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Piotr Kacprzyk (Warszawa)

LOCAL EXISTENCE OF SOLUTIONS OF THE FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF A MAGNETOHYDRODYNAMIC INCOMPRESSIBLE FLUID

Abstract. Local existence of solutions is proved for equations describing the motion of a magnetohydrodynamic incompressible fluid in a domain bounded by a free surface. In the exterior domain we have an electromagnetic field which is generated by some currents located on a fixed boundary. First by the Galerkin method and regularization techniques the existence of solutions of the linarized equations is proved; next by the method of successive aproximations the local existence is shown for the nonlinear problem.

1. Introduction. In this paper we prove the existence of local solutions to equations describing the motion of a magnetohydrodynamic incompressible fluid in a domain $\Omega_t \subset \mathbb{R}^3$ bounded by a free surface S_t. In the domain $D_t \subset \mathbb{R}^3$ which is exterior to Ω_t we have a gas under constant pressure p_0. Moreover in the domain D_t we have an electromagnetic field which is generated by some currents which are located on a fixed boundary B of D_t.

In the domain Ω_t the motion is described by the following problem,

$$
\begin{aligned}
& v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \mu_1 \overset{1}{H}\cdot\nabla\overset{1}{H} + \mu_1\nabla\overset{1}{H}{}^2 = f && \text{in } \widetilde{\Omega}^T,\\
& \operatorname{div} v = 0 && \text{in } \widetilde{\Omega}^T,\\
& \mu_1 \overset{1}{H}_t = -\operatorname{rot}\overset{1}{E} && \text{in } \widetilde{\Omega}^T,\\
& \operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}), && \text{in } \widetilde{\Omega}^T,\\
& \operatorname{div}(\mu_1\overset{1}{H}) = 0, && \text{in } \widetilde{\Omega}^T,
\end{aligned}
\tag{1.1}
$$

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where $\widetilde{\Omega}^T = \bigcup_{0\le t\le T} \Omega_t \times \{t\}$, $v = v(x,t)$ is the velocity of the fluid, $p = p(x,t)$ is the pressure, $\overset{1}{H} = \overset{1}{H}(x,t)$ is the magnetic field, $f = f(x,t)$ is the external force field per unit mass, μ_1 is the constant magnetic permeability, σ_1 is the constant electric conductivity, $\overset{1}{E} = \overset{1}{E}(x,t)$ is the electric field, and

$$\mathbb{T}(v,p) = \{\nu(\partial_{x_i} v_j + \partial_{x_j} v_i) - p\delta_{ij}\} \tag{1.2}$$

is the stress tensor, where ν is the viscosity of the fluid. Moreover, by

$$\mathbb{D}(v) = \{\nu(\partial_{x_i} v_j + \partial_{x_j} v_i)\} \tag{1.3}$$

we denote the dilatation tensor.

In the domain D_t which is a dielectric (gas) we assume that there is no fluid motion inside ($v = 0$). Therefore we have the electromagnetic field only described by the following system:

$$\begin{aligned} &\mu_2 \overset{2}{H}_t = -\operatorname{rot} \overset{2}{E} && \text{in } \widetilde{D}^T,\\ &\sigma_2 \overset{2}{E} = \operatorname{rot} \overset{2}{H} && \text{in } \widetilde{D}^T,\\ &\operatorname{div}(\mu_2 \overset{2}{H}) = 0 && \text{in } \widetilde{D}^T, \end{aligned} \tag{1.4}$$

where $\widetilde{D}^T = \bigcup_{0\le t\le T} D_t \times \{t\}$.

On $S_t = \partial\Omega_t \cap \partial D_t$ we assume the following transmission and boundary conditions:

$$\begin{aligned} &n \cdot \mathbb{T}(v,p) = -p_0 n && \text{on } \widetilde{S}^T,\\ &\frac{1}{\sigma_1}\overset{1}{H} = \frac{1}{\sigma_2}\overset{2}{H} && \text{on } \widetilde{S}^T,\\ &\overset{1}{E}\cdot\tau_\alpha = \overset{2}{E}\cdot\tau_\alpha, \quad \alpha = 1,2, && \text{on } \widetilde{S}^T\\ &v\cdot n = -\frac{\phi_t}{|\nabla\phi_t|} && \text{on } \widetilde{S}^T, \end{aligned} \tag{1.5}$$

where $\widetilde{S}^T = \bigcup_{0\le t\le T} S_t \times \{t\}$, n is the unit outward vector to Ω_t and normal to S_t, τ_α, $\alpha = 1,2$, is the tangent vector to S_t, $\phi(x,t) = 0$ describes S_t at least locally.

Next we assume the following boundary conditions on B:

$$\begin{aligned} &\overset{2}{H} = H_* \quad \text{on } B,\\ &\overset{2}{E} = E_* \quad \text{on } B, \end{aligned} \tag{1.6}$$

where H_* and E_* are connected by

$$\sigma_1 E_{*n} = \frac{1}{A_{\tau_1}A_{\tau_2}}\,(\partial_{\tau_1}(H_{*\tau_1}A_{\tau_2}) - \partial_{\tau_2}(H_{*\tau_1}A_{\tau_1})),$$

$$\mu_2\partial_t H_{*n} = \frac{1}{A_{\tau_1}A_{\tau_2}}\,(\partial_{\tau_2}(E_{*\tau_1}A_{\tau_1}) - \partial_{\tau_1}(E_{*\tau_2}A_{\tau_2})),$$

$$\begin{aligned}-\partial_t\partial_{\tau_1}(H_{*\tau_1}A_{\tau_2}A_n) - \partial_t\partial_{\tau_2}(H_{*\tau_2}A_{\tau_1}A_n) &= \partial_{\tau_1}\partial_{\tau_2}(E_{*n}A_n)\\ &\quad - \mu_2\partial_{\tau_1}(A_{\tau_2}A_{\tau_3}\partial_t H_{*\tau_1}) - \mu_2\partial_{\tau_2}(A_{\tau_1}A_{\tau_3}\partial_t H_{*\tau_2}) - \partial_{\tau_2}\partial_{\tau_1}(E_{*n}A_n),\end{aligned}$$

where (τ_1,τ_2,n) are curvilinear coordinates and $A_{\tau_1}, A_{\tau_2}, A_n$ the Lamé coefficients of the transformation $(\tau_1,\tau_2,n)\mapsto(x_1,x_2,x_3)$.

Finally, we assume the initial conditions

$$\begin{array}{llll} & \Omega_t|_{t=0} = \Omega, & S_t\big|_{t=0} = S, & D_t|_{t=0} = D,\\ (1.7) & v|_{t=0} = v_0, & \overset{1}{H}|_{t=0} = \overset{1}{H}_0 & \text{in } \Omega,\\ & \overset{2}{H}|_{t=0} = \overset{2}{H}_0 & & \text{in } D.\end{array}$$

To prove the existence of solutions to the above problem we introduce the Lagrangian coordinates $\xi\in\Omega$. The Lagrangian coordinates are connected with the velocity v as the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi\in\Omega. \tag{1.8}$$

Therefore $x_v(\xi,t) = \xi + \int_0^t \overline{v}(\xi,\tau)\,d\tau$, where

$$\overline{v}(\xi,t) = v(x_v(\xi,t),t).$$

To introduce the Lagrangian coordinates in D_t we extend v onto D_t. Let us denote the extended functions by v'. Then we define $\xi\in D$ to be the Cauchy data for the problem

$$\frac{dx}{dt} = v'(x,t), \quad x|_{t=0} = \xi\in D. \tag{1.9}$$

Therefore $x_{v'}(\xi,t) = \xi + \int_0^t \overline{v}'(\xi,\tau)\,d\tau$, where $\overline{v}'(\xi,t) = v'(x_{v'}(\xi,t),t)$. Then by $(1.1)_5$,

$$\Omega_t = \{x\in\mathbb{R}^3 : x = x_v(\xi,t),\ \xi\in\Omega\},$$
$$S_t = \{x\in\mathbb{R}^3 : x = x_v(\xi,t),\ \xi\in S\}.$$

Since S_t is determined at least locally by the equation $\phi(x,t)=0$, S is described by $\phi(x_v(\xi,t),t)\big|_{t=0} = 0$. Moreover we have

$$\overline{n}_v = n(x_v(\xi,t),t) = \frac{\nabla_x\phi(x,t)}{|\nabla_x\phi(x,t)|}\bigg|_{x=x_v(\xi,t)}.$$

We introduce the following notation:

$$\|u\|_{l,Q}=\|u\|_{H^l(Q)},\qquad Q\in\{\Omega,S,D,\Pi,B\},\ 0\le l\in\mathbb{Z},$$

$$\|u\|_{k,p,q,Q^T}=\|u\|_{L_q(0,T,W_p^k(Q))},\qquad Q\in\{\Omega,S,D,\Pi,B\},\quad p,q\in[1,\infty],\ 0\le k\in\mathbb{Z},$$

where $Q^t=Q\times(0,t)$, and

$$|u|_{p,Q}=\|u\|_{L_p(Q)},\qquad Q\in\{\Omega,S,D,\Pi,B\},\ p\in[1,\infty].$$

2. Weak solutions. Weak solutions to problem (1.1)–(1.7) are defined in Lagrangian coordinates. From the equality

$$\int_{\Omega_t}\operatorname{rot}H\psi\,dx=\int_{\Omega_t}H\operatorname{rot}\psi\,dx-\int_{S_t}(n\times H)\psi\,dx_{S_t}$$

and from (1.1)–(1.5) we get

$$\begin{aligned}\frac{1}{\sigma_2}\int_{S_t}(n\times\operatorname{rot}\overset{2}{H})\psi\,dx_{S_t}&-\frac{1}{\sigma_1}\int_{S_t}(n\times\operatorname{rot}\overset{1}{H})\psi\,dx_{S_t}\\&+\mu_1\int_{S_t}n\times(v\times\overset{1}{H})\psi\,dx_{S_t}=\int_{S_t}(n\times\overset{2}{E})\psi\,dx_{S_t}\\&-\int_{S_t}n\times(\overset{1}{E}+\mu_1v\times\overset{1}{H})\psi\,dx_{S_t}+\mu_1\int_{S_t}n\times(v\times\overset{1}{H})\psi\,dx_{S_t}=0.\end{aligned}$$

DEFINITION 2.1. By *weak solutions* to problem (1.1)–(1.7) we mean functions $\overline{v},\overline{H}$ which satisfy the integral identities

$$\begin{aligned}\int_0^T\int_\Omega(-\overline{v}\overline{\varphi}_t+\mathbb{D}_v(\overline{v})\mathbb{D}_v(\overline{\varphi}))\,d\xi\,dt&-\int_0^T\int_\Omega(\mu_1\overset{1}{\overline{H}}\nabla_v\overset{1}{\overline{H}}\cdot\overline{\varphi}-\mu_1\nabla_v\overset{1}{\overline{H}}{}^2\overline{\varphi})\,d\xi\,dt\\&=\int_0^T\int_\Omega\overline{f}\overline{\varphi}\,d\xi\,dt-\int_0^T\int_S p_0\overline{n}_v\varphi\,d\xi_S\,dt-\int_\Omega\overline{v}_0\overline{\varphi}(0)\,d\xi,\end{aligned}\tag{2.1}$$

$$\begin{aligned}\int_0^T\int_\Pi\bigg(-\mu\overline{H}\overline{\psi}_t-\mu\overline{v}\nabla_v\overline{H}\overline{\psi}+\frac{1}{\sigma}\operatorname{rot}_v\overline{H}\operatorname{rot}_v\overline{\psi}\bigg)d\xi\,dt&\\-\int_0^T\int_\Omega\mu_1(\overline{v}\times\overset{1}{\overline{H}})\operatorname{rot}_v\overline{\psi}\,d\xi\,dt&=\frac{1}{\sigma_2}\int_0^T\int_B(\overline{n}_v\times\overline{E}_*)\overline{\psi}\,d\xi_B\,dt\\-\mu\int_\Pi\overline{H}_0\overline{\psi}(0)\,d\xi,&\end{aligned}\tag{2.2}$$

where φ,ψ are sufficiently regular with $\varphi(x,T)=\psi(x,T)=0$, and $\overline{n}_v$ is the unit outward vector normal to S or B.

In (2.1), (2.2) we use the notation $\overline{A}(\xi,t)=A(x_v(\xi,t),t)$, $\overline{H}|_\Omega=\overset{1}{\overline{H}}$, $\overline{H}|_D=\overset{2}{\overline{H}}$, $\sigma|_\Omega=\sigma_1$, $\sigma|_D=\sigma_2$, $\Pi=\Omega\cup D$, $\mu|_\Omega=\mu_1$, $\mu|_D=\mu_2$; in (2.2) v is extended onto Π,

$$\mathbb{D}_v(\overline{v})=\{\nu(\partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_j+\partial_{x_j}\xi_k\nabla_{\xi_k}\overline{v}_i)\},\quad \operatorname{rot}_v\overline{v}=\nabla_v\times\overline{v},$$
$$\nabla_v=\partial_x\xi_i\nabla_{\xi_i},\quad \operatorname{div}_v\overline{v}=\nabla_v\cdot\overline{v}=\partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_i,\quad \partial_{\xi_i}=\nabla_{\xi_i}.$$

Let A be the Jacobi matrix of the transformation $x=x_v(\xi,t)$. Then $\det A=\exp(\int_0^t\operatorname{div}_v\overline{v}\,d\tau)=1$.

Moreover $x^i_{\xi_j}=\delta_{ij}+\int_0^t\partial_{\xi_j}\overline{v}^i(\xi,\tau)\,d\tau$ and $\xi_x=x_\xi^{-1}$. Then we get

$$\sup_{\xi\in\Omega}|x_\xi|\le 1+\sup_{\xi\in\Omega}\int_0^t|\overline{v}(\xi,\tau)|\,d\tau\le 1+c\int_0^t\|\overline{v}\|_{3,\Omega}\,d\tau$$
$$\le 1+c\sqrt{t}\Big(\int_0^t\|\overline{v}\|^2_{3,\Omega}\,d\tau\Big)^{1/2}\le 1+c\sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t}.$$

Thus $\sup_{x\in\Omega_t}|\xi_x|\le\varphi(a)$, where $a=\sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t}$ and φ is an increasing positive function.

To prove the existence of a solution to the above problem we introduce Lagrangian coordinates connected with a given divergence-free function u. Moreover we linearize the terms with v in (1.1) writing them in the form $\overline{u}\cdot\nabla\overline{v}$ and $\overline{u}\times\overset{1}{\overline{H}}$. Then from (2.1), (2.2) we get

$$\int_0^T\int_\Omega(-\overline{v}\overline{\varphi}_t+\mathbb{D}_u(\overline{v})\mathbb{D}_u(\overline{\varphi}))\,d\xi\,dt-\int_0^T\int_\Omega(\mu_1\overset{1}{\overline{H}}{}'\nabla_u\overset{1}{\overline{H}}{}'\overline{\varphi}-\mu_1\nabla_u\overset{1}{\overline{H}}{}'^2)\,d\xi\,dt$$
$$=\int_0^T\int_\Omega\overline{f}\overline{\varphi}\,d\xi\,dt-\int_0^T\int_S p_0\overline{n}_u\varphi\,d\xi_S\,dt-\int_\Omega\overline{v}_0\overline{\varphi}(0)\,d\xi,\tag{2.3}$$

$$\int_0^T\int_\Pi\Big(-\mu\overline{H}\overline{\psi}_t-\mu\overline{u}\nabla_u\overline{H}\overline{\psi}+\frac{1}{\sigma}\operatorname{rot}_u\overline{H}\operatorname{rot}_u\overline{\psi}\Big)\,d\xi\,dt$$
$$-\int_0^T\int_\Omega\mu_1(\overline{u}\times\overset{1}{\overline{H}})\operatorname{rot}_u\overline{\psi}\,d\xi\,dt$$
$$=\frac{1}{\sigma_2}\int_0^T\int_B(\overline{n}_u\times\overline{E}_*)\overline{\psi}\,d\xi_B\,dt-\mu\int_\Pi\overline{H}_0\overline{\psi}(0)\,d\xi,\tag{2.4}$$

where $\overset{1}{\overline{H}}{}'$ is a given function.

We have the following main theorem of this paper, which is proved in Section 5.

Main Theorem. *Assume that* $\overline{v}_0 \in H^2(\Omega)$; $\overline{v}_t(0), \overline{v}_{tt}(0) \in L_2(\Omega)$; $\overline{f}_t, \overline{f}_{tt} \in L_2(0,T,L_2(\Omega))$; $\overline{f} \in L_2(0,T,H^2(\Omega))$; $\overline{H}(0) \in H^2(\Pi)$; $\overline{H}_t(0) \in H^1(\Pi)$; $\overline{E}_* \in L_\infty(0,T,H^1(B))$; $E_{*t}, \overline{H}_{*tt} \in L_2(0,T,L_2(B))$; $E_{*t}, \overline{H}_{*tt} \in L_2(0,T,L_2(B))$; $\overline{H}_{*t} \in L_2(0,T,H^2(B))$; $\overline{H}_* \in L_2(0,T,H^3(B))$ *and* $S, B \in H^{5/2}$. *Then there exists* T^{**} *sufficiently small such that for* $T \le T^{**}$ *there exists a solution to problem* (1.1)–(1.7) *such that*

$$\begin{aligned}
&\overline{v} \in L_2(0,T,H^3(\Omega)) \cap L_\infty(0,T,H^1(\Omega));\\
&\overline{v}_t \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^2(\Omega));\\
&\overline{v}_{tt} \in L_\infty(0,T,L_2(\Omega)) \cap L_2(0,T,H^1(\Omega));\\
&\overline{p} \in L_2(0,T,H^2(\Omega)), \quad \overline{p}_t \in L_2(0,T,H^1(\Omega));\\
&\overline{H} \in L_2(0,T,H^3(\Pi)) \cap L_\infty(0,T,H^1(\Pi));\\
&\overline{H}_t \in L_\infty(0,T,H^1(\Pi)) \cap L_2(0,T,H^2(\Pi));\\
&\overline{H}_{tt} \in L_\infty(0,T,L_2(\Pi)) \cap L_2(0,T,H^1(\Pi))
\end{aligned}$$

and

$$\begin{aligned}
&\|\overline{v}_t\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{3,2,2,\Omega^T} + \|\overline{v}_t\|^2_{2,2,2,\Omega^T}\\
&\quad + \|\overline{v}_{tt}\|^2_{1,2,2,\Omega^T} + \|\overline{p}\|^2_{2,2,2,\Omega^T} + \|\overline{p}_t\|^2_{1,2,2,\Omega^T} + \|\overline{H}_t\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{1,2,\infty,\Pi^T}\\
&\quad + \|\overline{H}\|^2_{3,2,2,\Pi^T} + \|\overline{H}_t\|^2_{2,2,2,\Pi^T} + \|\overline{H}_{tt}\|^2_{1,2,2,\Pi^T} \le A,
\end{aligned}$$

where A *is a positive constant.*

3. Existence and regularity of solutions of the linearized problem (1.12). To prove the existence of a solution to the problems (1.12), (1.13) we use the Galerkin method. Take a basis $\{\overline{\varphi}_k\}$ in $L_2(\Omega)$ and $\{\overline{\psi}_k\}$ in $L_2(\Pi)$. Then we are looking for an approximate solution in the form

$$\overline{v}_n = \sum_{k=1}^n c_{kn}(t)\overline{\varphi}_k(\xi), \qquad \overline{H}_n = \sum_{k=1}^n d_{kn}(t)\overline{\psi}_k(\xi), \tag{3.1}$$

where the functions c_{kn}, d_{kn}, $k = 1, \ldots, n$, are solutions of the following system of ordinary differential equations:

$$\int_\Omega (\overline{v}_{nt}\overline{\varphi}_k + \mathbb{D}_u(\overline{v}_n)\mathbb{D}_u(\overline{\varphi}_k))\,d\xi - \mu_1 \int_\Omega (\overset{1}{\overline{H}}{}' \nabla_u \overset{1}{\overline{H}}{}' \overline{\varphi}_k - \nabla_u \overset{1}{\overline{H}}{}'^2 \overline{\varphi}_k)\,d\xi = \int_\Omega \overline{f}\overline{\varphi}_k\,d\xi - \int_S p_0 \overline{n}_u \overline{\varphi}_k\,d\xi_S, \tag{3.2}$$

$$\int_\Pi (\mu \overline{H}_{nt}\overline{\psi}_k - \mu\overline{u}\nabla_u \overline{H}_n \overline{\psi}_k + \frac{1}{\sigma}\operatorname{rot}_u \overline{H}_n \operatorname{rot}_u \overline{\psi}_k)\,d\xi - \int_\Omega \mu_1 (u \times \overset{1}{\overline{H}}) \operatorname{rot}_u \overline{\psi}_k\,d\xi = \frac{1}{\sigma_2}\int_B (\overline{n}_u \times \overline{E}_*)\overline{\psi}_k\,d\xi_B, \tag{3.3}$$

for $k=1,\dots,n$. The equations (3.2), (3.3) can be written in the form

$$(3.2)_1 \qquad \frac{d}{dt}c_{kn}+a_{ki}(t)c_{in}=f_k(t),$$

$$(3.3)_1 \qquad \frac{d}{dt}d_{nk}+b_{ki}(t)d_{in}=g_k(t),$$

where $k=1,\dots,n$ and summation over repeated indices is assumed. Then from (3.2), (3.3) we see that

$$\sum_{k,i}\int_0^T |a_{ki}(t)|\,dt \le \varphi(a)(\|\overset{1}{\overline{H}}{}'\|^2_{1,2,2,\Omega^T}+1),$$

$$\sum_{k,i}\int_0^T |b_{ki}(t)|\,dt \le \varphi(a)(\|\overline{u}\|^2_{1,2,2,\Omega^T}+1),$$

where $a=t^{1/2}\|\overline{u}\|_{3,2,2,\Omega^t}$ and φ is an increasing positive function. Assuming

$$(3.4) \qquad v_0=\sum_{k=1}^{\infty}c_k\overline{\varphi}_k, \qquad H_0=\sum_{k=1}^{\infty}d_k\overline{\psi}_k$$

we have the following initial conditions for solutions to the problem (3.2), (3.3):

$$(3.5) \qquad c_{kn}(0)=c_k, \qquad d_{kn}(0)=d_k, \qquad n\in\mathbb{N},\ k=1,\dots,n.$$

The existence and uniqueness of solution to the problem (3.2), (3.3), (3.5) follows from the theory of ordinary differential equations.

Next we have to assume that

$$(3.6) \qquad \sup_{t\in[0,T]}\sup_{\xi\in\Omega}|I-\xi_x|\le\delta,$$

where δ is sufficiently small and I is the unit matrix.

Now we obtain estimates for solutions of (3.2), (3.3).

LEMMA 3.1. *Assume that* $\overset{1}{\overline{H}}{}'\in L_\infty(0,T,L_2(\Omega))\cap L_2(0,T,H^3(\Omega))$; $\overset{1}{\overline{H}}{}'_t\in L_2(0,T,L_2(\Omega))$; $\overline{f}\in L_2(0,T,L_2(\Omega))$; $\overline{v}(0)\in L_2(\Omega)$; $\overline{u}\in L_2(0,T,H^3(\Omega))$. *Then*

$$(3.7) \qquad \|\overline{v}_n\|^2_{0,2,\infty,\Omega^t}+\|\overline{v}_n\|^2_{1,2,2,\Omega^t}$$
$$\le \alpha(t,a)[\|\overset{1}{\overline{H}}{}'\|^2_{0,2,\infty,\Omega^t}(\varepsilon\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t}+c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'(0)\|_{0,\Omega}+\|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t}))$$
$$+c\|\overline{f}\|^2_{0,2,2,\Omega^t}+cp_0^2+\|\overline{v}_n(0)\|^2_{0,\Omega}],$$

where α is an increasing positive function, $a=t^{1/2}\|\overline{u}\|_{3,2,2,\Omega^t}$ *and* $0\le t\le T$.

Proof. Multiplying (3.2) by c_{kn} and summing over k from 1 to n we get

$$\tag{3.8} \int_\Omega \left(\frac{1}{2}\,\frac{d}{dt}\overline{v}_n^2 + |\mathbb{D}_u(\overline{v}_n)|^2\right) d\xi = \mu_1 \int_\Omega (\overset{1}{\overline{H}}{}'\nabla_u \overset{1}{\overline{H}}{}'\overline{v}_n - \nabla_u \overset{1}{\overline{H}}{}'^2 \overline{v}_n)\, d\xi + \int_\Omega \overline{f}\,\overline{v}_n\, d\xi - \int_S p_0 \overline{n}_u \overline{v}_n\, d\xi_S.$$

Using in (3.8) the Hölder and Young inequalities together with the Korn inequality

$$\|\overline{v}_n\|_{1,\Omega}^2 \le c(\delta)(\|\mathbb{D}(\overline{v}_n)\|_{0,\Omega}^2 + \|\overline{v}_n\|_{0,\Omega}^2),$$

we get

$$\tag{3.9} \frac{d}{dt}\|\overline{v}_n\|_{0,\Omega}^2 + \|\overline{v}_n\|_{1,\Omega}^2 \le \varphi(a)\|\overset{1}{\overline{H}}{}'\|_{0,\Omega}^2 \|\overset{1}{\overline{H}}{}'\|_{2,\Omega}^2 + c\|p_0\overline{n}_u\|_{0,S}^2 + c\|\overline{f}\|_{0,\Omega}^2 + c\|\overline{v}_n\|_{0,\Omega}^2.$$

Integrating (3.9) with respect to time and using the Gronwall inequality we get (3.7).

We want to obtain more regular solutions of (3.2); therefore we show

LEMMA 3.2. *Let the assumptions of Lemma* 3.1 *be satisfied and* $\overline{v}(0) \in H^1(\Omega)$. *Then*

$$\tag{3.10} \begin{aligned} \|\overline{v}_{nt}\|_{0,2,2,\Omega^t}^2 + \|\overline{v}_n\|_{1,2,\infty,\Omega^t}^2 \le \alpha(t,a)[&\|\overset{1}{\overline{H}}{}'\|_{0,2,\infty,\Omega^t}^2 (\varepsilon\|\overset{1}{\overline{H}}{}'\|_{3,2,2,\Omega^t}^2 \\ &+ c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'(0)\|_{0,\Omega}^2 + \|\overset{1}{\overline{H}}{}'_t\|_{0,2,2,\Omega^t}^2)) + c\|\overline{f}\|_{0,2,2,\Omega^t}^2 + c\|\overline{v}_n(0)\|_{1,\Omega}^2 \\ &+ c\|\overline{v}_n\|_{0,2,2,\Omega^t}^2 + \varepsilon\|\overline{v}_{nt}\|_{1,2,2,\Omega^t}^2 + cp_0^2], \end{aligned}$$

where α *is an increasing positive function.*

Proof. Multiplying (3.2) by $\frac{d}{dt}c_{kn}$ and summing over k from 1 to n we get

$$\tag{3.11} \int_\Omega \left(\frac{1}{2}\,\overline{v}_{nt}^2 + \mathbb{D}_u(\overline{v}_n)\mathbb{D}_u(\overline{v}_{nt})\right) d\xi = \mu_1 \int_\Omega (\overset{1}{\overline{H}}{}'\nabla_u \overset{1}{\overline{H}}{}'\overline{v}_{nt} - \nabla_u \overset{1}{\overline{H}}{}'^2 \overline{v}_{nt})\, d\xi + \int_\Omega \overline{f}\,\overline{v}_{nt}\, d\xi - \int_S p_0 \overline{n}_u \overline{v}_{nt}\, d\xi_S.$$

Using in (3.11) the Hölder and Young inequalities we get

$$\tag{3.12} \begin{aligned} \|\overline{v}_{nt}\|_{0,\Omega}^2 + \frac{d}{dt}\|\mathbb{D}_u(\overline{v}_n)\|_{0,\Omega}^2 \le \varphi(a)\Big(&\|\overset{1}{\overline{H}}{}'\|_{0,\Omega}^2 \|\overset{1}{\overline{H}}{}'\|_{2,\Omega}^2 + \varepsilon\|\overline{v}_{nt}\|_{0,\Omega}^2 \\ &+ \int_\Omega |\overline{u}_\xi|\,|\mathbb{D}_u(\overline{v}_n)|\,|\overline{v}_{n\xi}|\, d\xi\Big) + c\|p_0\overline{n}_u\|_{0,S}^2 + c\|\overline{f}\|_{0,\Omega}^2 + \varepsilon\|\overline{v}_{nt}\|_{1,\Omega}^2. \end{aligned}$$

Integrating (3.12) with respect to time and applying the Korn and Gronwall inequalities we get (3.10).

To estimate $\|\overline{v}_{nt}\|^2_{1,2,2,\Omega^t}$ we need the following result.

LEMMA 3.3. *Let the assumptions of Lemma* 3.1 *be satisfied and* $\overline{f}_t \in L_2(0,T,L_2(\Omega))$; $\overset{1}{\overline{H}}{}', \overset{1}{\overline{H}}{}'_t \in L_\infty(0,T,H^1(\Omega))$; $\overline{v}_t(0) \in L_2(\Omega)$. *Then*

$$
\begin{aligned}
(3.13)\quad & \|\overline{v}_{nt}\|^2_{0,2,\infty,\Omega^t} + \|\overline{v}_{nt}\|^2_{1,2,2,\Omega^t} \le \alpha(a)[(\varepsilon\|\overline{u}\|^2_{3,2,2,\Omega^t} + c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Omega^t} \\
&\quad + \|\overline{u}(0)\|^2_{0,\Omega}))(\|\overline{v}_n\|^2_{1,2,\infty,\Omega^t} + cp_0^2 + \|\overset{1}{\overline{H}}{}'\|^4_{1,2,\infty,\Omega^t}) \\
&\quad + \|\overset{1}{\overline{H}}{}'_t\|^2_{1,2,\infty,\Omega^t}(\varepsilon\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} + c(\varepsilon)t(\|\overset{1}{\overline{H}}(0)\|^2_{0,\Omega} \\
&\quad + \|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t}))] + c\|\overline{v}_{nt}\|^2_{0,2,2,\Omega^t} + c\|\overline{f}_t\|^2_{0,2,2,\Omega^t} + \|\overline{v}_{nt}(0)\|^2_{0,\Omega},
\end{aligned}
$$

where α is an increasing positive function.

Proof. Differentiating (3.2) with respect to t, multiplying by $\frac{d}{dt}c_{kn}$, summing over k from 1 to n and using the Korn, Hölder and Young inequalities we get

$$
\begin{aligned}
(3.14)\quad \frac{d}{dt}\|\overline{v}_{nt}\|^2_{0,\Omega} + \|\overline{v}_{nt}\|^2_{1,\Omega} &\le \varphi(a)[\|\overline{u}_\xi\|_{\infty,\Omega}(\varepsilon\|\overline{v}_{nt}\|^2_{1,\Omega} + \|\overline{v}_n\|^2_{1,\Omega} \\
&\quad + \|\overset{1}{\overline{H}}{}'\|^4_{1,\Omega}) + \|\overset{1}{\overline{H}}{}'_t\|^2_{1,\Omega}\|\overset{1}{\overline{H}}{}'\|^2_{1,\Omega} + \varepsilon\|\overline{v}_{nt}\|^2_{1,\Omega}] \\
&\quad + c\|\overline{f}_t\|^2_{0,\Omega} + c\|(p_0\overline{n}_u)_t\|^2_{0,S} + \|\overline{v}_{nt}\|^2_{0,\Omega}.
\end{aligned}
$$

Integrating (3.14) with respect to time we get (3.13).

From (3.7), (3.10), (3.13) we have

LEMMA 3.4. *Let the assumptions of Lemmas* 3.1–3.3 *be satisfied. Then*

$$
\begin{aligned}
(3.15)\quad & \|\overline{v}_n\|^2_{1,2,2,\Omega^t} + \|\overline{v}_{nt}\|^2_{1,2,2,\Omega^t} + \|\overline{v}_n\|^2_{1,2,\infty,\Omega^t} + \|\overline{v}_{nt}\|^2_{0,2,\infty,\Omega^t} \\
&\le \alpha(a,t)[(\varepsilon\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega} + \|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t})) \\
&\quad \cdot (\|\overset{1}{\overline{H}}{}'\|^2_{0,2,\infty,\Omega^t} + \|\overset{1}{\overline{H}}{}'_t\|^2_{1,2,\infty,\Omega^t}) \\
&\quad + (\varepsilon\|\overline{u}\|^2_{3,2,2,\Omega^t} + c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Omega^t} + \|\overline{u}(0)\|^2_{0,\Omega}))(cp_0^2 + \|\overset{1}{\overline{H}}{}'\|^4_{1,2,\infty,\Omega^t}) \\
&\quad + c\|\overline{f}\|^2_{0,2,2,\Omega^t} + c\|\overline{f}_t\|^2_{0,2,2,\Omega^t} + \|\overline{v}_n(0)\|^2_{1,\Omega} + cp_0^2] + \|\overline{v}_{nt}(0)\|^2_{0,\Omega} \equiv \overline{F},
\end{aligned}
$$

where α is an increasing positive function.

Now choosing a subsequence and letting $n \to \infty$ we get

LEMMA 3.5. *Assume that* $\overline{u} \in L_2(0,T,H^3\Omega))$; $\overline{v}(0) \in H^1(\Omega)$; $\overline{v}_t(0) \in L_2(\Omega)$; $\overset{1}{\overline{H}}{}' \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^3(\Omega))$; $\overset{1}{\overline{H}}{}'_t \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,L_2(\Omega))$ *and* $\overline{f}_1, \overline{f}_t \in L_2(0,T,L_2(\Omega))$. *Then there exists a weak solution of problem* (3.2) *such that*

$$\overline{v} \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^1(\Omega)),$$
$$\overline{v}_t \in L_2(0,T,H^1(\Omega)) \cap L_\infty(0,T,L_2(\Omega))$$

and

$$\|\overline{v}\|^2_{1,2,2,\Omega^t} + \|\overline{v}_t\|^2_{1,2,2,\Omega^t} + \|\overline{v}\|^2_{1,2,\infty,\Omega^t} + \|\overline{v}_t\|^2_{0,2,\infty,\Omega^t} \le \overline{F}. \tag{3.16}$$

To show that $\overline{v} \in L_2(0,T,H^3(\Omega))$ we consider the following elliptic problem:

$$\begin{aligned} &-\operatorname{div}_u \mathbb{T}_u(\overline{v},\overline{p}) = -\overline{v}_t + \mu_1 \overset{1}{\overline{H}}{}' \nabla_u \overset{1}{\overline{H}}{}' + \mu_1 \nabla_u \overset{1}{\overline{H}}{}'^2 + \bar{f} && \text{in } \Omega^T,\\ &\operatorname{div}_u \overline{v} = 0 && \text{in } \Omega^T,\\ &\overline{n}_u \mathbb{T}_u(\overline{v},\overline{p}) = -p_0 \overline{n}_u && \text{on } S^T. \end{aligned} \tag{3.17}$$

Applying to (3.17) the regularization technique for elliptic problems (see [4]) and Lemma 3.5 we get

$$\begin{aligned} &\|\overline{v}\|^2_{3,\Omega} + \|\overline{p}\|^2_{2,\Omega}\\ &\quad \le \varphi(a)(\|\overset{1}{\overline{H}}{}'\|^4_{2,\Omega} + \|\overset{1}{\overline{H}}{}'\|^2_{\infty,\Omega} \|\overset{1}{\overline{H}}{}'\|^2_{2,\Omega}\\ &\qquad + \|\overset{1}{\overline{H}}{}'\|^2_{1,\Omega} \|\overset{1}{\overline{H}}{}'\|^2_{3,\Omega}) + \|\overline{v}_t\|^2_{1,\Omega} + \|\bar{f}\|^2_{1,\Omega} + \|p_0 \overline{n}_u\|^2_{3/2,S}. \end{aligned} \tag{3.18}$$

Integrating (3.18) with respect to time we get

LEMMA 3.6. *Let the asumptions of Lemma* 3.5 *be satisfied and* $\overset{1}{\overline{H}}{}' \in L_\infty(0,T,H^2(\Omega))$; $\bar{f} \in L_2(0,T,H^1(\Omega))$. *Then*

$$\begin{aligned} \|\overline{v}\|^2_{3,2,2,\Omega^t} + \|\overline{p}\|^2_{2,2,2,\Omega^t} &\le \alpha(a,t)[(\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t})\\ &\quad \cdot (\varepsilon \|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} + c(\varepsilon) t (\|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega} + \|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t}))\\ &\quad + c p_0^2 + \|\overline{v}_t\|^2_{1,2,2,\Omega^t} + \|\bar{f}\|^2_{1,2,2,\Omega^t}], \end{aligned} \tag{3.19}$$

where α *is an increasing positive function.*

To show that $\overline{v}_t \in L_2(0,T,H^2(\Omega))$ we differentiate (3.17) with respect to time. Then we get the following elliptic problem:

$$
(3.20)\qquad
\begin{aligned}
&-\operatorname{div}_u \mathbb{T}_u(\overline{v}_t,\overline{p}_t) = (\operatorname{div}_u)_t\mathbb{T}_u(\overline{v},\overline{p}) + \operatorname{div}_u(\mathbb{T}_u)_t(\overline{v},\overline{p}) - \overline{v}_{tt}\\
&\qquad + (\mu_1 \overset{1}{\overline{H}}{}'\nabla_u \overset{1}{\overline{H}}{}')_t - (\mu_1\nabla_u \overset{1}{\overline{H}}{}'^2)_t + \overline{f}_t \qquad \text{in } \Omega^T,\\
&\operatorname{div}_u \overline{v}_t = -(\operatorname{div}_u)_t\overline{v} \qquad \text{in } \Omega^T,\\
&\overline{n}_u\mathbb{T}_u(\overline{v}_t,\overline{p}_t) = -p_0(\overline{n}_u)_t - (\overline{n}_u\mathbb{T}_u)_t(\overline{v},\overline{p}), \quad \text{on } S^T.
\end{aligned}
$$

Applying to (3.20) the regularization technique for elliptic problems we get (see [4])

$$
(3.21)\qquad
\begin{aligned}
&\|\overline{v}_t\|^2_{2,\Omega} + \|\overline{p}_t\|^2_{1,\Omega}\\
&\quad\le \varphi(a)[\|\overline{u}\|^2_{2,\Omega}(\|\overline{v}\|^2_{3,\Omega} + \|\overline{p}\|^2_{2,\Omega}\\
&\qquad + \|\overset{1}{\overline{H}}{}'\|^4_{2,\Omega}) + \|\overset{1}{\overline{H}}{}'\|^2_{1,\Omega}\|\overset{1}{\overline{H}}{}'\|^2_{2,\Omega} + \|\overset{1}{\overline{H}}{}'_t\|^2_{2,\Omega}\|\overset{1}{\overline{H}}{}'\|^2_{1,\Omega}\\
&\qquad + cp_0^2\|\overline{u}\|^2_{1,\Omega}] + \|\overline{v}_{tt}\|^2_{0,\Omega} + \|\overline{f}_t\|^2_{0,\Omega}.
\end{aligned}
$$

Integrating (3.21) with respect to time and using Lemmas 3.5–3.6 we get

LEMMA 3.7. *Let the assumptions of Lemmas* 3.5–3.6 *be satisfied and* $\overline{u} \in L_\infty(0,T,H^2(\Omega))$; $\overset{1}{\overline{H}}{}'_t \in L_2(0,T,H^2(\Omega))$; $\overset{1}{\overline{H}}{}'_{tt}, \overline{f}_t \in L_2(0,T,L_2(\Omega))$; $\overset{1}{\overline{H}}{}'(0), \overset{1}{\overline{H}}{}'_t(0) \in L_2(\Omega)$. *Then*

$$
(3.22)\qquad
\begin{aligned}
&\|\overline{v}_t\|^2_{2,2,2,\Omega^t} + \|\overline{p}_t\|^2_{1,2,2,\Omega^t}\\
&\quad\le \alpha(a,t)[\|\overline{u}\|^2_{2,2,\infty,\Omega^t}(\|\overline{v}\|^2_{3,2,2,\Omega^t} + \|\overline{p}\|^2_{1,2,2,\Omega^t})\\
&\qquad + (\|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t}(1 + \|\overline{u}\|^2_{2,2,\infty,\Omega^t}) + \|\overset{1}{\overline{H}}{}'_t\|^2_{2,2,2,\Omega^t})\\
&\qquad \cdot (\varepsilon(\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'_t\|^2_{2,2,2,\Omega^t})\\
&\qquad + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'_{tt}\|^2_{0,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega}\\
&\qquad + \|\overset{1}{\overline{H}}{}'_t(0)\|^2_{0,\Omega})) + \|\overline{v}_{tt}\|^2_{0,2,2,\Omega^t} + cp_0^2 + \|\overline{f}_t\|^2_{0,2,2,\Omega^t}],
\end{aligned}
$$

where α is an increasing positive function.

Next we have to obtain an estimate for $\|\overline{v}_{tt}\|_{1,2,2,\Omega^t}$. Let

$$
\phi_n \in L_2(0,T,H^3(\Omega)) \cap L_\infty(0,T,H^1(\Omega)),
$$
$$
\phi_{nt} \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^2(\Omega))
$$

and suppose that

$$
\phi_n = \sum_{k=1}^{n} c_k(t)\varphi_k(\xi) \in A_n,
$$

where $\{\varphi_n\}$ is an orthonormal basis in $H^3(\Omega)$. Then $\bigcup_{n=1}^{\infty} A_n$ is dense in $L_2(0,T,H^3(\Omega))\cap L_\infty(0,T,H^1(\Omega))$ and $\bigcup_{n=1}^{\infty} A_{nt}$ is dense in $L_\infty(0,T,H^1(\Omega))$ $\cap\, L_2(0,T,H^2(\Omega))$. Let $\overline{v}_n \in \bigcup_{k=1}^{n} A_k$. Then from the weak solution of the linearized problem (1.12), where $\overline{n}_u\mathbb{T}(\overline{v}_n,\overline{q})=0$ on S^T and $\overline{q}=\overline{p}-p_0$ in Ω^T, if we let $n\to\infty$ we get

LEMMA 3.8. *Let the assumptions of Lemmas* 3.5–3.7 *be satisfied and* $\overline{u}_t \in L_2(0,T,H^2(\Omega))\cap L_\infty(0,T,H^1(\Omega))$; $\overset{1}{\overline{H}}{}'_{tt} \in L_2(0,T,H^1(\Omega))$; $\overline{v}_{tt}(0)\in L_2(\Omega)$ *and* $\overline{u}_{tt}, \bar f_{tt} \in L_2(0,T,L_2(\Omega))$. *Then*

$$
\begin{aligned}
(3.23)\qquad & \|\overline{v}_{tt}\|^2_{0,2,\infty,\Omega^t} + \|\overline{v}_{tt}\|^2_{1,2,2,\Omega^t} \\
&\le \alpha(a,t)[(\varepsilon\|\overline{v}\|^2_{3,2,2,\Omega^t} + c(\varepsilon)t(\|\overline{v}_t\|^2_{0,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{0,\Omega}))\|\overline{u}_t\|^2_{1,2,\infty,\Omega^t} \\
&\quad + \varepsilon\|\overset{1}{\overline{H}}{}'_{tt}\|^4_{1,2,2,\Omega^t} + ct^2\|\overset{1}{\overline{H}}{}'_t\|^4_{1,2,2,\Omega^t} + c\|\overset{1}{\overline{H}}{}'(0)\|^4_{2,\Omega} \\
&\quad + (\varepsilon\|\overset{1}{\overline{H}}{}'_t\|^2_{2,2,2,\Omega^t} + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'_{tt}\|^2_{0,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'_t(0)\|^2_{0,\Omega})) \\
&\quad \cdot (\|\overline{u}\|^2_{2,2,\infty,\Omega^t}\|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t} + \|\overset{1}{\overline{H}}{}'_t\|^2_{1,2,2,\Omega^t}) \\
&\quad + \|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t}(\varepsilon\|\overline{u}_t\|^2_{2,2,2,\Omega^t} + c(\varepsilon)t(\|\overline{u}_{tt}\|^2_{0,2,2,\Omega^t} + \|\overline{u}_t(0)\|^2_{0,\Omega})) \\
&\quad + \|\bar f_{tt}\|^2_{0,2,2,\Omega^t} + \|\overline{v}_{tt}(0)\|^2_{0,\Omega}],
\end{aligned}
$$

where α is an increasing positive function.

Now adding inequalities (3.22) and (3.23) for $p=q+p_0$ we get

LEMMA 3.9. *Let the assumptions of Lemmas* 3.5–3.8 *be satisfied. Then*

$$
\begin{aligned}
(3.24)\qquad & \|\overline{v}_{tt}\|^2_{0,2,\infty,\Omega^t} + \|\overline{v}_{tt}\|^2_{1,2,2,\Omega^t} + \|\overline{v}_t\|^2_{2,2,2,\Omega^t} + \|\overline{p}_t\|^2_{1,2,2,\Omega^t} \\
&\le \alpha(a,t)[(t\|\overline{u}_t\|^2_{2,2,2,\Omega^t} + \|\overline{u}(0)\|^2_{2,\Omega})(\|\overline{v}\|^2_{3,2,2,\Omega^t} + \|\overline{p}\|^2_{2,2,2,\Omega^t}) \\
&\quad + (\varepsilon\|\overline{v}\|^2_{3,2,2,\Omega^t} + c(\varepsilon)t(\|\overline{v}_t\|^2_{0,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{0,\Omega}))\|\overline{u}_t\|^2_{1,2,\infty,\Omega^t} \\
&\quad + (\varepsilon(\|\overset{1}{\overline{H}}{}'_t\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t}) + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'\|^2_{0,2,2,\Omega^t} \\
&\quad + \|\overset{1}{\overline{H}}{}'_{tt}\|^2_{0,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega} + \|\overset{1}{\overline{H}}{}'_t(0)\|^2_{0,\Omega})) \\
&\quad \cdot (\|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t}(1+\|\overline{u}\|^2_{2,2,\infty,\Omega^t}) + \|\overset{1}{\overline{H}}{}'_t\|^2_{2,2,2,\Omega^t}) \\
&\quad + \|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t}(\varepsilon\|\overline{u}_t\|^2_{2,2,2,\Omega^t} + c(\varepsilon)t(\|\overline{u}_{tt}\|^2_{0,2,2,\Omega^t} + \|\overline{u}_t(0)\|^2_{0,\Omega})) \\
&\quad + \|\bar f_{tt}\|^2_{0,2,2,\Omega^t} + \|\overline{v}_{tt}(0)\|^2_{0,\Omega} + cp_0^2 + \|\bar f_t\|^2_{0,2,2,\Omega^t}],
\end{aligned}
$$

where α is an increasing positive function.

From Lemmas 3.5–3.9 we get $\overline{v} \in L_2(0,T,H^3(\Omega)) \cap L_\infty(0,T,H^1(\Omega))$; $\overline{v}_t \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^2(\Omega))$; $\overline{v}_{tt} \in L_\infty(0,T,L_2(\Omega)) \cap L_2(0,T,H^1(\Omega))$ and $\overline{p} \in L_2(0,T,H^2(\Omega))$, $\overline{p}_t \in L_2(0,T,H^1(\Omega))$.

4. Existence and regularity of solutions of the linearized problem (1.13)

LEMMA 4.1. *Assume that* $\overline{E}_* \in L_2(0,T,L_2(B))$; $\overline{H}(0) \in L_2(\Pi)$; $H_* \in L_2(0,T,H^1(B))$. *Then for solutions of* (3.3) *the following inequality holds*:

$$\|\overline{H}_n\|^2_{0,2,\infty,\Pi^t} + \|\overline{H}_n\|^2_{1,2,2,\Pi^t} \le \alpha(a,t)[\|\overline{E}_*\|^2_{0,2,2,B^t} + \|\overline{H}_n(0)\|^2_{0,\Pi} + \|\overline{H}_*\|^2_{1,2,2,B^t}], \tag{4.1}$$

where α is an increasing positive function.

Proof. Multiplying (3.3) by d_{kn} and summing over k from 1 to n we get

$$\int_\Pi \left(\frac{1}{2}\mu\frac{d}{dt}\overline{H}^2_n + \frac{1}{\sigma}|\mathrm{rot}_u\,\overline{H}_n|^2\right)d\xi = \int_\Omega \mu(\overline{u}\times\overline{H}_n)\,\mathrm{rot}_u\,\overline{H}_n d\xi + \int_\Pi \mu\overline{u}\nabla_u\overline{H}_n\overline{H}_n\,d\xi + \frac{1}{\sigma_2}\int_B(\overline{n}_u\times\overline{E}_*)\overline{H}_n\,d\xi_B. \tag{4.2}$$

Applying to (4.2) the Hölder and Young inequalities and the inequality (see Appendix)

$$\|\overline{H}_n\|^2_{1,\Pi} \le c(\delta)(\|\mathrm{rot}_u\,\overline{H}_n\|^2_{0,\Pi} + \|\overline{H}_*\|^2_{1,B}), \tag{4.3}$$

we get

$$\frac{d}{dt}\|\overline{H}_n\|^2_{0,\Pi} + c\|\overline{H}_n\|^2_{1,\Pi} \le \varphi(a)(\|\overline{u}\|_{\infty,\Pi}\|\overline{H}_n\|^2_{0,\Pi} + \|\overline{E}_*\|^2_{0,B} + \|\overline{H}_*\|^2_{1,B}). \tag{4.4}$$

Integrating (4.4) with respect to time and using the Gronwall inequality we get (4.1).

LEMMA 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied and* $\overline{H}(0) \in H^1(\Pi)$; $\overline{H}_* \in L_\infty(0,T,H^1(B))$. *Then*

$$\|\overline{H}_{nt}\|^2_{0,2,2,\Pi^t} + \|\overline{H}_n\|^2_{1,2,\infty,\Pi^t} \le \alpha(a,t)[\varepsilon\|\overline{H}_{nt}\|^2_{1,2,2,\Pi^t} + \|\overline{E}_*\|^2_{0,2,2,B^t} + \|\overline{H}_n(0)\|^2_{1,\Pi} + \|\overline{H}_*\|_{1,2,\infty,B^t}], \tag{4.5}$$

where α is an increasing positive function.

Proof. Multiplying (3.3) by $\frac{d}{dt}d_{kn}$ and summing over k form 1 to n we get,

$$\int_\Pi \left(\frac{1}{2}\mu\overline{H}^2_{nt} + \frac{1}{\sigma}\mathrm{rot}_u\,\overline{H}_n\,\mathrm{rot}_u\,\overline{H}_{nt}\right)d\xi = \int_\Omega \mu(\overline{u}\times\overline{H}_n)\,\mathrm{rot}_u\,\overline{H}_{nt}\,d\xi + \int_\Pi \mu\overline{u}\nabla_u\overline{H}_nH_{nt}\,d\xi + \frac{1}{\sigma_2}\int_B(\overline{n}_u\times\overline{E}_*)\overline{H}_{nt}\,d\xi_B. \tag{4.6}$$

Applying to (4.6) the Hölder and Young inequalities we get

$$(4.7)\qquad \|\overline{H}_{nt}\|^2_{0,\Pi}+\frac{d}{dt}\left(\frac{1}{\sigma}\|\mathrm{rot}_u\,\overline{H}_n\|^2_{0,\Pi}\right) \le \varphi(a)(\|\overline{u}\|_{\infty,\Pi}\|\overline{H}_n\|^2_{1,\Pi}+\varepsilon\|\overline{H}_{nt}\|^2_{1,\Pi}+\|\overline{E}_*\|^2_{0,B}).$$

Integrating (4.7) with respect to time using (4.3) and the Gronwall inequality we get (4.5).

To estimate $\|\overline{H}_{nt}\|^2_{1,2,2,\Pi^t}$ we need the following result:

LEMMA 4.3. *Let the assumptions of Lemma* 4.1 *be satisfied and* $\overline{u}_t\in L_2(0,T,H^2(\Pi))$; $\overline{u}_{tt}\in L_2(0,T,L_2(\Pi))$; $\overline{u}\in L_2(0,T,H^3(\Pi))$; $\overline{E}_{*t}\in L_2(0,T,L_2(B))$; $\overline{E}_*\in L_\infty(0,T,H^1(B))$; $\overline{H}_t(0)\in L_2(\Pi)$; *and* $\overline{H}_{*t}\in L_2(0,T,H^1(B))$. *Then*

$$(4.8)\qquad \begin{aligned}\|\overline{H}_{nt}\|^2_{0,2,\infty,\Pi^t}+\|\overline{H}_{nt}\|^2_{1,2,2,\Pi^t} &\le \alpha(a)[(\varepsilon\|\overline{u}_t\|^2_{2,2,2,\Pi^t}\\ &\quad+c(\varepsilon)t(\|\overline{u}_{tt}\|^2_{0,2,2,\Pi^t}+\|\overline{u}_t(0)\|^2_{0,\Pi}))\|\overline{H}_n\|^2_{1,2,\infty,\Pi^t}\\ &\quad+(\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t}+c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Pi^t}+\|\overline{u}_t(0)\|^2_{0,\Pi}))\\ &\quad\cdot(\|\overline{H}_n\|^2_{1,2,\infty,\Pi^t}+\|\overline{H}_{nt}\|^2_{0,2,\infty,\Pi^t}+\|\overline{E}_*\|^2_{1,2,\infty,B^t})]\\ &\quad+c\|\overline{E}_{*t}\|^2_{0,2,2,B^t}+\|\overline{H}_{nt}(0)\|^2_{0,\Pi}+\|\overline{H}_{*t}\|^2_{1,2,2,B^t},\end{aligned}$$

where α *is an increasing positive function.*

Proof. Differentiating (3.3) with respect to time, multiplying by $\frac{d}{dt}d_{kn}$, summing over k from 1 to n and using the Korn, Hölder and Young inequalities we get

$$(4.9)\qquad \begin{aligned}&\frac{1}{2}\mu\frac{d}{dt}\|\overline{H}_{nt}\|^2_{0,\Pi}+\frac{1}{\sigma}\int_\Pi|\mathrm{rot}_u(\overline{H}_{nt})|^2\,d\xi\\ &\quad\le\varphi(a)[\|\overline{H}_n\|^2_{1,\Pi}(\|\overline{u}_t\|^2_{1,\Pi}+\|\overline{u}\|^2_{2,\Pi})\\ &\qquad+\varepsilon\|\overline{H}_{nt}\|^2_{1,\Pi}+\|\overline{u}_\xi\|^2_{\infty,\Pi}\|\overline{H}_n\|^2_{1,\Pi}+\|\overline{H}_{nt}\|^2_{0,\Pi}\|\overline{u}_\xi\|^2_{\infty,\Pi}]\\ &\qquad+c\|\overline{E}_{*t}\|^2_{0,B}+c\|\overline{E}_*\|^2_{\infty,B}\|\overline{u}\|^2_{2,\Pi}.\end{aligned}$$

Integrating (4.9) with respect to time using (4.3) we get (4.8). From (4.1), (4.5), (4.8) we have

LEMMA 4.4. *Let the assumptions of Lemmas* 4.1–4.3 *be satisfied. Then*

$$(4.10)\qquad \begin{aligned}&\|\overline{H}_n\|^2_{1,2,2,\Pi^t}+\|\overline{H}_{nt}\|^2_{1,2,2,\Pi^t}+\|\overline{H}_{nt}\|^2_{0,2,\infty,\Pi^t}+\|\overline{H}_n\|^2_{1,2,\infty,\Pi^t}\\ &\quad\le\alpha(a,t)[(\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t}+c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Pi^t}+\|\overline{u}(0)\|^2_{0,\Pi}))\|\overline{E}_*\|^2_{1,2,\infty,B^t}\\ &\qquad+\|\overline{H}_n(0)\|^2_{1,\Pi}+\|\overline{H}_{nt}(0)\|^2_{0,\Pi}+\|\overline{E}_{*t}\|^2_{0,2,2,B^t}+\|\overline{H}_*\|^2_{1,2,2,B^t}\\ &\qquad+\|\overline{H}_*\|^2_{1,2,\infty,B^t}+\|\overline{H}_{*t}\|^2_{1,2,2,B^t}]\equiv\overline{G},\end{aligned}$$

where α *is an increasing positive function.*

Now choosing a subsequence and letting $n \to \infty$ we get

LEMMA 4.5. *Assume that* $\overline{u} \in L_2(0,T,H^3(\Pi))$; $\overline{u}_t \in L_2(0,T,H^2(\Pi))$; $\overline{u}_{tt} \in L_2(0,T,L_2(\Pi))$; $\overline{H}_*, \overline{H}_{*t} \in L_2(0,T,H^1(B))$; $\overline{H}_*, \overline{E}_* \in L_\infty(0,T,H^1(B))$; $\overline{E}_{*t} \in L_2(0,T,L_2(B))$; $\overline{H}(0) \in H^1(\Pi)$; $\overline{H}_t(0) \in L_2(\Pi)$. *Then there exists a weak solution of problem* (3.3) *such that*

$$\overline{H} \in L_\infty(0,T,H^1(\Pi)) \cap L_2(0,T,H^1(\Pi)),$$
$$\overline{H}_t \in L_2(0,T,H^1(\Pi)) \cap L_\infty(0,T,L_2(\Pi))$$

and

$$\|\overline{H}\|^2_{1,2,2,\Pi^t} + \|\overline{H}_t\|^2_{1,2,2,\Pi^t} + \|\overline{H}_t\|^2_{0,2,\infty,\Pi^t} + \|\overline{H}\|^2_{1,2,\infty,\Pi^t} \le \overline{G}. \tag{4.11}$$

Next we introduce a partition of unity $(\widetilde{\Omega}_i, \{\xi_i\})$ with $\Pi = \Omega \cup D = \bigcup_i \widetilde{\Omega}_i$. Let $\widetilde{\Omega}$ be one of the $\widetilde{\Omega}_i$ and $\zeta(\xi) = \zeta_i(\xi)$ the corresponding function. If $\widetilde{\Omega} \subset \Omega$ or $\widetilde{\Omega} \subset D$, let $\widetilde{\omega}$ be such that $\overline{\widetilde{\omega}} \subset \overline{\widetilde{\Omega}}$ and $\zeta(\xi) = 1$ for $\xi \in \widetilde{\omega}$.

In a boundary subdomain $\overline{\widetilde{\Omega}} \cap B \ne \emptyset$, $\overline{\widetilde{\omega}} \cap B \ne \emptyset$, $\overline{\widetilde{\omega}} \subset \overline{\widetilde{\Omega}}$ we introduce local coordinates $\{y\}$ connected with $\{\xi\}$ by

$$y_k = \alpha_{kl}(\xi_l - \beta_l), \qquad \alpha_{3k} = n_k(\beta), \qquad k = 1,2,3, \tag{4.12}$$

where $\beta \in \overline{\widetilde{\omega}} \cap B \subset \overline{\widetilde{\omega}} \cap B$, $\widetilde{B} = \widetilde{\Omega} \cap B$ and $\{\alpha_{kl}\}$ is a constant orthogonal matrix such that $\widetilde{B}$ is determined by $y_3 = F(y_1, y_2)$, $F \in H^{5/2}$ and

$$\widetilde{\Omega} = \{y : |y_i| < d,\ i = 1,2,\ F(y') < y_3 < F(y') + d,\ y' = (y_1, y_2)\}.$$

Next we introduce a function v' by

$$v'_i(y) = \alpha_{ij} v_j(\xi)|_{\xi = \xi(y)}, \tag{4.13}$$

where $\xi = \xi(y)$ is the inverse transformation to (4.12).

Further, we introduce new variables by

$$z_i = y_i, \qquad i = 1,2, \qquad z_3 = y_3 - \widetilde{F}(y), \quad y \in \widetilde{\Omega}, \tag{4.14}$$

which will be denoted by $z = \phi(y)$, where $\widetilde{F}$ is an extension of F to $\widetilde{\Omega}$ with $\widetilde{F} \in H^{5/2}(\widetilde{\Omega})$. Let $\widehat{\Omega} = \phi(\widehat{\Omega}) = \{z : |z_i| < d,\ i = 1,2,\ 0 < z_3 < d\}$ and $\widehat{B} = \phi(\widetilde{B})$. Define

$$\widehat{v}(z) = v'(y)|_{y = \phi^{-1}(z)}, \tag{4.15}$$

Introduce $\widehat{\nabla}_k = \xi_{lx_k}(\xi) z_{i\xi_l} \nabla_{z_i}|_{\xi = \chi^{-1}(z)}$, where $\chi(\xi) = \phi(\psi(\xi))$ and $y = \psi(\xi)$ is described by (4.12).

If $\widetilde{\Omega} \cap S \ne \emptyset$ we similarly introduce local coordinates, but now $\widetilde{\Omega} = \{y : |y_i| < d,\ i = 1,2,\ |F(y') - y_3| < d,\ y' = (y_1, y_2)\}$ and $\beta \in S \cap \widetilde{\Omega} = \widetilde{S}$.

We also introduce the following notation:

$$\widetilde{v}(\xi) = \overline{v}(\xi)\zeta(\xi) \qquad \text{for } \widetilde{\Omega} \cap S = \emptyset \text{ and } \widetilde{\Omega} \cap B = \emptyset, \tag{4.16}$$

$$\widetilde{v}(z) = \widehat{v}(z)\widehat{\zeta}(z) \qquad \text{for } z \in \widehat{\Omega} = \phi(\widetilde{\Omega}), \quad \overline{\widetilde{\Omega}} \cap B \ne 0 \text{ or } \widetilde{\Omega} \cap S \ne \emptyset, \tag{4.17}$$

where $\widehat{\zeta}(z) = \zeta(\xi)_{\xi = \chi^{-1}(z)}$ and $z' = \tau$, $z_3 = n$.

From (1.1), (1.4) and (1.5) we get

$$\left(\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{H}-\mu_1 u\times\overset{1}{H}\right)\cdot\tau_\alpha=\left(\frac{1}{\sigma_2}\operatorname{rot}\overset{2}{H}\right)\cdot\tau_\alpha,\qquad \alpha=1,2,\ \text{on } \widetilde{S}^T. \tag{4.18}$$

In local z coordinates, (4.18) has the form

$$\left(\frac{1}{\sigma_1}\,\widehat{\operatorname{rot}}\overset{1}{\widehat{H}}-\mu_1\widehat{u}\times\overset{1}{\widehat{H}}\right)\cdot\widehat{\tau}_\alpha=\left(\frac{1}{\sigma_2}\,\widehat{\operatorname{rot}}\overset{2}{\widehat{H}}\right)\cdot\widehat{\tau}_\alpha,\qquad \alpha=1,2,\ \text{on } \widetilde{S}^T.$$

Thus we get

$$\begin{aligned}&\frac{1}{A_2A_3}\left(\frac{1}{\sigma_2}\partial_3(\overset{2}{\widehat{H}}_2A_2)-\frac{1}{\sigma_1}\partial_3(\overset{1}{\widehat{H}}_2A_2)\right)=\mu_1(\widehat{u}\times\overset{1}{\widehat{H}})\cdot\frac{\widehat{\tau}_1}{|\widehat{\tau}_1|}\quad\text{on } \widetilde{S}^T,\\ &\frac{1}{A_1A_2}\left(\frac{1}{\sigma_1}\partial_3(\overset{1}{\widehat{H}}_1A_1)-\frac{1}{\sigma_2}\partial_3(\overset{2}{\widehat{H}}_1A_1)\right)=\mu_1(\widehat{u}\times\overset{1}{\widehat{H}})\cdot\frac{\widehat{\tau}_2}{|\widehat{\tau}_2|}\quad\text{on } \widetilde{S}^T,\end{aligned} \tag{4.19}$$

where A_i, $i=1,2,3$, are the Lamé coefficients of the transformation $z\mapsto x$ and from $(1.1)_5$, $(1.4)_3$, $(1.5)_2$ we have

$$\frac{1}{\sigma_1}\,\partial_3\overset{1}{\widehat{H}}_3-\frac{1}{\sigma_2}\,\partial_3\overset{2}{\widehat{H}}_3=0. \tag{4.19_1}$$

Under the above notation problem (1.1), (1.4) has the following form in a boundary subdomain $\widetilde{\Omega}\cap S\neq\emptyset$:

$$\begin{aligned}\mu_1\overset{1}{\widetilde{H}}_1-\frac{1}{\sigma_1}\,\widehat{\nabla}_u^2\overset{1}{\widetilde{H}}&=\mu_1((\overset{1}{\widetilde{H}}\cdot\widehat{\nabla}_u\widehat{u}-(\widetilde{u}\cdot\widehat{\nabla}_u)\overset{1}{\widehat{H}})+\mu_1\widehat{\nabla}_u\overset{1}{\widehat{H}}\widetilde{u}\\ &\quad+\frac{1}{\sigma_1}\,((\widehat{\nabla}_u\zeta)\cdot\widehat{\nabla}_u\overset{1}{\widehat{H}}+\widehat{\nabla}_u\cdot(\overset{1}{\widehat{H}}\widehat{\nabla}_u\zeta))\quad\text{in } \Omega\cap\widehat{\Omega},\end{aligned} \tag{4.20}$$

$$\begin{aligned}\mu_2\overset{2}{\widetilde{H}}_t-\frac{1}{\sigma_2}\,\widehat{\nabla}_u^2\overset{2}{\widetilde{H}}&=\frac{1}{\sigma_2}\,((\widehat{\nabla}_u\cdot\zeta)\widehat{\nabla}_u\overset{2}{\widehat{H}}+\widehat{\nabla}_u\cdot(\overset{2}{\widehat{H}}\widehat{\nabla}_u\zeta))\\ &\quad+\mu_2\widehat{\nabla}_u\overset{2}{\widehat{H}}\cdot\widetilde{u}\quad\text{in } D\cap\widehat{\Omega}.\end{aligned} \tag{4.21}$$

Multiplying (4.20) by $\overset{1}{\widetilde{H}}$ and (4.21) by $\overset{2}{\widetilde{H}}$, integrating respectively over $\Omega\cap\widehat{\Omega}$ and $D\cap\widehat{\Omega}$, adding the resulting equalities, using the transmission conditions (4.19), assuming that $|\widehat{\Omega}|$ is sufficiently small, integrating with respect to time and using the Gronwall inequality we get

$$\begin{aligned}\|\widetilde{H}\|^2_{0,2,\infty,\widehat{\Omega}^t}+\|\widetilde{H}\|^2_{1,2,2,\widehat{\Omega}^t}&\le\alpha(\widehat{a},t)[(\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}\\ &\quad+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))(1+\|\widehat{u}\|^2_{0,2,\infty,\widehat{\Omega}^t})+\|\widetilde{H}(0)\|^2_{0,\widehat{\Omega}}].\end{aligned} \tag{4.22}$$

Here and below, α is an increasing positive function.

Now differentiating (4.20), (4.21) with respect to τ, multiplying respectively by $\overset{1}{\widetilde{H}}_\tau$, $\overset{2}{\widetilde{H}}_\tau$, integrating over $\Omega\cap\widehat{\Omega}$, $D\cap\widehat{\Omega}$, adding the resulting equalities and using the transmission conditions (4.19) we get

$$
\begin{aligned}
(4.23)\qquad \frac{d}{dt}\|\widetilde{H}_\tau\|^2_{0,\widehat{\Omega}}+\|\widetilde{H}_\tau\|^2_{1,\widehat{\Omega}} &\le \varphi(\widehat{a})(\varepsilon\|\widetilde{H}_\tau\|^2_{1,\widehat{\Omega}}+c\|\widehat{u}\|^2_{2,\widehat{\Omega}}+c\|\widehat{u}\|^2_{2,\widehat{\Omega}}\|\widetilde{H}\|^2_{1,\widehat{\Omega}}\\
&\quad+\|\widehat{H}\|^2_{1,\widehat{\Omega}}\|\widetilde{u}_\tau\|^2_{1,\widehat{\Omega}}+\varepsilon\|\widehat{H}_\tau\|^2_{1,\widehat{\Omega}}+c\|\widehat{H}_\tau\|^2_{0,\widehat{\Omega}}+c\widehat{a}^2\|\widetilde{H}\|^2_{3,\widehat{\Omega}}\\
&\quad+c\widehat{a}(\|\widetilde{H}\|^2_{2,\widehat{\Omega}}+\|\widetilde{H}_\tau\|^2_{1,\widehat{\Omega}})+\|\widehat{u}\|^2_{2,\widehat{\Omega}}\|\widetilde{H}_\tau\|^2_{0,\widehat{\Omega}}).
\end{aligned}
$$

Integrating (4.23) with respect to time, using (4.22) and the Gronwall inequality we get

$$
\begin{aligned}
(4.24)\qquad \|\widetilde{H}_\tau\|^2_{0,2,\infty,\widehat{\Omega}^t}+\|\widetilde{H}_\tau\|^2_{1,2,2,\widehat{\Omega}^t} &\le \alpha(\widehat{a},t)[((\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}\\
&\quad+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))(\|\widehat{u}\|^2_{0,2,\infty,\widehat{\Omega}^t}+1)\\
&\quad+\|\widetilde{H}(0)\|^2_{0,\widehat{\Omega}})(\|\widehat{u}\|^2_{0,2,\infty,\widehat{\Omega}^t}+\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t})+c\widehat{a}\|\widetilde{H}\|^2_{2,2,2,\widehat{\Omega}^t}\\
&\quad+c\widehat{a}^2\|\widetilde{H}\|^2_{3,2,2,\widehat{\Omega}^t}+\|\widetilde{H}_\tau(0)\|^2_{0,\widehat{\Omega}}].
\end{aligned}
$$

Now we estimate $\|\widetilde{H}_{nn}\|^2_{0,2,2,\widehat{\Omega}^t}$. From (4.20), (4.21) we get

$$
\begin{aligned}
(4.25)\qquad \|\widetilde{H}_{nn}\|^2_{0,\widehat{\Omega}} &\le \varphi(\widehat{a})(\widehat{a}\|\widetilde{H}\|^2_{2,\widehat{\Omega}}+\|\widetilde{H}_\tau\|^2_{1,\widehat{\Omega}}+\|\widetilde{H}\|^2_{1,\widehat{\Omega}}\|\widehat{u}\|^2_{2,\widehat{\Omega}}\\
&\quad+\|\widetilde{u}\|^2_{2,\widehat{\Omega}}\|\widehat{H}\|^2_{1,\widehat{\Omega}}+\|\widehat{u}\|^2_{1,\widehat{\Omega}}+\|\widehat{H}\|^2_{1,\widehat{\Omega}})+\|\widetilde{H}_t\|^2_{0,\widehat{\Omega}}.
\end{aligned}
$$

Integrating (4.25) with respect to time we get

$$
\begin{aligned}
(4.26)\qquad \|\widetilde{H}_{nn}\|^2_{0,2,2,\widehat{\Omega}^t} &\le \alpha(\widehat{a})(c\widehat{a}\|\widetilde{H}\|^2_{2,2,2,\widehat{\Omega}^t}+\|\widetilde{H}_\tau\|^2_{1,2,2,\widehat{\Omega}^t}\\
&\quad+\|\widetilde{H}\|^2_{1,2,2,\widehat{\Omega}^t}\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+\|\widetilde{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}\|\widehat{H}\|^2_{1,2,2,\widehat{\Omega}^t}\\
&\quad+\|\widehat{u}\|^2_{1,2,2,\widehat{\Omega}^t}+\|\widehat{H}\|^2_{1,2,2,\widehat{\Omega}^t})+\|\widetilde{H}_t\|^2_{0,2,2,\widehat{\Omega}^t}.
\end{aligned}
$$

Now we estimate $\|\widetilde{H}_t\|^2_{0,2,2,\widehat{\Omega}^t}$. Multiplying (4.20), (4.21) respectively by $\overset{1}{\widetilde{H}}_t$, $\overset{2}{\widetilde{H}}_t$, integrating over $\Omega\cap\widehat{\Omega}$, $D\cap\widehat{\Omega}$ using the transmission conditions (4.19) and integrating with respect to time we get

$$
\begin{aligned}
(4.27)\qquad \|\widetilde{H}_t\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widetilde{H}\|^2_{1,2,\infty,\widehat{\Omega}^t} &\le \varphi(\widehat{a},t)[(1+\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t})(\|\widehat{H}\|^2_{1,2,2,\widehat{\Omega}^t}\\
&\quad+\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))+\|\widetilde{H}(0)\|^2_{1,\widehat{\Omega}}].
\end{aligned}
$$

Then from (4.22), (4.26), (4.27) we get

$$
\begin{aligned}
(4.28)\qquad \|\widetilde{H}_{nn}\|^2_{0,2,2,\widehat{\Omega}^t} &\le \alpha(\widehat{a},t)[(\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}}+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))\\
&\quad\cdot(\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+1)+\|\widetilde{H}(0)\|^2_{0,\widehat{\Omega}})(t\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{2,\widehat{\Omega}}+1)\\
&\quad+c\widehat{a}\|\widetilde{H}\|^2_{2,2,2,\widehat{\Omega}^t}+c\widehat{a}^2\|\widetilde{H}\|^2_{3,2,2,\widehat{\Omega}^t}+\|\widetilde{H}(0)\|^2_{1,\widehat{\Omega}}].
\end{aligned}
$$

Now differentiating (4.20), (4.21) with respect to τ, multiplying respectively by $\overset{1}{\widetilde{H}}_{\tau\tau}$, $\overset{2}{\widetilde{H}}_{\tau\tau}$, integrating over $\widehat{\Omega}\cap\Omega$, $\widehat{\Omega}\cap D$ and integrating with respect to time using the transmission conditions (4.19) we get

$$
\begin{aligned}
(4.29)\quad & \|\widetilde{H}_{\tau\tau}\|^2_{0,2,\infty,\widehat{\Omega}^t}+\|\widetilde{H}_{\tau\tau}\|^2_{1,2,2,\widehat{\Omega}^t}\\
&\le \alpha(\widehat{a},t)[((1+\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t})(\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}^t}\\
&\quad+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))(\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+1)+\|\widetilde{H}(0)\|^2_{0,\widehat{\Omega}})(t\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}\\
&\quad+\|\widehat{u}(0)\|^2_{2,\Omega}+1)^2+c\widehat{a}^2\|\widetilde{H}\|^2_{3,2,2,\widehat{\Omega}^t}+c\widehat{a}\|\widetilde{H}\|^2_{2,2,2,\widehat{\Omega}^t}\\
&\quad+c\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+\|\widetilde{H}(0)\|^2_{1,\widehat{\Omega}}].
\end{aligned}
$$

Now we estimate $\widetilde{H}_{nn\tau}$. Differentiating (4.20), (4.21) with respect to τ, multiplying respectively by $\overset{1}{\widetilde{H}}_{nn\tau}$, $\overset{2}{\widetilde{H}}_{nn\tau}$, integrating over $\widehat{\Omega}\cap\Omega$, $\widehat{\Omega}\cap D$ and with respect to time we get

$$
\begin{aligned}
(4.30)\quad \|\widetilde{H}_{nn\tau}\|^2_{0,2,2,\widehat{\Omega}^t} &\le \alpha(\widehat{a},t)[((1+\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t})(\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}\\
&\quad+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))(\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+1)\\
&\quad+\|\widetilde{H}(0)\|^2_{0,\widehat{\Omega}})(t\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{2,\widehat{\Omega}}+1)^2\\
&\quad+c(\widehat{a}+\widehat{a}^2)\|\widetilde{H}\|^2_{3,2,2,\widehat{\Omega}^t}+c\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+\|\widetilde{H}(0)\|^2_{1,\widehat{\Omega}}\\
&\quad+\|\widetilde{H}_t\|^2_{1,2,2,\widehat{\Omega}^t}].
\end{aligned}
$$

Next we estimate $\|\widetilde{H}_t\|^2_{1,2,2,\widehat{\Omega}^t}$. Differentiating (4.20), (4.21) with respect to time, multiplying respectively by $\overset{1}{\widetilde{H}}_t$, $\overset{2}{\widetilde{H}}_t$, integrating over $\widehat{\Omega}\cap\Omega$, $\widehat{\Omega}\cap D$ and with respect to time, and using the transmission conditions (4.19) we get

$$
\begin{aligned}
(4.31)\quad & \|\widetilde{H}_t\|^2_{0,2,\infty,\widehat{\Omega}^t}+\|\widetilde{H}_t\|^2_{1,2,2,\widehat{\Omega}^t}\le\alpha(\widehat{a},t)[((1+\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}\\
&+\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}+\|\widehat{u}\|^4_{2,2,\infty,\widehat{\Omega}^t}+\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}+\|\widehat{u}_{tt}\|^2_{0,2,2,\widehat{\Omega}^t})\\
&\cdot(\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))(\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+1)\\
&+\|\widetilde{H}(0)\|^2_{0,\widehat{\Omega}})(t\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{2,\widehat{\Omega}}+1)^2\\
&+c(\widehat{a}+\widehat{a}^2)\|\widetilde{H}\|^2_{3,2,2,\widehat{\Omega}^t}+c\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+\varepsilon\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}\\
&+c(\varepsilon)t(\|\widehat{u}_{tt}\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}})+\|\widetilde{H}(0)\|^2_{1,\widehat{\Omega}}].
\end{aligned}
$$

Now we estimate $\|\widetilde{H}_{nnn}\|^2_{0,2,2,\widehat{\Omega}^t}$. Differentiating (4.20), (4.21) with respect to n and multiplying respectively by $\overset{1}{\widetilde{H}}_{nnn}$, $\overset{2}{\widetilde{H}}_{nnn}$, from the above inequalities we get

$$
\begin{aligned}
(4.32)\quad \|\widetilde{H}_{nnn}\|^2_{0,2,2,\widehat{\Omega}^t} &\le \alpha(\widehat{a},t)[((1+\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}\\
&+\|\widehat{u}\|^4_{2,2,\infty,\widehat{\Omega}^t}+\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t})(\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}^t}\\
&+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))(\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+1)+|\widetilde{H}(0)\|^2_{0,\widehat{\Omega}})(t\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}\\
&+\|\widehat{u}(0)\|^2_{2,\widehat{\Omega}}+1)^2+c(\widehat{a}+\widehat{a}^2)\|\widetilde{H}\|^2_{3,2,2,\widehat{\Omega}^t}+c\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}\\
&+\varepsilon\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}+c(\varepsilon)t(\|\widehat{u}_{tt}\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widehat{u}_t(0)\|^2_{0,\widehat{\Omega}})+\|\widetilde{H}(0)\|^2_{1,\widehat{\Omega}}].
\end{aligned}
$$

Adding the above inequalities we get the estimates

$$
\begin{aligned}
(4.33)\quad \|\widetilde{H}\|^2_{3,2,2,\widehat{\Omega}^t} &\le \alpha(\widehat{a},t)[((1+\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}\\
&+\|\widehat{u}\|^4_{2,2,\infty,\widehat{\Omega}^t}+\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t})(\varepsilon\|\widehat{u}\|^2_{3,2,2,\widehat{\Omega}^t}+c(\varepsilon)t(\|\widehat{u}_t\|^2_{0,2,2,\widehat{\Omega}^t}\\
&+\|\widehat{u}(0)\|^2_{0,\widehat{\Omega}}))(\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+1)+\|\widetilde{H}(0)\|^2_{0,\widehat{\Omega}})(t\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}\\
&+\|\widehat{u}(0)\|^2_{2,\widehat{\Omega}}+1)^2+c\|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t}+\varepsilon\|\widehat{u}_t\|^2_{2,2,2,\widehat{\Omega}^t}\\
&+c(\varepsilon)t(\|\widehat{u}_{tt}\|^2_{0,2,2,\widehat{\Omega}^t}+\|\widehat{u}_t(0)\|^2_{0,\widehat{\Omega}})+\|\widetilde{H}(0)\|^2_{1,\widehat{\Omega}}].
\end{aligned}
$$

Now we consider interior subdomains. Let $\widetilde{\Omega}\subset\Omega$. Then as above we prove

$$
\begin{aligned}
(4.34)\quad \|\overset{1}{\widetilde{H}}\|^2_{3,2,2,\widetilde{\Omega}^t} &\le \alpha(a,t)[((1+\|\overline{u}\|^2_{2,2,\infty\widetilde{\Omega}^t}+\|\overline{u}\|^2_{3,2,2,\widetilde{\Omega}^t}\\
&+\|\overline{u}\|^4_{2,2,\infty,\widetilde{\Omega}^t})(\varepsilon\|\overline{u}\|^2_{3,2,2,\widetilde{\Omega}^t}+c(\varepsilon)t(\|\overline{u}\|^2_{0,2,2,\widetilde{\Omega}^t}\\
&+\|\overline{u}(0)\|^2_{0,\widetilde{\Omega}}))(\|\overline{u}\|^2_{2,2,\infty,\widetilde{\Omega}^t}+1)+\|\overset{1}{\widetilde{H}}(0)\|^2_{0,\widetilde{\Omega}})(t\|\overline{u}_t\|^2_{2,2,2,\widetilde{\Omega}^t}\\
&+\|\overline{u}(0)\|^2_{2,\widetilde{\Omega}}+1)^2+c\|\overline{u}\|^2_{2,2,\infty,\widetilde{\Omega}^t}+\varepsilon\|\overline{u}_t\|^2_{2,2,2,\widetilde{\Omega}^t}\\
&+c(\varepsilon)t(\|\overline{u}_{tt}\|^2_{0,2,2,\widetilde{\Omega}^t}+\|\overline{u}_t(0)\|^2_{0,\widetilde{\Omega}})+\|\overset{1}{\widetilde{H}}(0)\|^2_{1,\widetilde{\Omega}}].
\end{aligned}
$$

Now let $\widetilde{\Omega}\subset D$. Then similarly to (4.32) we prove

$$
(4.35)\quad \|\overset{2}{\widetilde{H}}\|^2_{3,2,2,\widetilde{\Omega}^t}\le \alpha(a,t)[(1+\|\overline{u}\|^2_{2,2,\infty,\widetilde{\Omega}^t})\|\overset{2}{\widetilde{H}}(0)\|^2_{0,\widetilde{\Omega}}+\|\overset{2}{\widetilde{H}}(0)\|^2_{1,\widetilde{\Omega}}].
$$

Now we consider the following problem in boundary subdomains $\overline{\widetilde{\Omega}} \cap B \neq \emptyset$:

$$
(4.36)\qquad \begin{cases} \mu_2(\widetilde{H}_* + \widetilde{H}^*)_t - \dfrac{1}{\sigma_2}\,\widehat{\nabla}_u^2(\widetilde{H}_* + \widetilde{H}^*) \\ \qquad = \dfrac{1}{\sigma_2}\,(\widehat{\nabla}_u \cdot \zeta)\widehat{\nabla}(\widehat{H}_* + \widehat{H}^*) + \widehat{\nabla}_u \cdot ((\widehat{H}_* + \widehat{H}^*)\widehat{\nabla}_u \xi), \\ \widetilde{H}_* = \overset{2}{\widetilde{H}} - \widetilde{H}^*|_B = 0, \end{cases}
$$

where $\widetilde{H}^*$ is a given function. Then we get the system

$$
(4.37)\qquad \begin{cases} \mu_2 \widetilde{H}_{*t} - \dfrac{1}{\sigma_2}\,\widehat{\nabla}_u^2 \widetilde{H}_* = \dfrac{1}{\sigma_2}\,((\widehat{\nabla}_u \cdot \zeta)\widehat{\nabla}_u \widehat{H}_* + \widehat{\nabla}_u \cdot (\widehat{H}_* \widehat{\nabla}_u \zeta) \\ \qquad + \dfrac{1}{\sigma_2}\,(\widehat{\nabla}_u \cdot \zeta)\widehat{\nabla}_u \widehat{H}^* + \widehat{\nabla}_u \cdot (\widehat{H}^* \widehat{\nabla}_u \zeta)) - \mu_2 \widetilde{H}_t^* + \dfrac{1}{\sigma_2}\,\widehat{\nabla}_u^2 \widetilde{H}^*, \\ \widetilde{H}_*|_B = 0. \end{cases}
$$

Then as above we get

$$
(4.38)\qquad \begin{aligned} \|\widetilde{H}_*\|^2_{3,2,2,\widehat{\Omega}^t} &\le \alpha(\widehat{a}, t)[(1 + \|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t})(\|\widetilde{H}_*(0)\|^2_{0,\widehat{\Omega}} + \|\widetilde{H}^*\|^2_{3,2,2,\widehat{\Omega}^t}) \\ &\quad + \|\widetilde{H}_t^*\|^2_{2,2,2,\widehat{\Omega}^t} + \|\widetilde{H}_{tt}^*\|^2_{0,2,2,\widehat{\Omega}^t} \\ &\quad + \|\widetilde{H}_{*\tau\tau}\|^2_{0,2,2,\widehat{B}^t} + \|\widetilde{H}^*\|^2_{3,2,2,\widehat{B}^t}]. \end{aligned}
$$

Then from the trace theorem we get

$$
(4.39)\qquad \begin{aligned} \|\overset{2}{\widetilde{H}}\|^2_{3,2,2,\widehat{\Omega}^t} &\le \alpha(\widehat{a}, t)[(1 + \|\widehat{u}\|^2_{2,2,\infty,\widehat{\Omega}^t})(\|\overset{2}{\widetilde{H}}(0)\|^2_{0,\widehat{\Omega}} + \|\overset{2}{\widetilde{H}}\|^2_{3,2,2,\widehat{B}^t} \\ &\quad + \|\overset{2}{\widetilde{H}}(0)\|^2_{0,\widehat{B}}) + \|\overset{2}{\widetilde{H}}_t\|^2_{2,2,2,\widehat{B}^t} + \|\overset{2}{\widetilde{H}}_{tt}\|^2_{0,2,2,\widehat{B}^t}]. \end{aligned}
$$

Summing the inequalities (4.33), (4.34), (4.35), (4.39) and going back to the variables ξ we get

Lemma 4.6. *Let the assumptions of Lemma 4.5 be satisfied and $\overline{H}_* \in L_2(0,T,H^3(B))$; $\overline{H}_{*t} \in L_2(0,T,H^2(B))$; $\overline{H}_{*tt} \in L_2(0,T,L_2(B))$. Then*

$$
(4.40)\qquad \begin{aligned} \|\overline{H}\|^2_{3,2,2,\Pi^t} &\le \alpha(a,t)[((1 + \|\overline{u}\|^2_{2,2,\infty,\Pi^t} + \|\overline{u}\|^2_{3,2,2,\Pi^t} \\ &\quad + \|\overline{u}\|^4_{2,2,\infty,\Pi^t} + \|\overline{u}_t\|^2_{2,2,2,\Pi^t})(\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t} + c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Pi^t} \\ &\quad + \|\overline{u}(0)\|^2_{0,\Pi}))(\|\overline{u}\|^2_{2,2,\infty,\Pi^t} + 1) + \|\overline{H}(0)\|^2_{0,\Pi} + \|\overline{H}\|^2_{3,2,2,B^t} \\ &\quad + \|\overline{u}(0)\|^2_{2,\Pi} + 1)^2 + ct\|\overline{u}_t\|^2_{2,2,2,\Pi^t} + \|\overline{H}(0)\|^2_{0,B})(t\|\overline{u}_t\|^2_{2,2,2,\Pi^t} \\ &\quad + \|\overline{u}(0)\|^2_{2,\Pi} + \varepsilon\|\overline{u}_t\|^2_{2,2,2,\Pi^t} + c(\varepsilon)t(\|\overline{u}_{tt}\|^2_{0,2,2,\Pi^t} + \|\overline{u}_t(0)\|^2_{0,\Pi}) \\ &\quad + \|\overline{H}_t\|^2_{2,2,2,B^t} + \|\overline{H}_{tt}\|^2_{0,2,2,B^t} + \|\overline{H}(0)\|^2_{1,\Pi}]. \end{aligned}
$$

where α is an increasing positive function.

Next we have to obtain estimate for $\|\overline{H}_{tt}\|^2_{1,2,2,\varPi^t}$. Let $\phi_n \in L_2(0,T,H^3(\varPi)) \cap L_\infty(0,T,H^1(\varPi))$, $\phi_{nt} \in L_\infty(0,T,H^1(\varPi) \cap L_2(0,T,H^2(\varPi))$ and $\phi_n = \sum_{k=1}^n C_k(t)\varphi_k(\xi) \in A_n$ where $\{\varphi_n\}$ is an orthonormal basis in $H^3(\varPi)$. Then $\bigcup_{n=1}^\infty A_n$ is dense in $L_2(0,T,H^3(\varPi)) \cap L_\infty(0,T,H^1(\varPi))$ and $\bigcup_{n=1}^\infty A_{nt}$ is dense in $L_2(0,T,H^2(\varPi)) \cap L_\infty(0,T,H^1(\varPi))$.

Let $\overline{H}_n \in \bigcup_{k=1}^n A_k$. Then from the weak solution to the linearized problem (1.13) after letting $n \to \infty$ we get

LEMMA 4.7. *Let the assumptions of Lemmas* 4.5–4.6 *be satisfied and* $\overline{E}_*, \overline{E}_{*tt} \in L_2(0,T,L_2(B))$; $\overline{u}_{tt} \in L_2(0,T,H^1(\varPi))$; $\overline{u}_t \in L_\infty(0,T,H^1(\varPi))$; $\overline{u} \in L_\infty(0,T,H^2(\varPi))$ *and* $E_{*tt} \in L_2(0,T,L_2(B))$. *Then*

$$\begin{aligned}(4.41)\qquad \|\overline{H}_{tt}\|^2_{0,2,\infty,\varPi^t} &+ \|\overline{H}_{tt}\|^2_{1,2,2,\varPi^t} \le \alpha(a,t)[\varepsilon(\|\overline{u}_{tt}\|^4_{1,2,2,\varPi^t}\\
&+ \|\overline{u}_t\|^4_{1,2,\infty,\varPi^t}\|\overline{u}\|^4_{2,2,\infty,\varPi^t} + \|\overline{u}_t\|^4_{1,2,\infty,\varPi^t} + \|\overline{u}\|^4_{2,2,\infty,\varPi^t}\\
&+ \|\overline{u}_{tt}\|^4_{1,2,2,\varPi^t})(t^2\|\overline{H}_t\|^4_{1,2,2,\varPi^t} + \|\overline{H}(0)\|^4_{1,\varPi}) + (\varepsilon\|\overline{u}\|^2_{3,2,2,\varPi^t}\\
&+ c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\varPi^t} + \|\overline{u}(0)\|^2_{0,\varPi}))(\|\overline{u}_t\|^2_{1,2,\infty,\varPi^t} + 1)\|\overline{H}_{tt}(0)\|^2_{0,\varPi}\\
&+ \|\overline{u}_t\|^2_{1,2,\infty,\varPi^t}(\varepsilon\|\overline{H}\|^2_{3,2,2,\varPi^t} + c(\varepsilon)t(\|\overline{H}_t\|^2_{0,2,2,\varPi^t} + \|\overline{H}(0)\|^2_{0,\varPi}))\\
&+ \|E_*\|^2_{0,2,2,B^t} + \|E_{*tt}\|^2_{0,2,2,B^t}].\end{aligned}$$

where α is an increasing positive function.

Now we estimate $\|\overline{H}_t\|^2_{2,2,2,\varPi^t}$. As in (4.40) we prove

LEMMA 4.8. *Let the assumptions of Lemmas* 4.5–4.7 *be satisfied. Then*

$$\begin{aligned}(4.42)\qquad \|\overline{H}_t\|^2_{2,2,2,\varPi^t} &\le \alpha(a,t)[(\varepsilon(\|\overline{H}\|^2_{3,2,2,\varPi^t} + \|\overline{H}\|^2_{3,2,2,B^t} + \|\overline{u}_t\|^2_{2,2,2,\varPi^t})\\
&+ c(\varepsilon)t(\|\overline{H}_t\|^2_{0,2,2,\varPi^t} + \|\overline{H}_{tt}\|^2_{0,2,2,\varPi^t} + \|\overline{H}_t\|^2_{0,2,2,B^t} + \|\overline{H}_{tt}\|^2_{0,2,2,B^t}\\
&+ \|\overline{u}_t\|^2_{0,2,2,\varPi^t} + \|\overline{u}_{tt}\|^2_{0,2,2,\varPi^t} + \|\overline{u}(0)\|^2_{0,\varPi} + \|\overline{u}_t(0)\|^2_{0,\varPi} + \|\overline{H}(0)\|^2_{0,\varPi}\\
&+ \|\overline{H}_t(0)\|^2_{0,\varPi}))(\|\overline{u}_t\|^2_{2,2,2,\varPi^t} + \|\overline{u}_t\|^2_{1,2,\infty,\varPi^t} + \|\overline{u}(0)\|^2_{2,\varPi}\\
&+ \|\overline{H}(0)\|^2_{2,\varPi}) + \varepsilon\|\overline{u}_t\|^4_{2,2,2,\varPi^t} + \|\overline{H}_t\|^4_{1,2,2,\varPi^t} + \|\overline{H}_t\|^2_{1,2,\infty,B^t}\\
&+ \|\overline{H}_t\|^4_{1,2,2,B^t} + \|\overline{H}_t\|^2_{1,2,2,\varPi^t} + \|\overline{H}_t\|^2_{1,2,2,B^t} + (t\|\overline{u}_t\|^2_{2,2,2,\varPi^t}\\
&+ \|\overline{u}(0)\|^2_{2,\varPi})\|\overline{H}\|^2_{3,2,2,\varPi^t}].\end{aligned}$$

where α is an increasing positive function.

Then from Lemmas 4.5–4.8 we deduce that

$$\begin{aligned}&\overline{H} \in L_2(0,T,H^3(\varPi)) \cap L_\infty(0,T,H^1(\varPi));\\
&\overline{H}_t \in L_\infty(0,T,H^1(\varPi)) \cap L_2(0,T,H^2(\varPi));\\
&\overline{H}_{tt} \in L_\infty(0,T,L_2(\varPi)) \cap L_2(0,T,H^1(\varPi)).\end{aligned}$$

5. Existence of solution of (1.1)–(1.7). We prove the existence of a solution of (1.1)–(1.7) by the method of successive approximations determined by the problem

$$
\begin{aligned}
&\overline{v}_{m+1t} - \operatorname{div}_{\overline{v}_m} \mathbb{T}_{\overline{v}_m}(\overline{v}_{m+1}, \overline{p}_{m+1}) \\
&\qquad = \mu_1(\overset{1}{\overline{H}}_m \nabla_{\overline{v}_m} \overset{1}{\overline{H}}_m - \nabla_{\overline{v}_m} \overset{1}{\overline{H}}{}^2_m) + \overline{f} && \text{in } \Omega^T,\\
&\operatorname{div}_{\overline{v}_m} \overline{v}_{m+1} = 0 && \text{in } \Omega^T,\\
&\mu_1 \overset{1}{\overline{H}}_{m+1,t} = -\operatorname{rot}_{\overline{v}_m} \overset{1}{\overline{E}}_m + \mu_1 \nabla_{\overline{v}_m} \overset{1}{\overline{H}}_{m+1} \overline{v}_m && \text{in } \Omega^T,\\
&\operatorname{rot}_{\overline{v}_m} \overset{1}{\overline{H}}_{m+1} = \sigma_1(\overset{1}{\overline{E}}_m + \mu_1 \overline{v}_m \times \overset{1}{\overline{H}}_{m+1}) && \text{in } \Omega^T,\\
&\operatorname{div}_{\overline{v}_m}(\mu_1 \overset{1}{\overline{H}}_{m+1}) = 0 && \text{in } \Omega^T,
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&\overline{n}_{\overline{v}_n} \mathbb{T}_{\overline{v}_n}(\overline{v}_{m+1}, \overline{p}_{m+1}) = -p_0 \overline{n}_{\overline{v}_m} && \text{on } S^T,\\
&\frac{1}{\sigma_1} \overset{1}{\overline{H}}_{m+1} = \frac{1}{\sigma_2} \overset{2}{\overline{H}}_{m+1} && \text{on } S^T,\\
&\overset{1}{\overline{E}}_m \cdot \overline{\tau}_{\alpha \overline{v}_m} = \overset{2}{\overline{E}}_m \cdot \overline{\tau}_{\alpha \overline{v}_m},\ \alpha = 1, 2, && \text{on } S^T,\\
&\overline{v}_{m+1}|_{t=0} = \overline{v}_0,\ \overset{1}{\overline{H}}_{m+1}|_{t=0} = \overset{1}{\overline{H}}_0 && \text{in } \Omega,
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
&\mu_2 \overset{2}{\overline{H}}_{m+1,t} = -\operatorname{rot}_{\overline{v}_m} \overset{2}{\overline{E}}_m + \mu_2 \nabla_{\overline{v}_m} \overset{2}{\overline{H}}_{m+1} \overline{v}_m && \text{in } D^T,\\
&\sigma_2 \overset{2}{\overline{E}}_m = \operatorname{rot}_{\overline{v}_m} \overset{2}{\overline{H}}_{m+1} && \text{in } D^T,\\
&\operatorname{div}_{\overline{v}_m}(\mu_2 \overset{2}{\overline{H}}_{m+1}) = 0 && \text{in } D^T,
\end{aligned}
\tag{5.3}
$$

where in (5.3) $\overline{v}_m$ is an extension of the function $\overline{v}_m$ from (5.1) onto D^T, such that $\overline{v}_m \to 0$ as $\xi \to B$,

$$
\begin{aligned}
&\overset{2}{\overline{H}}_{m+1} = \overline{H}_*, \quad \overset{2}{\overline{E}}_m = \overline{E}_* \quad \text{on } B^T,\\
&\overset{2}{\overline{H}}_{m+1}|_{t=0} = \overset{2}{\overline{H}}_0 \quad \text{in } D.
\end{aligned}
\tag{5.4}
$$

First we show the boundedness of the sequence described by (5.1)–(5.4) in the norm

$$
\begin{aligned}
\beta_m(t) = {}& \|\overline{v}_{mt}\|^2_{1,\Omega} + \|\overline{v}_m\|^2_{1,\Omega} + \|\overline{v}_m\|^2_{3,2,2,\Omega^t} + \|\overline{v}_{mt}\|^2_{2,2,2,\Omega^t}\\
&+ \|\overline{v}_{mtt}\|^2_{1,2,2,\Omega^t} + \|\overline{p}_m\|^2_{2,2,2,\Omega^t} + \|\overline{p}_{mt}\|^2_{1,2,2,\Omega^t} + \|\overline{H}_{mt}\|^2_{1,\Pi}\\
&+ \|\overline{H}_m\|^2_{1,\Pi} + \|\overline{H}\|^2_{3,2,2,\Pi^t} + \|\overline{H}_{mt}\|^2_{2,2,2,\Pi^t} + \|\overline{H}_{mtt}\|^2_{1,2,2,\Pi^t}.
\end{aligned}
\tag{5.5}
$$

LEMMA 5.1. *There exists a positive function P and $A > 0$ such that $P(0,0,F_0) < A$, $\beta_m(0) < A$, and there exists T_* such that $\beta_m(t) \le A$ for $t \le T_*$, $m = 1,2,\ldots$, where F_0 is a function depending on $p_0, \overline{H}_*, \overline{E}_*, \overline{H}(0)$, $\overline{v}(0), \overline{u}(0), \overline{f}$ in the norms from Lemmas* 3.1–4.8.

Proof. From Lemmas 3.1–4.8 assuming that $\varepsilon = \sqrt{t}$ and $\overline{u} = \overline{v}_m$, $\overline{H}' = \overline{H}_m$, using the fact that

$$\|\overline{u}\|^2_{2,2,\infty,\Omega^t} \le ct(\|\overline{u}_t\|^2_{2,2,2,\Omega^t} + \|\overline{u}(0)\|^2_{2,\Omega}),$$
$$\|\overline{H}'\|^2_{2,2,\infty,\Pi^t} \le ct(\|\overline{H}'_t\|^2_{2,2,2,\Pi^t} + \|\overline{H}(0)\|^2_{2,\Pi}),$$

we get

$$\beta_{m+1}(t) \le P(t, t^\gamma \beta_m(t), F_0), \tag{5.6}$$

where $\gamma > 0$. Let A be such that $P(0,0,F_0) < A$. Since P is a continuous function there exists $T_* > 0$ such that for $t < T_*$ we get

$$P(t, t^\gamma A, F_0) \le A. \tag{5.7}$$

From (5.7) we see that if $\beta_m(t) \le A$ then $\beta_{m+1}(t) \le A$.

Now we prove the convergence of the sequence $\{\overline{v}_m, \overline{H}_m\}$. Set

$$\overline{\mathcal{V}}_m = \overline{v}_m - \overline{v}_{m-1}, \quad \overline{\mathcal{H}}_m = \overline{H}_m - \overline{H}_{m-1}, \quad \mathcal{P}_m = \overline{p}_m - \overline{p}_{m-1},$$

From (5.1)–(5.4), we obtain the following system of problems:

$$\begin{aligned}\overline{\mathcal{V}}_{m+1,t} &- \operatorname{div}_{\overline{v}_m} \mathbb{D}_{\overline{v}_m}(\overline{\mathcal{V}}_{m+1}) - \nabla_{\overline{v}_m}\overline{\mathcal{P}}_{m+1}\\
&= \operatorname{div}_{\overline{v}_m}[\mathbb{D}_{\overline{v}_m} - \mathbb{D}_{\overline{v}_{m-1}}](\overline{v}_m) + [\operatorname{div}_{\overline{v}_m} - \operatorname{div}_{\overline{v}_{m-1}}]\mathbb{D}_{\overline{v}_m}(\overline{v}_m)\\
&\quad + [\nabla_{\overline{v}_m} - \nabla_{\overline{v}_{m-1}}]\overline{P}_m - \mu_1 \overset{1}{\overline{H}}_m[\nabla_{\overline{v}_m} - \nabla_{\overline{v}_{m-1}}]\overset{1}{\overline{H}}_m\\
&\quad - \mu_1 \overset{1}{\overline{H}}_{m-1}\nabla_{\overline{v}_{m-1}}\overset{1}{\overline{\mathcal{H}}}_m - \mu_1\overset{1}{\overline{\mathcal{H}}}_m\nabla_{\overline{v}_{m-1}}\overset{1}{\overline{H}}_m\\
&\quad + \mu_1\nabla_{\overline{v}_m}\overset{1}{\overline{H}}_m\overset{1}{\overline{\mathcal{H}}}_m + \mu_1[\nabla_{\overline{v}_m} - \nabla_{\overline{v}_{m-1}}]\overset{1}{\overline{H}}_m\overset{1}{\overline{H}}_{m-1}\\
&\quad + \mu_1\nabla_{\overline{v}_{m-1}}\overset{1}{\overline{\mathcal{H}}}_{m-1}\overset{1}{\overline{H}}_{m-1} \equiv F^*,\end{aligned} \tag{5.8}$$

$$\begin{aligned}\mu\overline{\mathcal{H}}_{m+1,t} &- \frac{1}{\sigma}\operatorname{rot}_{\overline{v}_m}\operatorname{rot}_{\overline{v}_m}\overline{\mathcal{H}}_{m+1} = [\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}]\operatorname{rot}_{\overline{v}_m}\overline{H}_m\\
&+ \operatorname{rot}_{\overline{v}_{m-1}}[\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}]\overline{H}_m + \operatorname{rot}_{\overline{v}_m}(\overline{v}_m \times \overline{\mathcal{H}}_{m+1})\\
&+ \operatorname{rot}_{\overline{v}_m}(\overline{\mathcal{V}}_m \times \overline{H}_m) + [\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}](\overline{v}_{m-1} \times \overline{H}_m)\\
&+ [\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}]\operatorname{rot}_{\overline{v}_m}\overline{H}_m + \operatorname{rot}_{\overline{v}_{m-1}}[\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}]\overline{H}_m\\
&+ \nabla_{\overline{v}_m}\overline{\mathcal{H}}_{m+1}\overline{v}_m + (\nabla_{\overline{v}_m} - \nabla_{\overline{v}_{m-1}})\overline{H}_m\overline{\mathcal{V}}_m \equiv K^*\end{aligned} \tag{5.9}$$

and

$$(5.10)\quad \overline{n}_{\overline{v}_m}\mathbb{T}_{\overline{v}_m}(\mathcal{V}_{m+1},\overline{\mathcal{P}}_{m+1}) = [\overline{n}_{\overline{v}_m}-\overline{n}_{\overline{v}_{m-1}}]\mathbb{D}_{\overline{v}_m}(\overline{v}_m) + \overline{n}_{\overline{v}_{m-1}}[\mathbb{D}_{\overline{v}_m}-\mathbb{D}_{\overline{v}_{m-1}}]\overline{v}_m+[\overline{n}_{\overline{v}_m}-\overline{n}_{\overline{v}_{m-1}}]\overline{p}_m + p_0[\overline{n}_{\overline{v}_m}-\overline{n}_{\overline{v}_{m-1}}] \equiv G^*.$$

LEMMA 5.2. *Let the assumptions of Lemma 5.1 be satisfied. Then there exists T_{**} sufficiently small such that*

$$(5.11)\quad \|\overline{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Omega^t} + \|\overline{\mathcal{V}}_{m+1,t}\|^2_{0,2,2,\Omega^t} + \|\overline{\mathcal{V}}_{m+1}\|^2_{1,\Omega} + \|\mathcal{P}_{m+1}\|^2_{1,2,2,\Omega^t} + \|\overline{\mathcal{H}}_{m+1}\|^2_{0,\Pi} + \|\overline{\mathcal{H}}_{m+1}\|^2_{1,2,2,\Pi^t} \le \alpha(A,t)t(t+1)(\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Pi^t} + \|\overline{\mathcal{H}}_m\|^2_{1,2,2,\Pi^t}),$$

where $\alpha(A,t)t(t+1) < 1$ for $t < T_{**}$, and α is an increasing positive function.

Proof. Multiplying (5.8) by $\overline{\mathcal{V}}_{m+1}$ and integrating over Ω we get

$$(5.12)\quad \frac{1}{2}\int_\Omega \frac{d}{dt}\overline{\mathcal{V}}^2_{m+1}\,d\xi + \int_\Omega |\mathbb{D}_{\overline{v}_m}(\overline{\mathcal{V}}_{m+1})|^2\,d\xi = \int_\Omega F^*\overline{\mathcal{V}}_{m+1}d\xi + \int_S G^*\overline{\mathcal{V}}_{m+1}\,d\xi_S.$$

We have proved that

$$(5.13)\quad \int_{\Omega^t} |F^*\overline{\mathcal{V}}_{m+1}|\,d\xi\,dt \le \alpha(A)t(t+1)(\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}) + \varepsilon\|\overline{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t},$$

$$(5.14)\quad \int_{S^t} |G^*\overline{\mathcal{V}}_{m+1}|\,d\xi_S\,dt \le \alpha(A)t(t+1)\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \varepsilon\|\overline{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t}.$$

Applying to (5.12) the Korn inequality, integrating with respect to time and using (5.13), (5.14) we get

$$(5.15)\quad \|\overline{\mathcal{V}}_{m+1}\|^2_{0,\Omega} + \|\overline{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t} \le \alpha(A)t(t+1)(\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}).$$

Now using in (5.9) local z coordinates, and (4.19), as in (4.22) we get

$$(5.16)\quad \|\overline{\mathcal{H}}_{m+1}\|^2_{0,2,\infty,\Pi^t} + \|\overline{\mathcal{H}}_{m+1}\|^2_{1,2,2,\Pi^t} \le \alpha(A,t) + (t+1)\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Pi^t}.$$

Multiplying (5.8) by $\overline{\mathcal{V}}_{m+1,t}$, integrating over Ω and using Young and Hölder inequalities we get

$$(5.17)\quad \|\overline{\mathcal{V}}_{m+1,t}\|^2_{0,\Omega} + \frac{d}{dt}\int_\Omega |\mathbb{D}_{\overline{v}_m}(\overline{\mathcal{V}}_{m+1})|^2\,d\xi \le c\|\overline{v}_m\|^2_{3,\Omega}\int_\Omega |\mathbb{D}_{\overline{v}_m}(\overline{\mathcal{V}}_{m+1})|^2\,d\xi + \frac{d}{dt}\int_S G^*\overline{\mathcal{V}}_{m+1}\,d\xi_S + \|G^*_t\|^2_{0,S} + \|\overline{\mathcal{V}}_{m+1}\|^2_{1,\Omega} + \|F^*\|^2_{0,\Omega} + \|\overline{v}_m\|^2_{3,\Omega}\|G^*\|^2_{0,S}.$$

Integrating (5.17) with respect to time and using the Gronwall and Korn inequalities we get

$$(5.18)\quad \|\overline{\mathcal{V}}_{m+1t}\|^2_{0,2,2,\Omega^t}+\|\overline{\mathcal{V}}_{m+1}\|^2_{1,\Omega}\le\alpha(A,t)(\|G^*_t\|^2_{0,2,2,S^t}+\|\overline{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t} + \|F^*\|^2_{0,2,2,\Omega^t}+c\|G^*\|^2_{0,2,\infty,S^t}+\varepsilon\|\overline{\mathcal{V}}_{m+1}\|^2_{1,2,\infty,\Omega^t}).$$

We have proved that

$$(5.19)\quad \|F^*\|^2_{0,2,2,\Omega^t}\le\alpha(A)t(t+1)(\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t}+\|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}),$$

$$(5.20)\quad \|G^*\|^2_{0,S}\le\alpha(A)t(t+1)\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t},$$

$$(5.21)\quad \|G^*_t\|^2_{0,2,2,S^t}\le\alpha(A)t(t+1)\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t}.$$

Using (5.19)–(5.21) in (5.18) we get

$$(5.22)\quad \|\overline{\mathcal{V}}_{m+1,t}\|^2_{0,2,2,\Omega^t}+\|\overline{\mathcal{V}}_{m+1}\|^2_{1,\Omega}\le\alpha(A,t)[t(t+1)(\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t})+\|\overline{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t}+\varepsilon\|\overline{\mathcal{V}}_{m+1}\|^2_{1,2,\infty,\Omega^t}],$$

where α is an increasing positive function.

Now applying the regularization technique to the elliptic problem (cf. [4])

$$\begin{aligned} -\operatorname{div}_{\overline{v}_m}\mathbb{T}(\overline{\mathcal{V}}_{m+1},\overline{\mathcal{P}}_{m+1})&=F^*+\overline{\mathcal{V}}_{m+1,t} && \text{in } \Omega^T,\\ \overline{n}_{\overline{v}_m}\mathbb{T}_{\overline{v}_m}(\overline{\mathcal{V}}_{m+1},\overline{\mathcal{P}}_{m+1})&=G^* && \text{on } S^T,\\ \overline{\mathcal{V}}_{m+1}\big|_{t=0}&=0 && \text{in } \Omega,\end{aligned}$$

we get

$$(5.23)\quad \|\overline{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Omega^t}+\|\overline{\mathcal{P}}_{m+1}\|^2_{1,2,2,\Omega^t} \le c(\|F^*\|^2_{0,2,2,\Omega^t}+\|G^*\|^2_{1/2,2,2,S^t}+\|\overline{\mathcal{V}}_{m+1,t}\|^2_{0,2,2,\Omega^t}).$$

We have proved that

$$(5.24)\quad \|G^*\|^2_{1/2,2,2,S^t}\le\alpha(A)t(t+1)\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t}.$$

Now using in (5.23) inequalities (5.19), (5.24) we get

$$(5.25)\quad \|\overline{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Omega^t}+\|\overline{\mathcal{P}}_{m+1}\|^2_{1,2,2,\Omega^t} \le\alpha(A)t(t+1)(\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t}+\|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t})+c\|\overline{\mathcal{V}}_{m+1t}\|^2_{0,2,2,\Omega^t}.$$

Now summing up inequalities (5.15), (5.22), (5.25) we get

$$(5.26)\quad \|\overline{\mathcal{V}}_{m+1,t}\|^2_{0,2,2,\Omega^t}+\|\overline{\mathcal{V}}_{m+1}\|^2_{1,\Omega}+\|\overline{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Omega^t}+\|\overline{\mathcal{P}}_{m+1}\|^2_{1,2,2,\Omega^t} \le\alpha(A,t)t(t+1)[\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t}+\|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}].$$

Combining (5.16)–(5.26) we obtain (5.11).

From Lemmas 5.1 and 5.2 we have

THEOREM 5.1. *Let the assumptions of Lemmas* 5.1 *and* 5.2 *be satisfied. Then there exists* T^{**} *sufficiently small such that for* $T\le T^{**}$ *there exists* *a solution to problem* (1.1)–(1.7) *such that*

$$\overline{v} \in L_2(0,T,H^3(\Omega)) \cap L_\infty(0,T,H^1(\Omega));$$
$$\overline{v}_t \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^2(\Omega));$$
$$\overline{v}_{tt} \in L_2(T,0,H^1(\Omega)); \quad \overline{p} \in L_2(0,T,H^2(\Omega));$$
$$p_t \in L_2(0,T,H^1(\Omega)); \quad \overline{H} \in L_2(0,T,H^3(\Pi)) \cap L_\infty(0,T,H^1(\Pi));$$
$$\overline{H}_t \in L_\infty(0,T,H^1(\Pi)) \cap L_2(0,T,H^2(\Pi)); \quad \overline{H}_{tt} \in L_2(0,T,H^1(\Pi))$$

and

$$\begin{aligned}\|\overline{v}_t\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{3,2,2,\Omega^T} + \|\overline{v}_t\|^2_{2,2,2,\Omega^T} + \|\overline{v}_{tt}\|^2_{1,2,2,\Omega^T} \\ + \|\overline{p}\|^2_{2,2,2,\Omega^T} + \|\overline{p}_t\|^2_{1,2,2,\Omega^T} + \|\overline{H}_t\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{1,2,\infty,\Pi^T} \\ + \|\overline{H}\|^2_{3,2,2,\Pi^T} + \|\overline{H}_t\|^2_{2,2,2,\Pi^T} + \|\overline{H}_{tt}\|^2_{1,2,2,\Pi^T} \le A,\end{aligned}$$

where A is defined in Lemma 5.1*.*

Appendix

Lemma. *Let $\overset{1}{\overline{H}}(t) \in H^1(\Omega)$ and $\overset{2}{\overline{H}}(t) \in H^1(D)$ for $t \in [0,T]$ and suppose assumptions* (1.1)–(1.7) *are satisfied. Then*

$$\|\overline{H}\|_{1,\Pi} \le c(\|\operatorname{rot}\overline{H}\|_{0,\Pi} + \|\overline{H}_*\|_{1,B}). \tag{1}$$

Proof. Using the transmission conditions we get the problems

$$\begin{aligned}
&\operatorname{rot}\overset{1}{\overline{H}} = \operatorname{rot}\overset{1}{\overline{H}} && \text{in } \Omega,\\
&\operatorname{rot}\overset{2}{\overline{H}} = \operatorname{rot}\overset{2}{\overline{H}} && \text{in } D,\\
&\operatorname{div}\overset{1}{\overline{H}} = 0 && \text{in } \Omega,\\
&\operatorname{div}\overset{2}{\overline{H}} = 0 && \text{in } D,\\
&\frac{1}{\sigma_1}\,\overline{n}\overset{1}{\overline{H}}|_S = \frac{1}{\sigma_2}\,\overline{n}\overset{2}{\overline{H}}|_S && \text{on } S.
\end{aligned}\tag{2}$$

From [8, pp. 20–21] solutions of (2) will be sought in the form

$$\overset{1}{\overline{H}} = \nabla\overset{1}{\varphi} + \overset{1}{u}, \quad \overset{2}{\overline{H}} = \nabla\overset{2}{\varphi} + \overset{2}{u}, \tag{3}$$

where $\overset{1}{\varphi}$, $\overset{2}{\varphi}$ are solutions of the Neumann problems

$$\begin{aligned}
&\Delta\overset{1}{\varphi} = 0, && \Delta\overset{2}{\varphi} = 0,\\
&\frac{\partial\overset{1}{\varphi}}{\partial\overline{n}}\bigg|_S = \frac{\sigma_1}{\sigma_2}\,\overline{n}\overset{2}{\overline{H}}, && \frac{\partial\overset{2}{\varphi}}{\partial\overline{n}}\bigg|_S = \overline{n}\overset{1}{\overline{H}}, \qquad \frac{\partial\overset{2}{\varphi}}{\partial\overline{n}}\bigg|_B = \overline{n}\overline{H}_*
\end{aligned}\tag{4}$$

and $\overset{1}{u}$, $\overset{2}{u}$ are solutions of the problems

$$(5)\qquad \begin{aligned} &\operatorname{rot}\overset{1}{u} = \operatorname{rot}\overset{1}{\overline{\overline{H}}}, && \operatorname{rot}\overset{2}{u} = \operatorname{rot}\overset{2}{\overline{\overline{H}}},\\ &\operatorname{div}\overset{1}{u} = 0, && \operatorname{div}\overset{2}{u} = 0,\\ &\overline{n}\overset{1}{u}|_{\partial\Omega} = 0, && \overline{n}\overset{2}{u}|_{\partial D} = 0. \end{aligned}$$

There are vectors $\overset{1}{e}$, $\overset{2}{e}$ such that

$$(6)\qquad \overset{1}{u} = \operatorname{rot}\overset{1}{e}, \quad \operatorname{div}\overset{1}{e} = 0, \quad \overline{\tau}\cdot\overset{1}{e}|_S = 0,$$

where $\overline{\tau}$ is any tangent vector to S, and

$$(7)\qquad \overset{2}{u} = \operatorname{rot}\overset{2}{e}, \quad \operatorname{div}\overset{2}{e} = 0, \quad \overline{\tau}\cdot\overset{2}{e}|_{\partial D} = 0,$$

where $\overline{\tau}$ is any tangent vector to ∂D. Thus problems (6), (7) can be replaced by problems

$$(8)\qquad \begin{aligned} &-\Delta\overset{1}{e} = \operatorname{rot}\overset{1}{\overline{\overline{H}}}, && -\Delta\overset{2}{e} = \operatorname{rot}\overset{2}{\overline{\overline{H}}},\\ &\overline{\tau}\cdot\overset{1}{e}|_S = 0, && \overline{\tau}\cdot\overset{2}{e}|_{\partial D} = 0,\\ &\operatorname{div}\overset{1}{e}|_S = 0, && \operatorname{div}\overset{2}{e}|_{\partial D} = 0. \end{aligned}$$

From (2), (4) we get

$$(9)\qquad \varphi = \overset{1}{\varphi} + \overset{2}{\varphi} = \int_B \frac{\partial}{\partial n}(\overline{n}\overline{H}_*)\,\frac{1}{|\xi - x|}\,dx_B.$$

Using the partition of unity and local coordinates we write (8) in the form

$$(10)\qquad -\widehat{\Delta}_u\overset{k}{\widetilde{e}} = \widehat{\zeta}\,\widehat{\operatorname{rot}}_u\,\overset{k}{\overline{\overline{H}}} + 2\widehat{\nabla}_u\overset{k}{\widehat{e}}\,\widehat{\nabla}_u\widehat{\zeta} + \overset{k}{\widehat{e}}\,\widehat{\Delta}_u\widehat{\zeta},$$

$$(11)\qquad \overset{k}{\widetilde{e}}\cdot\widehat{\tau}|_{n=0} = 0, \quad \widehat{\operatorname{div}}_u\,\overset{k}{\widetilde{e}}|_{n=0} = \widehat{\nabla}_u\widehat{\zeta}\,\overset{k}{\widehat{e}}, \quad k = 1,2,$$

where $\overset{k}{\widetilde{e}}(z) = \overset{k}{\widehat{e}}(z)\widehat{\zeta}(z)$, $k = 1,2$.

Now applying to (10), (11) Theorem 10.2 of [1], summing the inequalities using Lemma 10.5 of [8] and next going back to variables ξ we get

$$(12)\qquad \|\overset{1}{e}\|_{2,\Omega} \le c\|\operatorname{rot}\overset{1}{\overline{\overline{H}}}\|_{0,\Omega}, \quad \|\overset{2}{e}\|_{2,D} \le c\|\operatorname{rot}\overset{2}{\overline{\overline{H}}}\|_{0,D}.$$

Now from (6), (7), we get

$$(13)\qquad \begin{aligned} &\|\overset{1}{u}\|_{1,\Omega} \le c\|\overset{1}{e}\|_{2,\Omega} \le c\|\operatorname{rot}\overset{1}{\overline{\overline{H}}}\|_{0,\Omega},\\ &\|\overset{2}{u}\|_{1,D} \le c\|\overset{2}{e}\|_{2,D} \le c\|\operatorname{rot}\overset{2}{\overline{\overline{H}}}\|_{0,D}. \end{aligned}$$

Then finally from (9), (13) we have

$$\|\overline{H}\|_{1,\Pi} = \|\nabla\varphi + u\|_{1,\Pi} \le c(\|\nabla\varphi\|_{1,\Pi} + \|u\|_{1,\Pi})$$
$$\le c(\|\nabla\varphi\|_{1,\Pi} + \|\operatorname{rot}\overline{H}\|_{0,\Pi}) \le c(\|\overline{H}_*\|_{1,B} + \|\operatorname{rot}\overline{H}\|_{0,\Pi}).$$

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Institute of Mathematics and Cryptology
Cybernetics Faculty
Military University of Technology
S. Kaliskiego 2
00-908 Warszawa, Poland
E-mail: p.kacprzyk@imbo.wat.edu.pl

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