{-\mu}^{l+1/2}(S)$ for some $l \in \mathbb{N}$, $\mu \in \mathbb{R}_+$, $\mu \notin \mathbb{Z}$. Assume the compatibility condition* (1.2) *holds. Then there exists a solution to problem* (1.6) *such that $\varphi \in H_{-\mu}^{l+2}(\Omega)$ and*

$$\|\varphi\|_{H_{-\mu}^{l+2}(\Omega)} \le c\|b\|_{H_{-\mu}^{l+1/2}(S)}. \tag{1.13}$$

Next we have

THEOREM 1.3. *Let $w \in H_{-\mu}^{l}(\Omega)$ for some $l \in \mathbb{N}$ and $\mu \in \mathbb{R}_+ \setminus \mathbb{Z}$. Assume the compatibility condition* $\operatorname{div} w = 0$ *holds. Then there exists a solution to problem* (1.10) *such that $e \in H_{-\mu}^{l+2}(\Omega)$ and*

$$\|e\|_{H_{-\mu}^{l+2}(\Omega)} \le c\|w\|_{H_{-\mu}^{l}(\Omega)}. \tag{1.14}$$

Theorems 1.2 and 1.3 imply Theorem 1.1.

Problem (1.1) is an important step in the proofs of existence of regular solutions to the Navier–Stokes equations. If we look for global regular solutions close to being axially symmetric, we need the existence of solutions to problem (1.1) in weighted Sobolev spaces. Hence, we exactly need Theorem 1.1. Therefore, Theorem 1.1 was used in [7–9, 11, 12].

2. Notation and auxiliary results. First we introduce weighted spaces. Let $\varrho(x) = \operatorname{dist}\{x, L\}$. Then for $\mu \in \mathbb{R}$ and $k \in \mathbb{N}$ we set

$$H_\mu^k(\Omega) = \Big\{u : \|u\|_{H_\mu^k(\Omega)} = \Big(\sum_{|\alpha|\le k} \int_\Omega |D_x^\alpha u|^2 \varrho^{2(\mu-k+|\alpha|)}(x)\,dx\Big)^{1/2} < \infty\Big\}.$$

For $k = 0$ we have

$$L_{2,\mu}(\Omega) = \Big\{u : \|u\|_{L_{2,\mu}(\Omega)} = \Big(\int_\Omega |u(x)|^2 \varrho^{2\mu}(x)\,dx\Big)^{1/2} < \infty\Big\}.$$

To examine regularity of solutions to problem (1.1) in a neighbourhood of L we introduce a local system of coordinates such that L is contained in the

x_3-axis and $0 \in L$. In these coordinates we introduce the Fourier transform

$$(F_1 u)(x', \xi) = \int_{\mathbb{R}} e^{-i\xi x_3} u(x', x_3)\, dx_3, \tag{2.1}$$

where $\xi \in \mathbb{R}$, $x' = (x_1, x_2)$.

Let $u = u(x', x_3)$ be given. Let r, φ be the polar coordinates such that $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$. Let $\tau = -\ln r$. Then for $\lambda \in \mathbb{C}$ we define the transform

$$(F_2 u)(\lambda, \varphi, x_3) = \int_{\mathbb{R}} e^{-i\lambda\tau} u(\tau, \varphi, x_3)\, d\tau. \tag{2.2}$$

Now we introduce a partition of unity. We distinguish four types of subdomains: $\Omega^{(1)}$, near an interior point of L; $\Omega^{(2)}$, near the points where L meets S; $\Omega^{(3)}$, near an interior point of $\Omega \setminus L$; $\Omega^{(4)}$, near a point of $S \setminus L$. To each subdomain $\Omega^{(k)}$ we attach a smooth function $\zeta^{(k)}$ which is equal to 1 in $\overline{w}^{(k)} \subset \Omega^{(k)}$ and vanishes outside $\Omega^{(k)}$, $k = 1, 2, 3, 4$. Let $\xi^{(k)} \in w^{(k)} \subset \Omega^{(k)}$, $k = 1, 3$, be any point of $\Omega^{(k)}$. Next $\xi^{(2)}$ is a point where L meets S, and $\xi^{(4)}$ is a point on S.

We shall examine problems (1.6) and (1.10) in subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ only, because restrictions of solutions to $\Omega^{(3)}$ and $\Omega^{(4)}$ are covered by Lemmas 1.2 and 1.3. We assume that $\Omega^{(1)}$ is a cylinder with axis L.

Let $\varphi^{(1)} = \varphi\zeta^{(1)}$, $e^{(1)} = e\zeta^{(1)}$. Then problems (1.6) and (1.10) take the form

$$\begin{aligned} &\Delta\varphi^{(1)} = 2\nabla\zeta^{(1)}\nabla\varphi + \varphi\Delta\zeta^{(1)} \equiv g^{(1)} && \text{in } \Omega^{(1)}, \\ &\varphi^{(1)} = 0 && \text{on } \partial\Omega^{(1)}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\Delta e^{(1)} = w^{(1)} + 2\nabla\zeta^{(1)}\nabla e + e\Delta\zeta^{(1)} \equiv h^{(1)} && \text{in } \Omega^{(1)}, \\ &e^{(1)} = 0 && \text{on } \partial\Omega^{(1)}. \end{aligned} \tag{2.4}$$

In view of Lemmas 1.2 and 1.3 and the Hardy inequality, we have $g^{(1)}, h^{(1)} \in L_{2,-\mu}(\Omega)$, $\mu \in (0, 1)$.

Problems (2.3) and (2.4) can be replaced by

$$\begin{aligned} &\Delta u^{(1)} = f^{(1)} + 2\nabla\zeta^{(1)}\nabla u + \Delta\zeta^{(1)} u \equiv g^{(1)} && \text{in } \Omega^{(1)}, \\ &u^{(1)} = 0 && \text{on } \partial\Omega^{(1)}, \end{aligned} \tag{2.5}$$

where $u \in H^2(\Omega)$ and replaces φ and e.

Problem (1.10) in a domain of type $\Omega^{(2)}$ implies

$$
(2.6)\qquad
\begin{aligned}
&-\Delta e^{(2)} = w^{(2)} - 2\nabla\zeta^{(2)}\nabla e - \Delta\zeta^{(2)} e \equiv f^{(2)},\\
&e^{(2)}\cdot\overline{\tau}|_{S\cap\partial\overline{\Omega}^{(2)}} = 0,\\
&(\overline{n}\cdot\nabla e_n^{(2)} + e_n^{(2)}\operatorname{div}\overline{n})|_{S\cap\partial\overline{\Omega}^{(2)}} = \overline{n}\cdot\nabla\zeta^{(2)} e_n|_{S\cap\partial\overline{\Omega}^{(2)}},\\
&e^{(2)}|_{\partial\Omega^{(2)}\setminus(S\cap\partial\overline{\Omega}^{(2)})} = 0,
\end{aligned}
$$

where $e^{(2)} = e\zeta^{(2)}$, $w^{(2)} = w\zeta^{(2)}$.

Let us introduce a local coordinate system $y = \{y_1, y_2, y_3\}$ with origin at a point where L meets S. Assume that L is on the y_3-axis and $y_3 > 0$ describes points inside Ω. Let $S^{(2)} = S\cap\overline{\Omega}^{(2)}$ be described by

$$
(2.7)\qquad y_3 = F(y_1, y_2).
$$

Then we introduce new coordinates

$$
(2.8)\qquad
\begin{aligned}
z_i &= y_i, \qquad i = 1, 2,\\
z_3 &= y_3 - F(y_1, y_2).
\end{aligned}
$$

Let us denote the mapping (2.8) by $z = \Phi(y)$.

Problem (1.10) is described in the coordinates $x = \{x_1, x_2, x_3\}$, so passage to the coordinates y can be achieved by a rotation and a translation. Let us denote the change of variables by

$$
y = Y(x).
$$

Hence,

$$
(2.9)\qquad z = (\Phi\circ Y)(x) \equiv \Psi(x), \qquad \widehat{\Omega} = \Psi(\Omega^{(2)}), \qquad \widehat{S} = \Psi(S^{(2)}).
$$

Introduce the notation

$$
\begin{aligned}
&\widetilde{e}^{(2)}(z) = e^{(2)}(\Psi^{-1}(z)), \qquad \widetilde{w}^{(2)}(z) = w^{(2)}(\Psi^{-1}(z)),\\
&\widetilde{e}(z) = e(\Psi^{-1}(z)), \qquad \widetilde{\zeta}(z) = \zeta(\Psi^{-1}(z)), \qquad \nabla_z = \partial_z,\\
&\nabla_\Psi = \frac{\partial z}{\partial x}\bigg|_{x=\Psi^{-1}(z)}\cdot\nabla_z = \Psi_x|_{x=\Psi^{-1}(z)}\cdot\nabla_z, \qquad \overline{n}_z = (0,0,1),\\
&\overline{n}_\Psi = (F_{y_1}, F_{y_2}, -1)|_{y=\Phi^{-1}(z)},\\
&\overline{\tau}_1 = (1,0,0), \qquad \overline{\tau}_2 = (0,1,0), \qquad \overline{\tau}_{1\Psi} = \overline{n}_\Psi\times\overline{\tau}_2, \qquad \overline{\tau}_{2\Psi} = \overline{n}_\Psi\times\overline{\tau}_1.
\end{aligned}
$$

Then problem (2.6) takes the form

$$
\begin{aligned}
&-\nabla_z^2\widetilde{e}^{(2)} = -(\nabla_z^2 - \nabla_\Psi^2)\widetilde{e}^{(2)} + \widetilde{f}^{(2)} \equiv k,\\
&\widetilde{e}^{(2)}|_{|z'|=R, z_3\in(0,a)} = 0,\\
&\widetilde{e}^{(2)}|_{z_3=a} = 0,\\
&\widetilde{e}_i^{(2)}|_{z_3=0} = \widetilde{e}^{(2)}\cdot\overline{\tau}_i|_{z_3=0} = \widetilde{e}^{(2)}\cdot(\overline{\tau}_i - \overline{\tau}_{i\Psi})|_{z_3=0} \equiv h_i, \quad i=1,2,\\
&\widetilde{e}_{3,z_3}^{(2)}|_{z_3=0} \equiv \overline{n}_z\cdot\nabla_z\widetilde{e}_n^{(2)}|_{z_3=0} = (\overline{n}_z\cdot\nabla_z\widetilde{e}_n^{(2)} - \overline{n}_\Psi\cdot\nabla_\Psi(\widetilde{e}^{(2)}\cdot\overline{n}_\Psi)\\
&\qquad - \widetilde{e}^{(2)}\cdot\overline{n}_\Psi\nabla_\Psi\cdot\overline{n}_\Psi + \overline{n}_\Psi\cdot\nabla_\Psi\zeta^{(2)}\widetilde{e}^{(2)}\cdot\overline{n}_\Psi)|_{z_3=0} \equiv h_3,
\end{aligned}
\tag{2.10}
$$

where we assume that $\widehat{\Omega}$ is the cylinder

$$
\begin{aligned}
&\widehat{\Omega} = \{z\in\mathbb{R}^3 : |r_z| < R,\ 0 < z_3 < a,\ \varphi_z\in[0,2\pi]\},\\
&r_z = \sqrt{z_1^2+z_2^2}, \qquad \varphi_z = \arctan\frac{z_2}{z_1}.
\end{aligned}
$$

Let us extend solutions to problem (2.10) to the cylinder

$$
\widehat{\Omega}' = \{z\in\mathbb{R}^3 : |r_z| < R,\ -a < z_3 < a,\ \varphi_z\in[0,2\pi]\}.
$$

For this purpose we construct a function $\eta = (\eta_1,\eta_2,\eta_3)$ such that

$$
\begin{aligned}
&\eta_i|_{z_3=0} = h_i, \quad i=1,2,\\
&\frac{\partial\eta_3}{\partial z_3}\bigg|_{z_3=0} = h_3,\\
&\eta_j|_{|z'|=R,\, z_3\in(0,a)} = 0,\\
&\eta_j|_{z_3=a} = 0, \quad i=1,2,3.
\end{aligned}
\tag{2.11}
$$

Then we introduce the function

$$
w = \widetilde{e}^{(2)} - \eta,
\tag{2.12}
$$

which is a solution to the problem

$$
\begin{aligned}
&-\nabla_z^2 w = k + \nabla_z^2\eta \quad \text{in } \widehat{\Omega},\\
&w_i|_{z_3=0} = 0, \qquad i=1,2,\\
&w_{3,z_3}|_{z_3=0} = 0,\\
&w_j|_{|z'|=R,\, z_3\in(0,a)} = 0,\\
&w_j|_{z_3=a} = 0, \quad j=1,2,3.
\end{aligned}
\tag{2.13}
$$

Now, we construct the following extension:

$$
\begin{aligned}
&w_i'(z_3) = -w_i(-z_3), \qquad i=1,2,\ z_3\in(-a,0),\\
&w_3'(z_3) = w_3(-z_3), \qquad z_3\in(-a,0),\\
&w_j'(z_3) = w_j(z_3), \qquad j=1,2,3,\ z_3\in(0,a).
\end{aligned}
\tag{2.14}
$$

In view of (2.14) problem (2.13) takes the form

$$
-\nabla_z^2 w' = k' + \nabla_z^2 \eta' \quad \text{in } \widehat{\Omega}', \qquad w' = 0 \quad \text{on } \partial\widehat{\Omega}'. \tag{2.15}
$$

Hence, problem (2.15) assumes the form of problem (2.4).

3. Regularity near the axis in the L_2-approach. To show the existence of solutions to problems (1.6) and (1.10) in a neighbourhood of L we introduce the cylinder

$$
C_{R,a} = \{x \in \mathbb{R}^3 : |x'| < R,\ |x_3| < a\}
$$

with the axis of symmetry the x_3-axis and with R, a given positive numbers.

Then we consider problems (2.5) and (2.15) in the form

$$
\begin{aligned}
&-\Delta u = f \quad \text{in } \mathbb{R}^3,\\
&\lim_{|x|\to\infty} u = 0, \qquad \operatorname{supp} f \subset C_{R,a},
\end{aligned} \tag{3.1}
$$

where $C_{R,a}$ replaces $\Omega^{(1)}$, $\widehat{\Omega}'$ and f vanishes outside $C_{R,a}$.

To prove the existence of solutions to problem (3.1) we consider first problem (3.1) in the form

$$
\begin{aligned}
&-\Delta u_\delta = f_\delta \quad \text{in } \mathbb{R}^3 \setminus \overline{C}_\delta,\\
&\lim_{|x|\to\infty} u_\delta = 0, \qquad u_\delta = 0 \quad \text{on } \partial C_\delta, \qquad \operatorname{supp} f_\delta \subset C_{R,a,\delta},
\end{aligned} \tag{3.2}
$$

where $\delta > 0$, $f_\delta = 0$ for $|x'| < \delta$ and

$$
C_{R,a,\delta} = \{x \in C_{R,a} : |x'| > \delta\}, \qquad C_\delta = \{x \in \mathbb{R}^3 : |x'| > \delta\}.
$$

Then we have (see [3, Ch. 4])

LEMMA 3.1. *Assume that $f_\delta \in H^l(C_{R,a,\delta})$. Then there exists a solution to problem* (3.2) *such that $u_\delta \in H^{l+2}(C_\delta)$ and*

$$
\|u_\delta\|_{H^{l+2}(C_\delta)} \le c\|f_\delta\|_{H^l(C_{R,a,\delta})}, \tag{3.3}
$$

where c does not depend on δ.

Let us express (3.2) in variables τ, φ, z, $\tau = -\ln r$:

$$
\begin{aligned}
&-(u_{\delta,\tau\tau} + u_{\delta,\varphi\varphi} + e^{-2\tau} u_{\delta,zz}) = e^{-2\tau} f_\delta,\\
&u_\delta|_{\tau=-\infty} = u_\delta|_{\tau=-\ln\delta} = 0,\\
&u_\delta|_{z=-\infty} = u_\delta|_{z=\infty} = 0.
\end{aligned} \tag{3.4}
$$

Let us introduce the function

$$
\overline{u}_\delta = \begin{cases} u_\delta & \text{for } \tau < -\ln\delta,\\ 0 & \text{for } \tau \ge -\ln\delta. \end{cases} \tag{3.5}
$$

It is shown in [5] that $\overline{u}_\delta$ is a solution to problem (3.1) (see also the proof of Lemma 3.6).

First, we examine the problem

$$
\begin{aligned}
&-\Delta' u = f + \nabla^2_{x_3} u \equiv g,\\
&u|_{|x'|=\infty} = 0.
\end{aligned}
\tag{3.6}
$$

LEMMA 3.2. *Assume that $g \in L_{2,-\mu}(\mathbb{R}^2)$ for some $\mu \in \mathbb{R} \setminus \mathbb{Z}$. Then any solution to problem* (3.6) *such that $u \in H^2_{-\mu}(\mathbb{R}^2)$ satisfies*

$$
\|u\|_{H^2_{-\mu}(\mathbb{R}^2)} \le c\|g\|_{L_{2,-\mu}(\mathbb{R}^2)}.
\tag{3.7}
$$

Proof. In polar coordinates problem (3.6) takes the form

$$
\begin{aligned}
&r\partial_r(r\partial_r u) + u_{,\varphi\varphi} = r^2 g \equiv h.\\
&u|_{r=\infty} = 0.
\end{aligned}
\tag{3.8}
$$

Let us introduce the new variable $\tau = -\ln r$, $r = e^{-\tau}$, $r\partial_r = -\partial_\tau$. Since u vanishes for large r, (3.8) can be considered in the form

$$
\begin{aligned}
&\partial^2_\tau u + \partial^2_\varphi u = h,\\
&u|_{\varphi=0} = u|_{\varphi=2\pi},\\
&\frac{\partial u}{\partial \varphi}\bigg|_{\varphi=0} = \frac{\partial u}{\partial \varphi}\bigg|_{\varphi=2\pi}.
\end{aligned}
\tag{3.9}
$$

Applying the Fourier transform F_2 denoted by $\widetilde{u} = F_2 u$, we obtain from (3.9) the problem

$$
\begin{aligned}
&-\lambda^2\widetilde{u} + \partial^2_\varphi \widetilde{u} = \widetilde{h},\\
&\widetilde{u}|_{\varphi=0} = \widetilde{u}|_{\varphi=2\pi},\\
&\frac{\partial \widetilde{u}}{\partial \varphi}\bigg|_{\varphi=0} = \frac{\partial \widetilde{u}}{\partial \varphi}\bigg|_{\varphi=2\pi}.
\end{aligned}
\tag{3.10}
$$

We have to underline that the eigenvalues of problem (3.10) are such that $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda \in \mathbb{Z}$ (see [10]).

The existence of solutions to problem (3.10) follows from the following construction. Let $\lambda = i\sigma$. Then we are looking for solutions to problem (3.10) in the form

$$
\widetilde{u} = \alpha \sin(\sigma\varphi) + \beta\cos(\sigma\varphi).
$$

By variation of constants we calculate α, β from the equations

$$
\begin{aligned}
&\frac{d\alpha}{d\varphi}\sin(\sigma\varphi) + \frac{d\beta}{d\varphi}\cos(\sigma\varphi) = 0,\\
&\frac{d\alpha}{d\varphi}\cos(\sigma\varphi) - \frac{d\beta}{d\varphi}\sin(\sigma\varphi) = \frac{1}{\sigma}\widetilde{h}.
\end{aligned}
$$

Solving the equations we obtain

$$
\frac{d\alpha}{d\varphi} = \frac{1}{\sigma}\cos(\sigma\varphi)\widetilde{h}, \qquad \frac{d\beta}{d\varphi} = -\frac{1}{\sigma}\sin(\sigma\varphi)\widetilde{h}.
$$

Hence

$$\alpha = \frac{1}{\sigma}\int_0^{\varphi} \cos(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi', \qquad \beta = -\frac{1}{\sigma}\int_0^{\varphi} \sin(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi'.$$

Then a general solution to (3.10) has the form

$$\widetilde{u} = \alpha\sin(\sigma\varphi) + \beta\cos(\sigma\varphi) + \frac{\sin(\sigma\varphi)}{\sigma}\int_0^{\varphi} \cos(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi' - \frac{\cos(\sigma\varphi)}{\sigma}\int_0^{\varphi} \sin(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi'. \tag{3.11}$$

The boundary conditions $(3.10)_{2,3}$ imply

$$\begin{aligned} -\sin(2\pi\sigma)\alpha + (1-\cos(2\pi\sigma))\beta &= \frac{\sin(2\pi\sigma)}{\sigma}\int_0^{2\pi} \cos(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi' \\ &\quad - \frac{\cos(2\pi\sigma)}{\sigma}\int_0^{2\pi} \sin(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi' \equiv A_1, \\ (1-\cos(2\pi\sigma))\alpha + \sin(2\pi\sigma)\beta &= \frac{\cos(2\pi\sigma)}{\sigma}\int_0^{2\pi} \cos(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi' \\ &\quad + \frac{\sin(2\pi\sigma)}{\sigma}\int_0^{2\pi} \sin(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi' \equiv A_2. \end{aligned} \tag{3.12}$$

Solving (3.12) yields

$$\begin{aligned} \alpha &= \frac{-A_1\sin(2\pi\sigma) + A_2(1-\cos(2\pi\sigma))}{2(1-\cos(2\pi\sigma))}, \\ \beta &= \frac{A_1(1-\cos(2\pi\sigma)) + A_2\sin(2\pi\sigma)}{2(1-\cos(2\pi\sigma))}, \end{aligned} \tag{3.13}$$

so α, β are defined for

$$1 \neq \cos(2\pi\sigma) \quad \text{so} \quad \sigma \notin \mathbb{Z}. \tag{3.14}$$

Let

$$B_1 = \frac{1}{\sigma}\int_0^{2\pi} \sin(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi', \qquad B_2 = \frac{1}{\sigma}\int_0^{2\pi} \cos(\sigma\varphi')\widetilde{h}(\varphi')\,d\varphi'.$$

Then

$$\begin{aligned} A_1 &= -\cos(2\pi\sigma)B_1 + \sin(2\pi\sigma)B_2, \\ A_2 &= \sin(2\pi\sigma)B_1 + \cos(2\pi\sigma)B_2 \end{aligned}$$

and

$$\alpha = [B_1 \sin(2\pi\sigma) - B_2(1 - \cos(2\pi\sigma))]/[2(1 - \cos(2\pi\sigma))],$$
$$\beta = [B_1(1 - \cos(2\pi\sigma)) + B_2 \sin(2\pi\sigma)]/[2(1 - \cos(2\pi\sigma))].$$

Using α and β in (3.11) yields

$$\begin{aligned} \widetilde{u} &= [B_1 \sin(2\pi\sigma) - B_2(1 - \cos(2\pi\sigma))] \frac{\sin(\sigma\varphi)}{2(1 - \cos(2\pi\sigma))} \\ &\quad + [B_1(1 - \cos(2\pi\sigma)) + B_2 \sin(2\pi\sigma)] \frac{\cos(\sigma\varphi)}{2(1 - \cos(2\pi\sigma))} \\ &\quad + \sin(\sigma\varphi) B_2 - \cos(\sigma\varphi) B_1 \\ &= \frac{B_1}{2(1 - \cos(2\pi\sigma))} [\cos((2\pi - \varphi)\sigma) - \cos(\sigma\varphi)] \\ &\quad + \frac{B_2}{2(1 - \cos(2\pi\sigma))} [\sin(\sigma\varphi) + \sin((2\pi - \varphi)\sigma)] \\ &= \frac{\sin(\pi\sigma)}{1 - \cos(2\pi\sigma)} [-B_1 \sin((\pi - \varphi)\sigma) + B_2 \cos((\pi - \varphi)\sigma)]. \end{aligned} \tag{3.15}$$

Finally, the solution to problem (3.10) has the form

$$\begin{aligned} \widetilde{u} = \frac{\sin(\pi\sigma)}{\sigma(1 - \cos(2\pi\sigma))} \Big[&- \sin((\pi - \varphi)\sigma) \int_0^{2\pi} \sin(\sigma\varphi') \widetilde{h}(\varphi')\, d\varphi' \\ &+ \cos((\pi - \varphi)\sigma) \int_0^{2\pi} \cos(\sigma\varphi') \widetilde{h}(\varphi')\, d\varphi' \Big]. \end{aligned} \tag{3.16}$$

Now we obtain the estimate (3.7). Multiplying $(3.10)_1$ by $\overline{\widetilde{u}}$, where $\overline{v}$ is the complex conjugate to v, integrating with respect to φ and by parts yields

$$\int_0^{2\pi} (\lambda^2 |\widetilde{u}|^2 + |\partial_\varphi \widetilde{u}|^2)\, d\varphi = \int_0^{2\pi} \widetilde{h} \overline{\widetilde{u}}\, d\varphi. \tag{3.17}$$

Let $\lambda = \lambda_r + i\lambda_i$, $\lambda_r, \lambda_i \in \mathbb{R}$. Then

$$|\lambda|^2 = \lambda_r^2 + \lambda_i^2, \qquad \lambda^2 = \lambda_r^2 - \lambda_i^2 + 2i\lambda_r\lambda_i.$$

Hence (3.17) takes the form

$$\int_0^{2\pi} (|\lambda|^2 |\widetilde{u}|^2 + |\partial_\varphi \widetilde{u}|^2)\, d\varphi = 2(\lambda_i^2 - i\lambda_r\lambda_i) \int_0^{2\pi} |\widetilde{u}|^2\, d\varphi + \int_0^{2\pi} \widetilde{h} \overline{\widetilde{u}}\, d\varphi. \tag{3.18}$$

Integrating (3.18) with respect to λ from $-\infty+ih_0$ to $+\infty+ih_0$ and applying the Hölder and Young inequalities yields

$$\begin{aligned}(3.19)\qquad &\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \int\limits_0^{2\pi} d\varphi\,(|\lambda|^2|\widetilde{u}|^2 + |\partial_\varphi \widetilde{u}|^2) \\ &\le \frac{\varepsilon_1}{2}\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\,\lambda_r^2 \int\limits_0^{2\pi} d\varphi\,|\widetilde{u}|^2 + \int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \int\limits_0^{2\pi} d\varphi\, 2h_0^2\Big(1+\frac{1}{\varepsilon_1}\Big)|\widetilde{u}|^2 \\ &\quad + \int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \int\limits_0^{2\pi} d\varphi \left(\frac{\varepsilon_2}{2}\, h_0^2|\widetilde{u}|^2 + \frac{1}{2\varepsilon_2 h_0^2}\,|\widetilde{h}|^2\right).\end{aligned}$$

Assuming $\varepsilon_1 = \varepsilon_2 = 1$ and multiplying the result by 2 gives

$$\begin{aligned}(3.20)\qquad &\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \int\limits_0^{2\pi} d\varphi\,(|\lambda|^2|\widetilde{u}|^2 + |\partial_\varphi \widetilde{u}|^2) \\ &\le 8h_0^2 \int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \int\limits_0^{2\pi} d\varphi\,|\widetilde{u}|^2 + \frac{1}{h_0^2}\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \int\limits_0^{2\pi} d\varphi\,|\widetilde{h}|^2.\end{aligned}$$

From (3.18) we also have the inequality

$$\begin{aligned}(3.21)\qquad &\int\limits_0^{2\pi} (|\lambda|^2|\widetilde{u}|^2 + |\partial_\varphi \widetilde{u}|^2)\,d\varphi \le 2(|\lambda_i|^2 + |\lambda_r|\,|\lambda_i|)\int\limits_0^{2\pi} |\widetilde{u}|^2\,d\varphi \\ &\quad + \frac{\varepsilon_1}{2}|\lambda|^2 \int\limits_0^{2\pi} |\widetilde{u}|^2\,d\varphi + \frac{1}{2\varepsilon_1|\lambda|^2}\int\limits_0^{2\pi} |\widetilde{h}|^2\,d\varphi.\end{aligned}$$

Multiplying (3.21) by $|\lambda|^2$, integrating the result with respect to λ from $-\infty + ih_0$ to $+\infty + ih_0$ and applying the Hölder and Young inequalities yields

$$\begin{aligned}(3.22)\qquad &\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \int\limits_0^{2\pi} d\varphi\,|\lambda|^2(|\lambda|^2|\widetilde{u}|^2 + |\partial_\varphi \widetilde{u}|^2) \\ &\le 2\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \left(\frac{\varepsilon_2}{2}\,|\lambda|^4 + \frac{1}{2\varepsilon_2}\,|h_0|^4 + \frac{\varepsilon_3^{4/3}}{4/3}\,|\lambda|^4 + \frac{1}{4\varepsilon_3^4}\,|h_0|^4\right)\int\limits_0^{2\pi} |\widetilde{u}|^2\,d\varphi \\ &\quad + \frac{\varepsilon_1}{2}\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\,|\lambda|^4 \int\limits_0^{2\pi} |\widetilde{u}|^2\,d\varphi + \frac{1}{2\varepsilon_1}\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda \int\limits_0^{2\pi} |\widetilde{h}|^2\,d\varphi.\end{aligned}$$

Assuming $\frac{\varepsilon_1}{2}+\varepsilon_2+\frac{3}{2}\varepsilon_3^{4/3}\le\frac{1}{2}$ we obtain from (3.22) the inequality

$$\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi}|\lambda|^2(|\lambda|^2|\widetilde{u}|^2+|\partial_\varphi\widetilde{u}|^2)\,d\varphi \le c_1\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi}|\widetilde{u}|^2d\varphi+c_2\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi}|\widetilde{h}|^2d\varphi. \tag{3.23}$$

From (3.20) and (3.23) we have

$$\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi}(|\lambda|^4|\widetilde{u}|^2+|\lambda|^2|\widetilde{u}|^2+|\lambda|^2|\partial_\varphi\widetilde{u}|^2+|\partial_\varphi\widetilde{u}|^2)\,d\varphi \le c_3\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi}|\widetilde{u}|^2d\varphi+c_4\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi}|\widetilde{h}|^2d\varphi. \tag{3.24}$$

In virtue of (3.24) and $(3.10)_1$ we obtain

$$\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi(|\lambda|^4|\widetilde{u}|^2+|\lambda|^2|\widetilde{u}|^2+|\lambda|^2|\partial_\varphi\widetilde{u}|^2+|\partial_\varphi\widetilde{u}|^2+|\partial_\varphi^2\widetilde{u}|^2) \le c_5\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,|\widetilde{u}|^2+c_6\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,|\widetilde{h}|^2. \tag{3.25}$$

Let $a>0$ be such that $a^4=2c_5$. Then the part of the first integral on the r.h.s. of (3.25) for $|\lambda_r|\ge a$ is absorbed by 1/2 of the l.h.s. integral. Hence

$$\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,(|\lambda|^4|\widetilde{u}|^2+|\lambda|^2|\widetilde{u}|^2+|\lambda|^2|\partial_\varphi\widetilde{u}|^2+|\partial_\varphi\widetilde{u}|^2+|\partial_\varphi^2\widetilde{u}|^2) \le 2c_5\int\limits_{-a+ih_0}^{a+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,|\widetilde{u}|^2+2c_6\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,|\widetilde{h}|^2. \tag{3.26}$$

To estimate the first integral on the r.h.s. of (3.26) we use the explicit formula (3.16) for solutions of (3.10). Since $\sigma=-i\lambda=-i(\lambda_r+ih_0)=-i\lambda_r+h_0$ and since $|\lambda_r|\le a$ and h_0 is a fixed number we obtain from (3.16) the inequality

$$\int\limits_{-a+ih_0}^{a+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,|\widetilde{u}|^2\le c_7\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,|\widetilde{h}|^2. \tag{3.27}$$

Estimates (3.26) and (3.27) imply

$$\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,(|\lambda|^4|\widetilde{u}|^2+|\lambda|^2|\widetilde{u}|^2+|\lambda|^2|\partial_\varphi\widetilde{u}|^2+|\partial_\varphi\widetilde{u}|^2+|\partial_\varphi^2\widetilde{u}|^2) \le c_8\int\limits_{-\infty+ih_0}^{+\infty+ih_0} d\lambda\int\limits_0^{2\pi} d\varphi\,|\widetilde{h}|^2. \tag{3.28}$$

Setting $h_0 = 1 + \mu$ for some $\mu \notin \mathbb{Z}$ we infer that the constructed solution (3.16) belongs to $H^2_{-\mu}(\mathbb{R}^2)$ and estimate (3.7) holds. This concludes the proof.

Now we consider problem (3.1).

LEMMA 3.3. *Assume that $f \in L_{2,-\mu}(\mathbb{R}^3)$ for some $\mu \in (\mathbb{R}_+ \cup (0,1)) \setminus \mathbb{Z}_+$. Then any solution to problem* (3.1) *such that $u \in H^2_{-\mu}(\mathbb{R}^3)$ satisfies*

$$\|u\|_{H^2_{-\mu}(\mathbb{R}^3)} \le c\|f\|_{L_{2,-\mu}(\mathbb{R}^3)}. \tag{3.29}$$

Proof. Let $\widetilde{u} = F_1 u$ (see (2.1)). Then problem (3.1) takes the form

$$-\Delta'\widetilde{u} + \xi^2\widetilde{u} = \widetilde{f} \quad \text{in } \mathbb{R}^2, \tag{3.30}$$

where $\Delta' = \partial^2_{x_1} + \partial^2_{x_2}$.

If $\mu \in (0,1)$, then multiplying (3.30) by $\overline{\widetilde{u}}|x'|^{-2\mu}$ and integrating over $\mathbb{R}^2$ yields

$$\begin{aligned}\int_{\mathbb{R}^2} (|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)|x'|^{-2\mu}\,dx' \\ = 2\mu\int_{\mathbb{R}^2} \nabla'\widetilde{u}\overline{\widetilde{u}}|x'|^{-2\mu-1}\nabla|x'|\,dx' + \int_{\mathbb{R}^2} \widetilde{f}\overline{\widetilde{u}}|x'|^{-2\mu}\,dx',\end{aligned} \tag{3.31}$$

where the first integral on the r.h.s. is estimated by

$$\frac{\varepsilon_1}{2}\int_{\mathbb{R}^2} |\nabla'\widetilde{u}|^2|x'|^{-2\mu}\,dx' + \frac{2\mu^2}{\varepsilon_1}\int_{\mathbb{R}^2} |\widetilde{u}|^2|x'|^{-2\mu-2}\,dx'$$

and the second by

$$\frac{\varepsilon_2}{2}\int_{\mathbb{R}^2} \xi^2|\widetilde{u}|^2|x'|^{-2\mu}\,dx' + \frac{1}{2\varepsilon_2}\frac{1}{\xi^2}\int_{\mathbb{R}^2} |\widetilde{f}|^2|x'|^{-2\mu}\,dx'.$$

Multiplying (3.31) by ξ^2 and assuming $\varepsilon_1 = \varepsilon_2 = 1$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} (\xi^2|\nabla'\widetilde{u}|^2 + \xi^4|\widetilde{u}|^2)|x'|^{-2\mu}\,dx' \le 4\mu^2\int_{\mathbb{R}^2} \xi^2|\widetilde{u}|^2|x'|^{-2\mu-2}\,dx' \\ + \int_{\mathbb{R}^2} |\widetilde{f}|^2|x'|^{-2\mu}\,dx'.\end{aligned} \tag{3.32}$$

From Lemma 3.2 we have

$$\|\widetilde{u}\|^2_{H^2_{-\mu}(\mathbb{R}^2)} \le c\xi^4\|\widetilde{u}\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + c\|\widetilde{f}\|^2_{L_{2,-\mu}(\mathbb{R}^2)}. \tag{3.33}$$

From (3.32) and (3.33) we get

$$\begin{aligned}\|\widetilde{u}\|^2_{H^2_{-\mu}(\mathbb{R}^2)} + \int_{\mathbb{R}^2} (\xi^2|\nabla'\widetilde{u}|^2 + \xi^4|\widetilde{u}|^2)|x'|^{-2\mu}\,dx' \\ \le c\int_{\mathbb{R}^2} \xi^2|\widetilde{u}|^2|x'|^{-2\mu-2}\,dx' + c\|\widetilde{f}\|^2_{L_{2,-\mu}(\mathbb{R}^2)}.\end{aligned} \tag{3.34}$$

Now we have to examine the first integral on the r.h.s. of (3.34). For this purpose we introduce

$$
\begin{aligned}
&Q_1 = \{(\xi, x') : |\xi|\,|x'| \le a_1\},\\
&Q_2 = \{(\xi, x') : |\xi|\,|x'| \ge a_2\},\\
&Q_3 = \{(\xi, x') : a_1 \le |\xi|\,|x'| \le a_2\}.
\end{aligned}
\tag{3.35}
$$

The numbers a_i, $i = 1, 2$, will be determined later.

Let us examine the first integral on the r.h.s. of (3.34). In view of (3.35), we express it in the form

$$
\int d\xi \int_{\mathbb{R}^2} \xi^2|\widetilde{u}|^2|x'|^{-2\mu-2}\,dx' = \sum_{i=1}^{3}\int_{Q_i} d\xi\,dx'\,|\xi|^2|\widetilde{u}|^2|x'|^{-2\mu-2} \equiv \sum_{i=1}^{3} I_i.
\tag{3.36}
$$

From the properties of the sets Q_i, $i = 1, 2, 3$, we have

$$
\begin{aligned}
&I_1 \le a_1^2 \int d\xi \int_{\mathbb{R}^2} |\widetilde{u}|^2|x'|^{-2\mu-4}\,dx',\\
&I_2 \le \frac{1}{a_2^2}\int d\xi \int_{\mathbb{R}^2} dx'\,|\xi|^4|\widetilde{u}|^2|x'|^{-2\mu},\\
&I_3 \le \frac{1}{a_1^{2(1+\mu)}}\int d\xi \int_{Q_3} dx'\,|\xi|^{4+2\mu}|\widetilde{u}|^2 \equiv I.
\end{aligned}
\tag{3.37}
$$

To estimate I we introduce the sets

$$
\begin{aligned}
&d_1(\xi) = \{x' \in \mathbb{R}^2 : |\xi|\,|x'| \le a_1\},\\
&d_2(\xi) = \{x' \in \mathbb{R}^2 : |\xi|\,|x'| \ge a_2\},\\
&d_3(\xi) = \{x' \in \mathbb{R}^2 : a_1 \le |\xi|\,|x'| \le a_2\}.
\end{aligned}
$$

Moreover, for $\lambda > 0$ we have

$$
\begin{aligned}
&\Omega^\lambda = \{(x', \xi) : \lambda|\xi|\,|x'| \le 1\},\\
&w^\lambda(\xi) = \{x' \in \mathbb{R}^2 : \lambda|\xi|\,|x'| \le 1\}.
\end{aligned}
$$

We see that $Q_3 \subset \Omega^\lambda$ for $\lambda \in (0, a_2^{-1})$. Let us introduce a smooth function $\chi = \chi(t)$ such that $\chi(t) = 1$ for $t \le 1$ and $\chi(t) = 0$ for $t \ge 2$, $0 \le \chi(t) \le 1$, $|\chi'(t)| \le 2$.

Set $\chi_\lambda(x', \xi) = \chi(\lambda|\xi|\,|x'|)$. Then $\chi_\lambda(x', \xi) \ne 0$ for $\lambda^{-1} \le |\xi|\,|x'| \le 2\lambda^{-1}$.

Multiplying (3.30) by $\overline{\widetilde{u}}\chi_\lambda^2$ and integrating over $\mathbb{R}^2$ we obtain

$$
\int_{\mathbb{R}^2} (|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)\chi_\lambda^2\,dx' = -2\int_{\mathbb{R}^2} \nabla'\widetilde{u}\overline{\widetilde{u}}\nabla\chi_\lambda\chi_\lambda\,dx' + \int_{\mathbb{R}^2} \widetilde{f}\,\overline{\widetilde{u}}\chi_\lambda^2\,dx'.
\tag{3.38}
$$

The first term on the r.h.s. of (3.38) is estimated by

$$
\frac{\varepsilon_1}{2}\int_{\mathbb{R}^2} |\nabla'\widetilde{u}|^2\chi_\lambda^2\,dx' + \frac{2}{\varepsilon_1}\int_{\mathbb{R}^2} |\widetilde{u}|^2|\nabla'\chi_\lambda|^2\,dx'
$$

and the second by

$$\frac{\varepsilon_2}{2}\int\limits_{\mathbb{R}^2}|\widetilde{u}|^2|\xi|^{2+2\mu}|x'|^{2\mu}\chi_\lambda^2\,dx' + \frac{1}{2\varepsilon_2}\frac{1}{|\xi|^{2+2\mu}}\int\limits_{\mathbb{R}^2}|\widetilde{f}|^2|x'|^{-2\mu}\chi_\lambda^2\,dx' \equiv J_1.$$

On supp χ_λ we have $|x'|^{2\mu}|\xi|^{2\mu} \le (2/\lambda)^{2\mu}$, so the first term in J_1 is bounded by

$$\frac{\varepsilon_2}{2}\left(\frac{2}{\lambda}\right)^{2\mu}\int\limits_{\mathbb{R}^2}|\widetilde{u}|^2|\xi|^2\chi_\lambda^2\,dx'.$$

Assuming $\varepsilon_1 = 1$ and $\varepsilon_2(2/\lambda)^{2\mu} = 1$ we obtain from (3.38) the inequality

$$\frac{1}{2}\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)\chi_\lambda^2\,dx' \le 2\int\limits_{\mathbb{R}^2}|\widetilde{u}|^2|\nabla'\chi_\lambda|^2\,dx' + \frac{1}{2}\left(\frac{2}{\lambda}\right)^{2\mu}\frac{1}{|\xi|^{2+2\mu}}\int\limits_{\mathbb{R}^2}|\widetilde{f}|^2|x'|^{-2\mu}\chi_\lambda^2\,dx'.$$

Multiplying the above inequality by $2|\xi|^{2+2\mu}$ and integrating with respect to ξ yields

$$\int d\xi\,|\xi|^{2+2\mu}\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde{u}|^2+\xi^2|\widetilde{u}|^2)\chi_\lambda^2\,dx' \le 4\int d\xi\,|\xi|^{2+2\mu}\int\limits_{\mathbb{R}^2}|\widetilde{u}|^2|\nabla'\chi_\lambda|^2\,dx' + \left(\frac{2}{\lambda}\right)^{2\mu}\int d\xi\int\limits_{\mathbb{R}^2}|\widetilde{f}|^2|x'|^{-2\mu}\,dx'. \tag{3.39}$$

Using the estimate $|\nabla'\chi_\lambda| \le 2\lambda|\xi|$ we obtain from (3.39) the inequality

$$\int d\xi\,|\xi|^{2+2\mu}\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)\chi_\lambda^2\,dx' \le 16\lambda^2\int\limits_{\mathbb{R}^3\cap\operatorname{supp}\nabla'\chi_\lambda}d\xi\,dx'\,|\xi|^{4+2\mu}|\widetilde{u}|^2 + \left(\frac{2}{\lambda}\right)^{2\mu+2}\int d\xi\int\limits_{\mathbb{R}^2}|\widetilde{f}|^2|x'|^{-2\mu}\chi_\lambda^2\,dx', \tag{3.40}$$

where $\lambda \le 2$ is utilized.

Now, $\nabla'\chi_\lambda \ne 0$ for $\lambda^{-1} \le |\xi|\,|x'| \le 2\lambda^{-1}$ implies that $\operatorname{supp}\nabla'\chi_\lambda \subset w^{\lambda/2}(\xi)\setminus w^\lambda(\xi)$ for any $\xi \in \mathbb{R}$. Multiplying (3.40) by $(\lambda/2)^{2\mu+2}$ yields

$$\left(\frac{\lambda}{2}\right)^{2\mu+2}\int d\xi\,|\xi|^{2+2\mu}\int\limits_{w^\lambda(\xi)}(|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)\,dx' \le 64\cdot 4^\mu\lambda^2\left(\frac{\lambda/2}{2}\right)^{2\mu+2}\int d\xi\,|\xi|^{4+2\mu}\int\limits_{w^{\lambda/2}(\xi)\setminus w^\lambda(\xi)}|\widetilde{u}|^2\,dx' + \int d\xi\int\limits_{\mathbb{R}^2}|\widetilde{f}|^2|x'|^{-2\mu}\,dx'. \tag{3.41}$$

Let $4^{3+\mu}\lambda^2 \le 1/2$. Then iterating (3.41) up to order k we obtain

$$\begin{aligned}(3.42)\qquad \Big(\frac{\lambda}{2}\Big)^{2\mu+2}\int d\xi\,|\xi|^{4+2\mu}\int\limits_{w^\lambda(\xi)}|\widetilde u|^2\,dx' &\le \frac{1}{2^k}\Big(\frac{\lambda/2^k}{2}\Big)^{2\mu+2}\int d\xi\,|\xi|^{4+2\mu}\int\limits_{w^{\lambda/2^{k+1}}(\xi)\setminus w^{\lambda/2^k}(\xi)}|\widetilde u|^2\,dx'\\ &\quad+2\int d\xi\int\limits_{\mathbb{R}^2}|\widetilde f|^2|x'|^{-2\mu}\,dx',\end{aligned}$$

where

$$\begin{aligned}(3.43)\qquad w^{\lambda/2^{k+1}}(\xi)\setminus w^{\lambda/2^k}(\xi) &= \Big\{x'\in\mathbb{R}^2 : \frac{2}{\lambda/2^k}\le|\xi|\,|x'|\le\frac{2}{\lambda/2^{k+1}}\Big\}\\ &= \Big\{x'\in\mathbb{R}^2 : \frac{2^{k+1}}{\lambda}\le|\xi|\,|x'|\le\frac{2^{k+2}}{\lambda}\Big\}.\end{aligned}$$

On the set (3.43) we have

$$|\xi|\le 2\,\frac{2^{k+1}}{\lambda}\,|x'|^{-1},$$

so the first term on the r.h.s. of (3.42) is estimated by

$$\begin{aligned}(3.44)\qquad \frac{1}{2^k}\Big(\frac{\lambda}{2^{k+1}}\Big)^{2\mu+2}\int d\xi\,\xi^2\int\limits_{w^{\lambda/2^{k+1}}(\xi)\setminus w^{\lambda/2^k}(\xi)}|\widetilde u|^2 2^{2\mu+2}\Big(\frac{2^{k+1}}{\lambda}\Big)^{2\mu+2}|x'|^{-2\mu-2}\\ = \frac{2^{2\mu+2}}{2^k}\int d\xi\,\xi^2\int\limits_{w^{\lambda/2^{k+1}}(\xi)\setminus w^{\lambda/2^k}(\xi)}|\widetilde u|^2|x'|^{-2\mu-2}\,dx'.\end{aligned}$$

In view of (3.36), (3.37), (3.42) and (3.44) we have

$$\begin{aligned}(3.45)\qquad \int d\xi\int\limits_{\mathbb{R}^2}\xi^2|\widetilde u|^2|x'|^{-2\mu-2}\,dx' &\le a_1^2\int d\xi\int\limits_{\mathbb{R}^2}|\widetilde u|^2|x'|^{-2\mu-4}\,dx'\\ &\quad+\frac{1}{a_2^2}\int d\xi\int\limits_{\mathbb{R}^2}\xi^4|\widetilde u|^2|x'|^{-2\mu}\,dx'+\frac{2^{2\mu+2}}{2^k a_1^{2\mu+2}}\int d\xi\int\limits_{\mathbb{R}^2}\xi^2|\widetilde u|^2|x'|^{-2\mu-2}\,dx'.\end{aligned}$$

Using (3.45) in (3.34) integrated with respect to ξ and assuming that a_1 is sufficiently small and a_2, k are sufficiently large and applying the Hardy inequality

$$\int\limits_{\mathbb{R}^2}|\widetilde u|^2|x'|^{-2\mu-2}\,dx'\le c\int\limits_{\mathbb{R}^2}|\nabla'\widetilde u|^2|x'|^{-2\mu}\,dx'$$

we obtain (3.29). This concludes the proof.

Now, we shall improve the regularity of solutions to problem (3.1).

LEMMA 3.4. *Assume that $g \in H^l_{-\mu}(\mathbb{R}^2)$ for some $\mu \in \mathbb{R}\setminus\mathbb{Z}$ and $l \in \mathbb{Z}_+$. Then any solution to problem* (3.6) *such that $u \in H^{l+2}_{-\mu}(\mathbb{R}^2)$ satisfies*

$$\|u\|_{H^{l+2}_{-\mu}(\mathbb{R}^2)} \le c\|g\|_{H^l_{-\mu}(\mathbb{R}^2)}. \tag{3.46}$$

Proof. To show higher regularity of solutions to problem (3.9) we consider it in the following form:

$$\begin{aligned} &\partial_\tau^2 u + \partial_\varphi^2 u = h,\\ &\partial_\varphi^\sigma u|_{\varphi=0} = \partial_\varphi^\sigma u|_{\varphi=2\pi}, \qquad \sigma \le l+1, \end{aligned} \tag{3.47}$$

where u vanishes sufficiently fast as τ converges to $-\infty$. Applying the Fourier transform F_2 to (3.47) yields

$$\begin{aligned} &-\lambda^2\widetilde{u} + \partial_\varphi^2\widetilde{u} = \widetilde{h},\\ &\partial_\varphi^\sigma \widetilde{u}|_{\varphi=0} = \partial_\varphi^\sigma u|_{\varphi=2\pi}, \qquad \sigma \le l+1. \end{aligned} \tag{3.48}$$

Differentiating $(3.48)_1$ twice with respect to φ, multiplying by $\overline{\partial_\varphi^2\widetilde{u}}$ and integrating with respect to φ we get

$$\int_0^{2\pi} (\lambda^2|\widetilde{u}_{\varphi\varphi}|^2 + |\partial_\varphi^3\widetilde{u}|^2)\,d\varphi = \int_0^{2\pi} \partial_\varphi^2\widetilde{h}\partial_\varphi^2\widetilde{u}\,d\varphi = -\int_0^{2\pi} \partial_\varphi\widetilde{h}\partial_\varphi^3\widetilde{u}\,d\varphi. \tag{3.49}$$

Since $\lambda^2 = |\lambda|^2 - 2\lambda_i^2 + 2i\lambda_i\lambda_r$ we obtain from (3.49) the inequality

$$\int_0^{2\pi} (|\lambda|^2|\widetilde{u}_{\varphi\varphi}|^2 + |\partial_\varphi^3\widetilde{u}|^2)\,d\varphi \le 2(|\lambda_i|^2 + |\lambda_i|\,|\lambda_r|)\int_0^{2\pi}|\widetilde{u}_{\varphi\varphi}|^2\,d\varphi + c\int_0^{2\pi}|\partial_\varphi\widetilde{h}|^2\,d\varphi. \tag{3.50}$$

Continuing, we obtain

$$\int_0^{2\pi} (|\lambda|^2|\widetilde{u}_{\varphi\varphi}|^2 + |\partial_\varphi^3\widetilde{u}|^2)\,d\varphi \le c|\lambda_i|^2\int_0^{2\pi} |\widetilde{u}_{,\varphi\varphi}|^2\,d\varphi + c\int_0^{2\pi} |\partial_\varphi\widetilde{h}|^2\,d\varphi. \tag{3.51}$$

Differentiating $(3.48)_2$ with respect to φ, multiplying by $\overline{\lambda}^2\overline{\widetilde{u}}_{\varphi\varphi}$ and integrating with respect to φ we have

$$\int_0^{2\pi} (|\lambda|^4|\widetilde{u}_\varphi|^2 + \overline{\lambda}^2\widetilde{u}_{\varphi\varphi}\overline{\widetilde{u}}_\varphi)\,d\varphi = \int_0^{2\pi} \overline{\lambda}^2\widetilde{h}_\varphi\overline{\widetilde{u}}_\varphi\,d\varphi = -\int_0^{2\pi} \overline{\lambda}^2\widetilde{h}\overline{\widetilde{u}}_{\varphi\varphi}\,d\varphi.$$

Continuing, we get

$$\begin{aligned} \int_0^{2\pi} |\lambda|^4|\widetilde{u}_\varphi|^2\,d\varphi \le \varepsilon\int_0^{2\pi} |\lambda|^2|\widetilde{u}_{\varphi\varphi}|^2\,d\varphi + c(1/\varepsilon)\int_0^{2\pi} |\lambda|^2|\widetilde{u}_\varphi|^2\,d\varphi&\\ + c\int_0^{2\pi} |\lambda|^2|\widetilde{h}|^2\,d\varphi.& \end{aligned} \tag{3.52}$$

In view of (3.51) and sufficiently small ε inequalities (3.51), (3.52) imply

$$\begin{aligned}(3.53)\qquad &\int_0^{2\pi}(|\lambda|^4|\widetilde{u}_\varphi|^2+|\lambda|^2|\widetilde{u}_{\varphi\varphi}|^2+|\partial_\varphi^3\widetilde{u}|^2)\,d\varphi\\ &\le c|\lambda_i|^2\int_0^{2\pi}|\widetilde{u}_{\varphi\varphi}|^2\,d\varphi+c\int_0^{2\pi}|\lambda|^2|\widetilde{u}_\varphi|^2\,d\varphi+c\int_0^{2\pi}(|\widetilde{h}_{,\varphi}|^2+|\lambda|^2|\widetilde{h}|^2)\,d\varphi.\end{aligned}$$

Multiplying $(3.48)_1$ by $\overline{\lambda}^2|\lambda|^2\overline{\widetilde{u}}$ and integrating with respect to φ yields

$$\begin{aligned}\int_0^{2\pi}|\lambda|^6|\widetilde{u}|^2\,d\varphi&\le\int_0^{2\pi}|\lambda|^4|\widetilde{u}_\varphi|^2\,d\varphi+\int_0^{2\pi}|\lambda|^4|\widetilde{h}|\,|\widetilde{u}|\,d\varphi\\ &\le\varepsilon_1\int_0^{2\pi}|\lambda|^6|\widetilde{u}|^2\,d\varphi+c\int_0^{2\pi}|\lambda|^4|\widetilde{u}_\varphi|^2\,d\varphi+c(1/\varepsilon_1)\int_0^{2\pi}|\lambda|^2|\widetilde{h}|^2\,d\varphi.\end{aligned}$$

Hence, in view of (3.53) and for sufficiently small ε_1, we have

$$\begin{aligned}(3.54)\qquad &\int_0^{2\pi}(|\lambda|^6|\widetilde{u}|^2+|\lambda|^4|\widetilde{u}_\varphi|^2+|\lambda|^2|\widetilde{u}_{\varphi\varphi}|^2+|\partial_\varphi^3\widetilde{u}|^2)\,d\varphi\\ &\le c|\lambda_i|^2\int_0^{2\pi}|\widetilde{u}_{\varphi\varphi}|^2\,d\varphi+c\int_0^{2\pi}|\lambda|^2|\widetilde{u}_\varphi|^2\,d\varphi+c\int_0^{2\pi}(|\widetilde{h}_{,\varphi}|^2+|\lambda|^2|\widetilde{h}|^2)\,d\varphi.\end{aligned}$$

Integrating (3.54) with respect to λ from $-\infty+ih_0$ to $+\infty+ih_0$ and using (3.28) we obtain

$$\begin{aligned}(3.55)\qquad &\int_{-\infty+ih_0}^{+\infty+ih_0}d\lambda\int_0^{2\pi}d\varphi\sum_{i=0}^{3}|\lambda|^{2(3-i)}|\partial_\varphi^i\widetilde{u}|^2\\ &\le c\int_{-\infty+ih_0}^{+\infty+ih_0}d\lambda\int_0^{2\pi}d\varphi\sum_{i=0}^{1}|\lambda|^{2(1-i)}|\partial_\varphi^i\widetilde{h}|^2.\end{aligned}$$

Continuing, we get

$$\begin{aligned}(3.56)\qquad &\int_{-\infty+ih_0}^{+\infty+ih_0}d\lambda\int_0^{2\pi}\sum_{i=0}^{l+2}|\lambda|^{2(l+2-i)}|\partial_\varphi^i\widetilde{u}|^2\,d\varphi\\ &\le c\int_{-\infty+ih_0}^{+\infty+ih_0}d\lambda\int_0^{2\pi}\sum_{i=0}^{l}|\lambda|^{2(l-i)}|\partial_\varphi^i\widetilde{h}|^2\,d\varphi.\end{aligned}$$

Choosing $h_0=1+l+\mu$, we find that $u\in H^{l+2}_{-\mu}(\mathbb{R}^2)$ and (3.46) holds. This concludes the proof.

Now we consider problem (3.1).

LEMMA 3.5. *Assume that $f \in H^l_{-\mu}(\mathbb{R}^3)$ for some $\mu \in (\mathbb{R}_+ \cup (0,1)) \setminus \mathbb{Z}_+$ and $l \in \mathbb{Z}_+$. Then any solution to problem (3.1) such that $u \in H^{l+2}_{-\mu}(\mathbb{R}^3)$ satisfies*

$$\|u\|_{H^{l+2}_{-\mu}(\mathbb{R}^3)} \le c\|f\|_{H^l_{-\mu}(\mathbb{R}^3)}. \tag{3.57}$$

Proof. Multiplying $(3.30)_1$ by $\xi^2\overline{\widetilde{u}}|x'|^{-2\mu}$ and integrating over $\mathbb{R}^2$ yields

$$\begin{aligned}\int_{\mathbb{R}^2} \xi^2(|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)|x'|^{-2\mu}\,dx' \\ = 2\mu \int_{\mathbb{R}^2} \xi^2\nabla'\widetilde{u}\overline{\widetilde{u}}|x'|^{-2\mu-1}\nabla|x'|dx' + \int_{\mathbb{R}^2} \xi^2\widetilde{f}\,\overline{\widetilde{u}}|x'|^{-2\mu}\,dx',\end{aligned} \tag{3.58}$$

where the first integral on the r.h.s. of (3.58) is estimated by

$$\frac{\varepsilon_1}{2}\int_{\mathbb{R}^2} \xi^2|\nabla'\widetilde{u}|^2|x'|^{-2\mu}\,dx' + \frac{2\mu^2}{\varepsilon_1}\int_{\mathbb{R}^2} \xi^2|\widetilde{u}|^2|x'|^{-2\mu-2}\,dx',$$

and the second by

$$\frac{\varepsilon_2}{2}\int_{\mathbb{R}^2} \xi^4|\widetilde{u}|^2|x'|^{-2\mu}\,dx' + \frac{1}{2\varepsilon_2}\int_{\mathbb{R}^2} |\widetilde{f}|^2|x'|^{-2\mu}\,dx'.$$

Hence for $\varepsilon_1 = \varepsilon_2 = 1$ inequality (3.58) yields

$$\begin{aligned}\int_{\mathbb{R}^2} \xi^2(|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)|x'|^{-2\mu}\,dx' \le 4\mu^2 \int_{\mathbb{R}^2} \xi^2|\widetilde{u}|^2|x'|^{-2\mu-2}\,dx' \\ + \int_{\mathbb{R}^2} |\widetilde{f}|^2|x'|^{-2\mu}\,dx'.\end{aligned} \tag{3.59}$$

Next, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \xi^4(|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)|x'|^{-2\mu}\,dx' \le 4\mu^2 \int_{\mathbb{R}^2} \xi^4|\widetilde{u}|^2|x'|^{-2\mu-2}\,dx' \\ + \int_{\mathbb{R}^2} \xi^2|\widetilde{f}|^2|x'|^{-2\mu}\,dx'.\end{aligned} \tag{3.60}$$

Applying Lemma 3.4 to (3.30) gives

$$\|\widetilde{u}\|^2_{H^3_{-\mu}(\mathbb{R}^2)} \le c\xi^4\|\widetilde{u}\|^2_{H^1_{-\mu}(\mathbb{R}^2)} + c\|\widetilde{f}\|^2_{H^1_{-\mu}(\mathbb{R}^2)} \tag{3.61}$$

and

$$\xi^2\|\widetilde{u}\|^2_{H^2_{-\mu}(\mathbb{R}^2)} \le c\xi^6\|\widetilde{u}\|^2_{L_{2,-\mu}(\mathbb{R}^2)} + c\xi^2\|\widetilde{f}\|^2_{L_{2,-\mu}(\mathbb{R}^2)}. \tag{3.62}$$

From (3.60)–(3.61) we have, after integration with respect to ξ,

$$\begin{aligned}&\int_{\mathbb{R}} d\xi\,(\|\widetilde{u}\|^2_{H^3_{-\mu}(\mathbb{R}^2)} + \xi^2\|\widetilde{u}\|^2_{H^2_{-\mu}(\mathbb{R}^2)} + \xi^4\|\widetilde{u}\|^2_{H^1_{-\mu}(\mathbb{R}^2)} + \xi^6\|\widetilde{u}\|^2_{L_{2,-\mu}(\mathbb{R}^2)}) \\ &\le c\int_{\mathbb{R}}\int_{\mathbb{R}^2} \xi^4|\widetilde{u}|^2|x'|^{-2\mu-2}\,dx'd\xi + c\int_{\mathbb{R}}(\|\widetilde{f}\|^2_{H^1_{-\mu}(\mathbb{R}^2)} + \xi^2\|\widetilde{f}\|^2_{L_{2,-\mu}(\mathbb{R}^2)})d\xi.\end{aligned} \tag{3.63}$$

Repeating the considerations leading to (3.45) with $\widetilde{u}$ replaced by $\xi\widetilde{u}$, we obtain, for k sufficiently large,

$$\int_{\mathbb{R}} d\xi \int_{\mathbb{R}^2} \xi^4|\widetilde{u}|^2|x'|^{-2\mu-2}\,dx' \le ca_1^2\int_{\mathbb{R}} d\xi \int_{\mathbb{R}^2} |\widetilde{u}|^2|x'|^{-2\mu-4}\,dx' + \frac{c}{a_2^2}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}^2} \xi^4|\widetilde{u}|^2|x'|^{-2\mu}\,dx'. \tag{3.64}$$

Using (3.64) in (3.63) and assuming that a_1 is sufficiently small and a_2 sufficiently large we obtain

$$\sum_{i=0}^{3}\int_{\mathbb{R}} d\xi\,|\xi|^{2i}\|\widetilde{u}\|^2_{H^{3-i}_{-\mu}(\mathbb{R}^2)} \le c\sum_{i=0}^{1}\int_{\mathbb{R}} d\xi\,|\xi|^{2i}\|\widetilde{f}\|^2_{H^{1-i}_{-\mu}(\mathbb{R}^2)}. \tag{3.65}$$

Continuing, we get

$$\sum_{i=0}^{l+2}\int_{\mathbb{R}} d\xi\,|\xi|^{2i}\|\widetilde{u}\|^2_{H^{l+2-i}_{-\mu}(\mathbb{R}^2)} \le c\sum_{i=0}^{l}\int_{\mathbb{R}} d\xi\,|\xi|^{2i}\|\widetilde{f}\|^2_{H^{l-i}_{-\mu}(\mathbb{R}^2)}. \tag{3.66}$$

From (3.66) we obtain (3.57). This concludes the proof.

In view of Lemma 3.5 the solution $\overline{u}_\delta$ of problem (3.1) satisfies the estimate

$$\|\overline{u}_\delta\|_{H^{l+2}_{-\mu}(\mathbb{R}^3)} \le c\|f_\delta\|_{H^l_{-\mu}(\mathbb{R}^3)}. \tag{3.67}$$

We have to underline that (3.67) holds if $\overline{u}_\delta$ and f_δ belong to the corresponding spaces. Letting $\delta\to 0$ and using Lemma 3.1 yields

Lemma 3.6. *Assume that $f\in H^l_{-\mu}(\mathbb{R}^3)$ for some $l\in\mathbb{N}$ and $\mu\in\mathbb{R}_+\setminus\mathbb{Z}$. Then there exists a solution to problem* (3.1) *such that $u\in H^{l+2}_{-\mu}(\mathbb{R}^3)$ and estimate* (3.57) *holds.*

Proof. Since we have estimates for solutions to problem (3.1) (see Lemmas 3.3, 3.5) our aim now is to prove their existence. But we have the existence of solutions to problem (3.2) (see Lemma 3.1). Hence, we have to show that $\overline{u}_\delta$, defined by (3.5), is a solution to (3.1). For this purpose we examine problem (3.4) which is exactly problem (3.2) in coordinates τ,φ,z. Now we have to examine the behaviour of the derivatives of $\overline{u}_\delta$ with respect to τ in a neighbourhood of $\tau=-\ln\delta$. For $\tau=-\ln\delta$ we have $\overline{u}_\delta=0$, so $(3.4)_1$ takes the form

$$-\overline{u}_{\delta,\tau\tau}=e^{-2\tau}f_\delta \quad \text{for } \tau=-\ln\delta. \tag{3.68}$$

Assuming that

$$f_\delta|_{\tau=-\ln\delta}=0 \tag{3.69}$$

we obtain

$$\lim_{\tau\to-\ln\delta^+}\overline{u}_{\delta,\tau\tau}=0 \tag{3.70}$$

For $\tau>-\ln\delta$ we have $\overline{u}_\delta=0$, so

$$\lim_{\tau\to-\ln\delta^-}\overline{u}_{\delta,\tau\tau}=0, \tag{3.71}$$

because

$$\overline{u}_\delta|_{\tau=-\ln\delta}=0.$$

Assuming that f_δ is a smooth function with respect to φ and z we deduce from Lemma 3.1 that $\overline{u}_{\delta,\tau}$, $\overline{u}_\delta$ are also smooth with respect to φ and z. We show that $\overline{u}_{\delta,\tau}$ is continuous at $\tau=-\ln\delta$ in [5].

In view of the above considerations we see that $\overline{u}_\delta$ is a solution to problem (3.1) under the assumption that f_δ satisfies (3.69).

Since $\overline{u}_\delta$ is a solution to problem (3.1) and belongs to $H^2_{-\mu}(\mathbb{R}^3)$ we obtain from Lemmas 3.1 and 3.3 the existence in $H^2_{-\mu}(\mathbb{R}^3)$ and the estimate

$$\|\overline{u}_\delta\|_{H^2_{-\mu}(\mathbb{R}^3)}\le c\|f_\delta\|_{L_{2,-\mu}(\mathbb{R}^3)}, \tag{3.72}$$

with the constant c independent of δ.

Letting $\delta\to0$ and using the density of smooth $\overline{u}_\delta$ and f_δ in the corresponding spaces we obtain the existence of solutions to problem (3.1) in $H^2_{-\mu}(\mathbb{R}^3)$ and estimate (3.29).

To show the existence of solutions to problem (3.1) in $H^{l+2}_{-\mu}(\mathbb{R}^3)$ we have to examine the behaviour of $\partial_\tau^{l+2}\overline{u}_\delta$ in a neighbourhood of $\tau=-\ln\delta$. Repeating the above considerations we can see that $\partial_\tau^{j+2}\overline{u}_\delta|_{\tau=-\ln\delta}=0$ for all j up to $j=l-1$ under the assumption that $\partial_\tau^j f_\delta|_{\tau=-\ln\delta}=0$.

Thus, as $\overline{u}_\delta$ is a solution to problem (3.1) and belongs to $H^{l+2}_{-\mu}(\mathbb{R}^3)$, estimate (3.57) holds in the form

$$\|\overline{u}_\delta\|_{H^{l+2}_{-\mu}(\mathbb{R}^3)}\le c\|f_\delta\|_{H^l_{-\mu}(\mathbb{R}^3)}.$$

Applying a density argument and letting $\delta\to0$ we obtain the existence of solutions to (3.1) in $H^{l+2}_{-\mu}(\mathbb{R}^3)$ and estimate (3.57). This concludes the proof.

4. Existence in a bounded domain. In this section we prove Theorems 1.2. The proof of Theorem 1.3 is similar. Finally, Theorem 1.1 follows from Theorems 1.2 and 1.3. Consider a cylinder C_δ of radius δ such that near the points where L meets S, the cylinder is orthogonal to S. We assume the boundary condition $\varphi=0$ on the boundary of the cylinder. Let us denote $\Omega_\delta=\Omega\setminus\overline{C}_\delta$. In the domain Ω_δ we have the existence of solutions to problem (1.6) (by using a partition of unity and [2, Ch. 4; 3, Ch. 4]) such that $\varphi_\delta\in H^{l+2}(\Omega_\delta)$ and

$$\|\varphi_\delta\|_{H^{l+2}(\Omega_\delta)}\le c\|b_\delta\|_{H^{l+1/2}(S_\delta)}, \tag{4.1}$$

where $S_\delta=S\setminus\overline{C}_\delta$ and b_δ is the restriction $b_\delta=b|_{S_\delta}$ and $b_\delta=0$ for $r\le\delta$.

Repeating the proof of Lemma 3.6 and assuming that

$$\partial_\tau^j b_\delta|_{\tau=-\ln\delta} = 0, \quad j \le l-1,$$

we will show that solutions to problem (1.6) such that $\varphi_\delta = \varphi|_{\Omega_\delta}$, $\varphi_\delta = 0$ for $r < \delta$, satisfy the estimate

$$\|\varphi_\delta\|_{H^{l+2}_{-\mu}(\Omega)} \le c\|b_\delta\|_{H^{l+1/2}_{-\mu}(S)}, \tag{4.2}$$

where c does not depend on δ. Letting $\delta \to 0$ proves Theorem 1.2.

Finally, we prove estimate (4.2) for solutions to problem (1.6). Note that Lemma 3.6 gives an estimate of type (4.2) only locally near L.

Let $\{\zeta^{(k)}\}$ be a partition of unity as in Section 2. Take $\zeta^{(k)}$ such that $\operatorname{supp}\zeta^{(k)} \cap L \ne \emptyset$. Let $\varphi^{(k)} = \varphi\zeta^{(k)}$, $f^{(k)} = f\zeta^{(k)}$. Then problem (1.6) takes the form (where f follows from the extension of boundary condition $(1.6)_2$)

$$\begin{aligned} &\Delta\varphi^{(k)} = f^{(k)} + 2\nabla\zeta^{(k)}\nabla\varphi + \Delta\zeta^{(k)}\varphi,\\ &\overline{n}\cdot\nabla\varphi^{(k)} = \varphi\overline{n}\cdot\nabla\zeta^{(k)}. \end{aligned} \tag{4.3}$$

For $k = 1$ problem (4.3) transforms into

$$\begin{aligned} &\Delta\varphi^{(1)} = f^{(1)} + 2\nabla\zeta^{(1)}\nabla\varphi + \Delta\zeta^{(1)}\varphi,\\ &\varphi^{(1)}|_{\partial\Omega^{(1)}} = 0. \end{aligned} \tag{4.4}$$

In the case $k = 2$ and after the change of variables (2.9) problem (4.3) assumes the form

$$\begin{aligned} &-\nabla_z^2\widetilde{\varphi}^{(2)} = -(\nabla_z^2 - \nabla_\Psi^2)\widetilde{\varphi}^{(2)} + 2\nabla_\Psi\widetilde{\zeta}^{(2)}\nabla_\Psi\widetilde{\varphi} + \nabla_\Psi^2\widetilde{\zeta}^{(2)}\widetilde{\varphi} + f^{(2)},\\ &\overline{n}_z\cdot\nabla_z\widetilde{\varphi}^{(2)} = (\overline{n}_z - \overline{n}_\Psi)\cdot\nabla\widetilde{\varphi}^{(2)} + \widetilde{\varphi}\overline{n}_\Psi\cdot\nabla_\Psi\widetilde{\zeta}^{(2)}, \end{aligned} \tag{4.5}$$

where we use the notation introduced before (2.10). Since $\overline{n}_z = (0,0,1)$ problem (4.5) is considered in the half-space $z_3 > 0$.

Let us choose a function $\widetilde{\eta}^{(2)}$ such that

$$\frac{\partial}{\partial z_3}\widetilde{\eta}^{(2)}\bigg|_{z_3=0} = (\overline{n}_z - \overline{n}_\Psi)\cdot\nabla_z\widetilde{\varphi}^{(2)} + \widetilde{\varphi}\overline{n}_\Psi\cdot\nabla_\Psi\zeta^{(2)}. \tag{4.6}$$

Then the function

$$\widetilde{\psi}^{(2)} = \widetilde{\varphi}^{(2)} - \widetilde{\eta}^{(2)} \tag{4.7}$$

is a solution to the problem

$$\begin{aligned} -\nabla_z^2\widetilde{\Psi}^{(2)} &= \nabla_z^2\widetilde{\eta}^{(2)} - (\nabla_z^2 - \nabla_\Psi^2)\widetilde{\varphi}^{(2)} + \widetilde{f}^{(2)}\\ &\quad + 2\nabla_\Psi\widetilde{\zeta}^{(2)}\nabla_\Psi\widetilde{\varphi} + \nabla_\Psi^2\widetilde{\zeta}^{(2)}\widetilde{\varphi} \equiv \widetilde{F}, \quad z_3 > 0,\\ \frac{\partial\widetilde{\psi}^{(2)}}{\partial z_3}\bigg|_{z_3=0} &= 0. \end{aligned} \tag{4.8}$$

After reflection with respect to the plane $z_3 = 0$, problem (4.8) assumes the form of problem (4.4),

$$
\begin{aligned}
&-\nabla_z^2 \psi'^{(2)} = \widetilde{F}', \\
&\widetilde{\psi}'^{(2)}|_{\partial\Omega'^{(2)}} = 0,
\end{aligned} \tag{4.9}
$$

where η' means that $\eta'(z', z_3) = \eta(z', z_3)$ for $z_3 > 0$, $z' = (z_1, z_2)$ and $\eta'(z', z_3) = \eta(z', -z_3)$ for $z_3 < 0$. In view of Lemma 3.5, for solutions to problems (4.3) and (4.9) we obtain the estimate

$$
\begin{aligned}
&\|\varphi^{(k)}\|_{H^{l+2}_{-\mu}(\mathbb{R}^3)} \\
&\quad \le c(\|f^{(k)}\|_{H^l_{-\mu}(\mathbb{R}^3)} + \|\nabla\varphi\|_{H^l_{-\mu}(\mathbb{R}^3 \cap \operatorname{supp} \nabla\zeta^{(k)})} + \|\varphi\|_{H^l_{-\mu}(\mathbb{R}^3 \cap \operatorname{supp} \nabla\zeta^{(k)})}),
\end{aligned} \tag{4.10}
$$

where k is either 1 or 2 and in the case of problem (4.9) we have used (4.7) and the fact that $\operatorname{diam} \operatorname{supp} \Omega^{(2)}$ is sufficiently small.

To estimate the last two terms on the r.h.s. of (4.10) we need the estimate

$$
\|\varphi\|_{H^{l+2}(\Omega)} \le c\|f\|_{H^l(\Omega)}, \tag{4.11}
$$

which is well known for solutions to problem (1.6), where $f = \Delta\widetilde{b}$ and $\widetilde{b}$ is an extension of the boundary data such that $\overline{n} \cdot \nabla\widetilde{b}|_S = b$.

Let us consider the case $\mu \in (0, 1)$. Using the Hardy inequality we estimate the last two terms on the r.h.s. of (4.10) by $c\|f\|_{H^l(\Omega)}$. Using the boundedness of Ω we obtain from (4.10) the estimate

$$
\|\varphi^{(k)}\|_{H^{l+2}_{-\mu}(\mathbb{R}^3)} \le c\|f\|_{H^l_{-\mu}(\Omega)}, \tag{4.12}
$$

where k is either 1 or 2. Choosing now $\zeta^{(k)}$ such that $\operatorname{dist}\{\operatorname{supp}\zeta^{(k)}, L\} > 0$, we obtain problems similar to (4.3), (4.5), where k is either 3 or 4. Using (4.11) for solutions of these problems we obtain the estimate

$$
\|\varphi^{(k)}\|_{H^{l+2}(\mathbb{R}^3)} \le c\|f\|_{H^l(\Omega)}, \tag{4.13}
$$

where k is either 3 or 4. Summing up inequalities (4.12) and (4.13) over all admissible k we obtain

$$
\|\varphi\|_{H^{l+2}_{-\mu}(\Omega)} \le c\|f\|_{H^l_{-\mu}(\Omega)}. \tag{4.14}
$$

Let us now consider the case $\mu \in (1, 2)$. Then to estimate the last two terms on the r.h.s. of (4.10) we use (4.14) for $\mu \in (0, 1)$ and the Hardy inequality. Repeating the above considerations we obtain (4.14) for $\mu \in (1.2)$.

Continuing the above considerations and assuming that $\varphi \in H^{l+2}_{-\mu}(\Omega)$, $\mu \in (k-1, k)$, we obtain (4.14) for $\mu \in (k, k+1)$, $k \in \mathbb{N}$.

The existence at each step can be proved by applying the construction of φ_δ. This concludes the proof of Theorem 1.2.

Similarly, we prove Theorem 1.3. Theorems 1.2 and 1.3 imply Theorem 1.1.

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