APPLICATIONES MATHEMATICAE
31,2 (2004), pp. 175–199

Jarosław L. Bojarski (Warszawa)

GENERAL METHOD OF REGULARIZATION. I: FUNCTIONALS DEFINED ON BD SPACE

Abstract. The aim of this paper is to prove that the relaxation of the elastic-perfectly plastic energy (of a solid made of a Hencky material) is the lower semicontinuous regularization of the plastic energy. We find the integral representation of a non-locally coercive functional. In part II, we will show that the set of solutions of the relaxed problem is equal to the set of solutions of the relaxed problem proposed by Suquet. Moreover, we will prove the existence theorem for the limit analysis problem.

1. Introduction. In this paper we investigate the convex functional

$$BD \ni \mathbf{u} \mapsto \mathbb{B}(\boldsymbol{\varepsilon}(\mathbf{u})) = \int_{\Omega} h(x, \boldsymbol{\varepsilon}(\mathbf{u})) \tag{1.1}$$

with constraints on the boundary of Ω, where $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetrized gradient of $\mathbf{u}$ and $BD(\Omega)$ is the space of bounded deformations (cf. (2.1) and (2.2)). Moreover, we assume that $\mathbb{B}(\boldsymbol{\varepsilon}(\mathbf{u})) = \infty$ if $\boldsymbol{\varepsilon}(\mathbf{u}) \notin L^1$. In [8] we find the *lower semicontinuous* (l.s.c.) relaxation of $\mathbb{B}$, and we show that the relaxation is a l.s.c. function (in the weak* BD topology), not greater than $\mathbb{B}$. Here we prove that this relaxation is the largest l.s.c. minorant less than $\mathbb{B}$, i.e. it is the l.s.c. regularization of $\mathbb{B}$ (cf. [18, p. 10]). If the volume forces are equal to 0, then we can omit the assumption of global coercivity of the functional considered (cf. Theorem 14 and Assumption 7).

The l.s.c. regularization (in the L^1_{loc}-topology) of functionals defined on the space $BV(\Omega)$ is investigated in many papers ([2], [3], [5], [20]), but their authors do not consider problems with constraints on the boundary of Ω.

2000 *Mathematics Subject Classification*: Primary 49J45; Secondary 49K99, 47N10, 47H04, 26B30, 74C05.

Key words and phrases: regularization (relaxation), functions of bounded deformation, Hencky plasticity.

Supported by Committee for Scientific Research (Poland) grant No. 2P03A00518.

In some contributions (cf. [5]) only a relaxation of the original problem is found, i.e. a l.s.c. minorant of the original functional.

Fonseca and Müller [20] find the l.s.c. regularization (in L^1) of a quasiconvex functional on $BV(\Omega)$. However, they neglect kinematic boundary conditions and assume local coercivity (if $f(x, \mathbf{u}, \mathbf{A}_1) = 0$ for $\mathbf{A}_1 \neq \mathbf{0}$ then $f(x, \mathbf{u}, \mathbf{A}) = 0$ for every $\mathbf{A}$; moreover, the function g in their condition (H3) is continuous).

In [6] an integral representation for the regularization in $SBV(\Omega, \mathbb{R}^m)$ of the functional

$$\mathbf{u} \mapsto \int_{\Omega} f(x, \nabla \mathbf{u}(x))\, dx + \int_{\Sigma(u)} \varphi(x, [\mathbf{u}](x), \boldsymbol{\nu}(x))\, dH_{N-1}(x) \tag{1.2}$$

with respect to the BV weak* convergence is obtained, where

$$c\|\mathbf{A}\| \le f(x, \mathbf{A}) \le C(1 + \|\mathbf{A}\|), \qquad c_1|\boldsymbol{\xi}| \le \varphi(x, \boldsymbol{\xi}, \boldsymbol{\nu}) \le C_1|\boldsymbol{\xi}| \tag{1.3}$$

for every x, $\mathbf{A}$, $\boldsymbol{\xi}$, with constants $C \ge c > 0$, $C_1 \ge c_1 > 0$. The kinematic boundary conditions are ignored.

In [7], the global method of relaxation (cf. [10]) is applied to l.s.c. regularization of symmetric-quasiconvex functionals, defined on $SBD(\Omega)$. The authors ignore the kinematic boundary condition (i.e. the Dirichlet condition). The essential assumption of the method is the local coercivity of the density of elastic-plastic energy (with work of external forces) (see assumption (1.3) above and [10, formula (2.3′), Theorems 3.7 and 3.12]). Note that the existence theorem is proved in the space $BD(\Omega)$, larger than $SBD(\Omega)$.

In [11], the global method of relaxation (cf. [10]) is applied to l.s.c. regularization of quasiconvex functionals with constraints (Dirichlet condition). These functionals are defined on $BV(\Omega)$. The constraints considered do not describe the relaxation proposed by Suquet (see [26] and part II of the paper). Here, similarly to [7], the essential assumption of the method is the local coercivity of the density of elastic-plastic energy (with work of external forces) (cf. assumption (1.3)).

Kohn and Temam [23] solve the existence problem for an elastic-perfectly plastic solid made of a homogeneous Hencky material. To prove that the functional of the total potential energy is weak* l.s.c. on $BD(\Omega)$, they use the method of relaxation of the kinematic boundary condition. They do not show that the relaxed problem is the l.s.c. regularization of the original problem. Indeed, in Theorem 6.1 of [27, Chapter 2] and Theorem 6.1 of [27, Chapter 1] only the equality of the infima of the relaxed and original problems is shown. But it is not proved that for every solution $\widehat{\mathbf{u}}$ of the relaxed problem there exists a sequence $\{\mathbf{u}_m\}_{m\in\mathbb{N}}$ which minimizes the original problem and $\mathbf{u}_m \rightharpoonup \widehat{\mathbf{u}}$ as $m \to \infty$.

The existence problem for an anisotropic elastic-plastic solid made of a non-homogeneous Hencky material with the Signorini constraints on the boundary (i.e. with the unilateral contact condition) is solved in [8]. The Signorini problem for an isotropic homogeneous body made of a Hencky material (with the von Mises plastic yield condition) is solved in [29].

In [14] the l.s.c. regularization of the elastic-plastic energy of a homogeneous Hencky material with the von Mises (or Tresca) yield condition is found. The work of external forces is neglected. The local coercivity of the relevant functional is assumed. Moreover, the kinematic boundary conditions are not studied in [14].

Here we prove that the relaxation (established in [8]) is the l.s.c. regularization of elastic-plastic energy if the volume force is equal to 0 and if Assumption 5 is satisfied (see Theorem 14). In this case we do not assume that the functional considered is coercive. Therefore, a body with cavities can be described by such a functional. Moreover, we can assume that the density of elastic-plastic energy has nonlinear growth at infinity, on a ray, and has linear growth on the complementary ray of the same straight line (cf. mechanics of soil).

It seems that this paper is the first one where the problem of regularization of a non-coercive functional, with the Dirichlet condition, is solved. Here the density of energy is not bounded from below.

In the special case when the integral of the total elastic-plastic energy is coercive, the relaxation is the l.s.c. regularization of the total energy in the weak* BD topology (cf. (5.5) and Theorem 18). That is, we prove that for every solution $\widehat{\mathbf{u}}$ of the relaxed problem $(RP^{**}_{\lambda,j})$, there exists a generalized sequence (net) $\{\mathbf{u}_m\}_{m\in H}$ which minimizes the original problem $(P_{\lambda,j})$ and $\mathbf{u}_m \rightharpoonup \widehat{\mathbf{u}}$ in weak* $BD(\Omega)$ topology (see (3.9)–(3.11), (5.1), (5.3), (5.4) and (5.8)).

We show that the set of solutions of the relaxed problem is equal to the set of solutions of the relaxed problem proposed by Suquet (see [26] and Theorem 11 in part II).

In [15] and [16] Christiansen has found the solution for the limit analysis problem, associated to the relaxed problem proposed by Suquet. But the limit analysis problem is not explicitly formulated in [15]. Also, the relation between solutions of the relaxed problem and solutions of the relaxed problem proposed by Suquet is not considered.

In Section 3 of part II, we obtain the existence theorem for the limit analysis problem, associated to the relaxed problem proposed by Suquet. In Corollary 10 of part II, we obtain a criterion of coercivity of the original problem $(P_{\lambda,j})$, or the relaxed problem $(RP^{**}_{\lambda,j})$ (see (3.9)–(3.11), (5.1), (5.3), (5.4) and (5.8)).

In the Appendix of part II, we show the scheme of duality in convex optimization in the case of Hencky plasticity.

2. Some basic definitions and theorems. Let Ω be a bounded, open, connected set of class C^1 in $\mathbb{R}^n$. The space of continuous functions with compact support is denoted by C_c. Let $C^\infty(\Omega,\mathbb{R}^m)$ be the space of $\mathbb{R}^m$-valued, infinitely differentiable functions. Moreover, the space of infinitely differentiable functions equal to 0 at the boundary $\operatorname{Fr}\Omega$ of Ω is denoted by $C_0^\infty(\Omega)$. Finally, $\mathbb{M}_b(\Omega,\mathbb{R}^m)$ is the space of $\mathbb{R}^m$-valued, Radon, bounded, regular measures on Ω, with the norm $\|\cdot\|_{\mathbb{M}_b(\Omega,\mathbb{R}^m)}$.

We will use the dual pairs $(\mathbb{M}_r, C_c)$ or $(\mathbb{M}_b, C_0)$, where $\mathbb{M}_r$ is the space of regular measures. The duality pairing will be denoted by $\langle\cdot,\cdot\rangle$, and the scalar product of $\mathbf{z},\mathbf{z}^*\in\mathbb{R}^n$ by $\mathbf{z}\cdot\mathbf{z}^*$ or $\mathbf{z}\mathbf{z}^*$. The scalar product of $\mathbf{w}$, $\mathbf{w}^*\in\mathbb{R}^{n\times n}$ is denoted by $\mathbf{w}:\mathbf{w}^* = w^{ij}w^*_{ij}$. Let $\mathbf{g}=(g_1,\ldots,g_m)\in C(\overline{\Omega},\mathbb{R}^m)$ and $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_m)\in\mathbb{M}_b(\Omega,\mathbb{R}^m)$. Then $\int_\Omega \mathbf{g}\cdot\boldsymbol{\mu}=\int_\Omega \mathbf{g}\boldsymbol{\mu}\equiv\sum_{i=1}^m\int_\Omega g_i\mu_i$. If $F: Y\to\mathbb{R}\cup\{\infty\}$, then F^* denotes its polar function (see [18]) $F^*(y^*)=\sup\{\langle y^*,y\rangle - F(y)\mid y\in Y\}$, and $\operatorname{dom}F=\{y\in Y\mid F(y)<\infty\}$ is the effective domain of F. If Q is a subset of Y, then $I_Q(\cdot)$ stands for its indicator function (taking the value 0 in Q and ∞ outside), and $I_Q^*(\cdot)$ stands for its support function.

Finally, we need the following notations. Let V be a metric space. Then $B_V(\Xi,r)$ is the closed ball in V with center Ξ and radius r. Furthermore, $\operatorname{cl}_V(Z)$ stands for the closure of $Z\subset V$ in the topology of the space V; analogously, $\operatorname{cl}_{\|\cdot\|}(Z)$ is the closure of the set Z in the norm $\|\cdot\|$. Similarly $\operatorname{int}Z$ denotes the interior of Z. We will also consider the spaces $\mathbf{E}^n$ of real $n\times n$ matrices and $\mathbf{E}^n_s$ of symmetric real $n\times n$ matrices. We set $\|[e_{ij}]\|_{\mathbf{E}^n}\equiv\sum_{i,j=1}^n|e_{ij}|$ and $\|\cdot\|_{\mathbf{E}^n_s}\equiv\|\cdot\|_{\mathbf{E}^n}$, where $[e_{ij}]\in\mathbf{E}^n$. We denote by $\otimes$ (resp. $\otimes_s$) the tensor product (resp. symmetric tensor product). Let $\mathcal{L}^0(\Omega,\mathbb{R}^m)_\mu$ be the set of μ-measurable functions from Ω into $\mathbb{R}^m$. If $\tau\subset 2^X$ is a linear topology in a vector space X, then $[X,\tau]$ denotes the topological space and $[X,\tau]^*$ is the space dual to $[X,\tau]$. We define the following Banach spaces (see [23], [27], [28]):

$$LD(\Omega)\equiv\Big\{\mathbf{u}\in L^1(\Omega,\mathbb{R}^n)\;\Big|\; \varepsilon_{ij}(\mathbf{u})\equiv\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)\in L^1(\Omega),\ i,j=1,\ldots,n\Big\}, \tag{2.1}$$

$$BD(\Omega)\equiv\{\mathbf{u}\in L^1(\Omega,\mathbb{R}^n)\mid \varepsilon_{ij}(\mathbf{u})\in\mathbb{M}_b(\Omega),\ i,j=1,\ldots,n\}, \tag{2.2}$$

with the natural norms

$$(2.3)\quad \|\mathbf{u}\|_{LD} = \|\mathbf{u}\|_{L^1} + \sum_{i,j=1}^{n} \|\varepsilon_{ij}(\mathbf{u})\|_{L^1},\ \|\mathbf{u}\|_{BD} = \|\mathbf{u}\|_{L^1} + \sum_{i,j=1}^{n} \|\varepsilon_{ij}(\mathbf{u})\|_{\mathbb{M}_b}.$$

$\mathcal{R}_0 \equiv \{\mathbf{u} \in BD(\Omega) \mid \boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}\}$ denotes the space of rigid motions in $\mathbb{R}^n$.

PROPOSITION 1 (see [27]). *Let $BD(\Omega)$ and $L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ be endowed with the norm topologies. There exists a continuous surjective linear trace $\boldsymbol{\gamma}_B$ from $BD(\Omega)$ into $L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ such that $\boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{u}_{|\mathrm{Fr}\,\Omega}$ for all $\mathbf{u} \in BD(\Omega) \cap C(\overline{\Omega}, \mathbb{R}^n)$.* ■

We define the spaces

$$(2.4)\qquad X \equiv C_c(\Omega, \mathbb{R}^n) \times C_c(\Omega, \mathbf{E}^n_s), \qquad X_0 \equiv \{(\mathbf{g}, \mathbf{h}) \in X \mid \mathbf{g} = \operatorname{div} \mathbf{h}\},$$

endowed with the natural norm

$$(2.5)\qquad \|\mathbf{g}\|_{C(\Omega,\mathbb{R}^n)} + \|\mathbf{h}\|_{C(\Omega,\mathbf{E}^n_s)} \equiv \sup\{\|\mathbf{g}(x)\|_{\mathbb{R}^n} \mid x \in \Omega\} + \sup\{\|\mathbf{h}(x)\|_{\mathbf{E}^n_s} \mid x \in \Omega\}.$$

Then $BD(\Omega)$ is isomorphic to the dual of $[X/X_0, \|\cdot\|_{C(\Omega,\mathbb{R}^n)} + \|\cdot\|_{C(\Omega,\mathbf{E}^n_s)}]$ (see [28]). The topology $\sigma((X/X_0)^*, X) = \sigma(BD(\Omega), C_c(\Omega, \mathbb{R}^n) \times C_c(\Omega, \mathbf{E}^n_s))$ is called the *weak* BD topology*. A net $\{\mathbf{u}_\delta\}_{\delta\in D} \subset BD(\Omega)$ is convergent to $\mathbf{u}_0 \in BD(\Omega)$ in this topology if and only if for all $(\mathbf{g}, \mathbf{h}) \in X$,

$$(2.6)\qquad \int_\Omega \mathbf{g} \cdot (\mathbf{u}_0 - \mathbf{u}_\delta)\, dx + \int_\Omega \mathbf{h} : \boldsymbol{\varepsilon}(\mathbf{u}_0 - \mathbf{u}_\delta) \to 0.$$

For every $\boldsymbol{\varphi} \in L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$, the set $\{\mathbf{u} \in BD(\Omega) \mid \boldsymbol{\gamma}_B(\mathbf{u}) = \boldsymbol{\varphi}\}$ is dense in the space $[BD(\Omega)$, weak* topology] (see [8, Proposition 2.5]). The trace operator $\boldsymbol{\gamma}_B$ is not continuous on $[BD(\Omega)$, weak* topology] if the space $L^1(\mathrm{Fr}\Omega, \mathbb{R}^n)$ is endowed with a Hausdorff topology (or a T_1-topology, see [19, Chapter I, Section 5] and [27]).

DEFINITION 1 (see [27] and [19, Chapter I, Section 6]). A net $\{\mathbf{u}_\delta\}_{\delta\in D}$ converges to $\mathbf{u}_0$ (in the topology (2.7)–(2.8)) if

$$(2.7)\quad \mathbf{u}_\delta \to \mathbf{u}_0 \quad \text{in } \|\cdot\|_{L^p(\Omega,\mathbb{R}^n)} \text{ for all } p \text{ such that } 1 \le p < q = n/(n-1) \text{ and weakly in } L^q(\Omega, \mathbb{R}^n)\ (q = \infty \text{ if } n = 1),$$

$$(2.8)\qquad \boldsymbol{\varepsilon}(\mathbf{u}_\delta) \to \boldsymbol{\varepsilon}(\mathbf{u}_0) \quad \text{weak}^* \text{ in } \mathbb{M}_b(\Omega, \mathbf{E}^n_s).$$

PROPOSITION 2 (cf. [8] and [9, Proposition 2]). *The weak* $BD(\Omega)$ topology and the topology* (2.7)–(2.8) *are equivalent on bounded subsets of $BD(\Omega)$.* ■

The injection of $[BD(\Omega)$, weak*] into $[L^p(\Omega, \mathbb{R}^n)$, weak topology] is continuous on bounded subsets of $BD(\Omega)$, where $1 \le p \le q = n/(n-1)$ ($q = \infty$ if $n = 1$).

We define the Banach space of measurable functions

$$(2.9)\qquad W^n(\Omega, \operatorname{div}) \equiv \{\boldsymbol{\sigma} \in L^\infty(\Omega, \mathbf{E}^n_s) \mid \operatorname{div} \boldsymbol{\sigma} \in L^n(\Omega, \mathbb{R}^n)\}$$

endowed with the natural norm $\|\boldsymbol{\sigma}\|_{W^n(\Omega,\mathrm{div})} = \|\boldsymbol{\sigma}\|_{L^\infty(\Omega,\mathbf{E}^n_s)} + \|\mathrm{div}\,\boldsymbol{\sigma}\|_{L^n(\Omega,\mathbb{R}^n)}$ (cf. [27, Chapter II, Section 7] and [8]). The distribution $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})$, where $\boldsymbol{\sigma} \in W^n(\Omega,\mathrm{div})$, $\mathbf{u} \in BD(\Omega)$, defined (for every $\varphi_1 \in C^\infty_c(\Omega)$) by

$$\langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}), \varphi_1\rangle_{D'\times D} = -\int_\Omega (\mathrm{div}\,\boldsymbol{\sigma})\cdot \mathbf{u}\varphi_1\,dx - \int_\Omega \boldsymbol{\sigma} : (\mathbf{u}\otimes \mathrm{grad}\,\varphi_1)\,dx, \tag{2.10}$$

is a bounded measure on Ω, and it is absolutely continuous with respect to $|\boldsymbol{\varepsilon}(\mathbf{u})|$ (see [27]).

ASSUMPTION 1. Ω and Ω_1 are bounded open connected sets of class C^1 in $\mathbb{R}^n$. Moreover, $\Omega \subset\subset \Omega_1$. ■

THEOREM 3 (cf. [27]). *There exists a continuous, linear, surjective, open map $\boldsymbol{\beta}_B$ from $[W^n(\Omega,\mathrm{div}), \|\cdot\|_{W^n(\Omega,\mathrm{div})}]$ onto $[L^\infty(\mathrm{Fr}\,\Omega,\mathbb{R}^n), \|\cdot\|_{L^\infty}]$ such that for every $\boldsymbol{\sigma} \in C(\overline{\Omega},\mathbf{E}^n_s)$, $\boldsymbol{\beta}_B(\boldsymbol{\sigma}) = \boldsymbol{\sigma}_{|\mathrm{Fr}\,\Omega}\cdot \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ denotes the exterior unit vector normal to $\mathrm{Fr}\,\Omega$. Furthermore, for all $\mathbf{u} \in BD(\Omega)$ and all $\boldsymbol{\sigma} \in W^n(\Omega,\mathrm{div})$, the following Green formula holds*:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) + \int_\Omega (\mathrm{div}\,\boldsymbol{\sigma})\cdot\mathbf{u}\,dx = \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\beta}_B(\boldsymbol{\sigma})\cdot \boldsymbol{\gamma}_B(\mathbf{u})\,ds. \tag{2.11}$$

3. Auxiliary theorems and spaces. In this paper, the Lebesgue and Hausdorff measures on Ω and $\mathrm{Fr}\,\Omega$ are denoted by dx and ds, respectively. Let Γ_0 and Γ_1 $(= \overline{\Gamma}_1)$ be Borel subsets of $\mathrm{Fr}\,\Omega$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $ds(\mathrm{Fr}\,\Omega - (\Gamma_0\cup\Gamma_1)) = 0$. We will consider an elastic-perfectly plastic body, occupying the given set Ω. We first introduce some functions. Let $\mathcal{K} : \overline{\Omega} \to 2^{\mathbf{E}^n_s}$ be a multifunction.

ASSUMPTION 2 (cf. [8]). $\mathcal{K}(x)$ is a convex closed subset of $\mathbf{E}^n_s$, for all $x \in \overline{\Omega}$. Moreover, there exists $\mathbf{z}_0 \in C^1(\overline{\Omega},\mathbf{E}^n_s)$ such that

$$\mathbf{z}_0(x) \in \mathcal{K}(x) \text{ for every } x \in \overline{\Omega} \tag{3.1}$$

and the following conditions hold:

(i) if $\mathbf{z}(x) \in \mathcal{K}(x)$ for dx-almost every (dx-a.e.) $x \in \Omega$, $\mathbf{z} \in C(\overline{\Omega},\mathbf{E}^n_s)$ and $\mathbf{z}_{|\mathrm{int}\,\Omega} \in W^n(\Omega,\mathrm{div})$, then $\mathbf{z}(y) \in \mathcal{K}(y)$ for every $y \in \overline{\Omega}$;

(ii) for every $y \in \overline{\Omega}$ and every $\mathbf{w} \in \mathcal{K}(y)$ there exists $\mathbf{z} \in C(\overline{\Omega},\mathbf{E}^n_s)$ such that $\mathbf{z}_{|\mathrm{int}\,\Omega} \in W^n(\Omega,\mathrm{div}), \mathbf{z}(y) = \mathbf{w}$ and $\mathbf{z}(x) \in \mathcal{K}(x)$ for every $x \in \overline{\Omega}$.

Conditions (i) and (ii) are equivalent to the condition that for every $y \in \overline{\Omega}$,

$$\mathcal{K}(y) = \{\mathbf{z}(y) \mid \mathbf{z} \in C(\overline{\Omega},\mathbf{E}^n_s),\ \mathbf{z}_{|\mathrm{int}\,\Omega} \in W^n(\Omega,\mathrm{div}),\ \mathbf{z}(x) \in \mathcal{K}(x) \text{ for } dx\text{-a.e. } x \in \Omega\}. \ ■ \tag{3.2}$$

Definition 2. Let $j^* : \Omega \times \mathbf{E}^n_s \to \mathbb{R} \cup \{\infty\}$ be a convex normal integrand, i.e.

(a) the function $\mathbf{E}^n_s \ni \mathbf{w}^* \mapsto j^*(x, \mathbf{w}^*)$ is convex and l.s.c. for dx-a.e. $x \in \Omega$;

(b) there exists a Borel function $\widetilde{j^*} : \Omega \times \mathbf{E}^n_s \to \mathbb{R} \cup \{\infty\}$ such that $\widetilde{j^*}(x, \cdot) = j^*(x, \cdot)$ for dx-a.e. $x \in \Omega$

(cf. [18, Chapter 8, p. 232]). Moreover, assume

$$\{\mathbf{w}^* \in \mathbf{E}^n_s \mid j^*(x, \mathbf{w}^*) < \infty\} = \mathcal{K}(x) \quad \text{for } dx\text{-a.e. } x \in \Omega. \tag{3.3}$$

Assumption 3. For every $\widehat{r} > 0$ there exists $c_{\widehat{r}}$ such that

$$\sup\Big\{ \int_\Omega j^*(x, \mathbf{z}^*)\, dx \;\Big|\; \mathbf{z}^* \in L^\infty(\Omega, \mathbf{E}^n_s),\ \|\mathbf{z}^*\|_{L^\infty} < \widehat{r} \text{ and } \mathbf{z}^*(x) \in \mathcal{K}(x) \text{ for } dx\text{-a.e. } x \in \Omega \Big\} < c_{\widehat{r}} < \infty. \tag{3.4}$$

Assumption 4. There exist $\mathbf{u}^e \in LD(\Omega)$ and $q \in L^1(\Omega, \mathbb{R})$ such that

$$j^*(x, \mathbf{w}^*) \geq \boldsymbol{\varepsilon}(\mathbf{u}^e)(x) : \mathbf{w}^* + q(x) \tag{3.5}$$

for dx-a.e. $x \in \Omega$ and every $\mathbf{w}^* \in \mathbf{E}^n_s$, and $\boldsymbol{\gamma}_B(\mathbf{u}^e) = \mathbf{0}$ on $\operatorname{Fr}\Omega$. ■

The set $\mathcal{K}(x)$ denotes the elasticity convex domain at the point x.

Define

$$j(x, \mathbf{w}) \equiv j^{**}(x, \mathbf{w}) \equiv \sup\{\mathbf{w} : \mathbf{w}^* - j^*(x, \mathbf{w}^*) \mid \mathbf{w}^* \in \mathbf{E}^n_s\} \tag{3.6}$$

for dx-a.e. $x \in \Omega$ and all $\mathbf{w} \in \mathbf{E}^n_s$. Then j is a convex normal integrand (cf. [18, Chapter 8, Proposition 1.2]). Define $j_\infty : \overline{\Omega} \times \mathbf{E}^n_s \to \mathbb{R} \cup \{\infty\}$ by

$$j_\infty(x, \mathbf{w}) \equiv \sup\{\mathbf{w} : \mathbf{w}^* - I_{\mathcal{K}(x)}(\mathbf{w}^*) \mid \mathbf{w}^* \in \mathbf{E}^n_s\} \tag{3.7}$$

for $x \in \overline{\Omega}$ and $\mathbf{w} \in \mathbf{E}^n_s$.

Let $\mathbf{f} \in L^n(\Omega, \mathbb{R}^n)$ and $\mathbf{g} \in L^\infty(\Gamma_1, \mathbb{R}^n)$. In this paper we consider the functional

$$BD(\Omega) \ni \mathbf{u} \mapsto P_{\lambda,j}(\mathbf{u}) = \lambda F(\mathbf{u}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u})), \tag{3.8}$$

where

$$\lambda F(\mathbf{u}) \equiv -\lambda L(\mathbf{u}) + I_{C_a(\mathbf{u}^0)}(\mathbf{u}), \quad L(\mathbf{u}) \equiv \int_\Omega \mathbf{f} \cdot \mathbf{u}\, dx + \int_{\Gamma_1} \mathbf{g} \cdot \boldsymbol{\gamma}_B(\mathbf{u})\, ds, \tag{3.9}$$

$\mathbf{u}^0 \in L^1(\Gamma_0, \mathbb{R}^n)$ and

$$C_a(\mathbf{u}^0) \equiv \{\mathbf{u} \in BD(\Omega) \mid \boldsymbol{\gamma}_B(\mathbf{u})_{|\Gamma_0} = \mathbf{u}^0\}. \tag{3.10}$$

The functional $G_j : \mathbb{M}_b(\Omega, \mathbf{E}^n_s) \to \mathbb{R} \cup \{\infty\}$ is given by

$$G_j(\boldsymbol{\mu}) \equiv \begin{cases} \int\limits_{\Omega} j(x, \boldsymbol{\mu})\,dx & \text{if } \boldsymbol{\mu} \in L^1(\Omega, \mathbf{E}^n_s), \text{ i.e. } \boldsymbol{\mu} \text{ is absolutely} \\ & \text{continuous with respect to } dx, \\ \infty & \text{otherwise.} \end{cases} \tag{3.11}$$

The formula (3.8) describes the total elastic-perfectly plastic energy of a body occupying the given subset Ω of $\mathbb{R}^n$. This body is subjected to volume forces $\mathbf{f} \in L^n(\Omega, \mathbb{R}^n)$ and boundary forces $\mathbf{g} \in L^\infty(\Gamma_1, \mathbb{R}^n)$. The constant $\lambda \geq 0$, $\lambda < \infty$ is the load multiplier (see [27, Chapter I, Section 4]). The set $C_a(\mathbf{u}^0)$ consists of the kinematically admissible displacement fields for the body clamped on Γ_0 (see [8] and [27]).

Assumption 5. There exists $\boldsymbol{\sigma}_0 \in C(\overline{\Omega}, \mathbf{E}^n_s)$ such that $\boldsymbol{\sigma}_{0|\mathrm{int}\,\Omega} \in W^n(\Omega, \mathrm{div})$, $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) = \lambda\mathbf{g}$ on Γ_1 and $\boldsymbol{\sigma}_0(x) \in \mathcal{K}(x)$ for dx-a.e. $x \in \Omega$. ■

By Assumption 5, the boundary force $\mathbf{g} \in L^\infty(\Gamma_1, \mathbb{R}^n)$ is a regular function.

Proposition 4 (see [27, p. 255]). *If* $\mathbf{u} \in BD(\Omega_1)$, *then*

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega} + \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega_1 - \overline{\Omega}} + (\boldsymbol{\gamma}^O_B(\mathbf{u}) - \boldsymbol{\gamma}^I_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}\,ds, \tag{3.12}$$

where the inside trace $\boldsymbol{\gamma}^I_B : BD(\Omega) \to L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ *and outside trace* $\boldsymbol{\gamma}^O_B : BD(\Omega_1 - \overline{\Omega}) \to L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ *are given by* $\boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{u}_{|\mathrm{Fr}\,\Omega}$ *for* $\mathbf{u} \in BD(\Omega) \cap C(\overline{\Omega}, \mathbb{R}^n)$ *and* $\boldsymbol{\gamma}^O_B(\mathbf{u}) = \mathbf{u}_{|\mathrm{Fr}\Omega}$ *for* $\mathbf{u} \in BD(\Omega_1 - \overline{\Omega}) \cap C(\Omega_1 - \Omega, \mathbb{R}^n)$, *respectively, and where* $\otimes_s$ *denotes the symmetric tensor product*: $(\mathbf{p} \otimes_s \boldsymbol{\nu})_{ij} \equiv (p_i\nu_j + p_j\nu_i)/2$.

Definition 3 (see [22]). A Borel set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a *Caccioppoli set* if $\sup\{\int_{\mathcal{C}} \mathrm{div}\,\widetilde{f}\,dx \mid \widetilde{f} \in C^1_0(\Omega_2, \mathbb{R}^n), \|\widetilde{f}(x)\|_{\mathbb{R}^n} \leq 1\ \forall x \in \Omega_2\} < \infty$ for all bounded open subsets Ω_2 of $\mathbb{R}^n$.

Remark 1. For every $\boldsymbol{\sigma} \in W^n(\Omega_1, \mathrm{div})$ and $\mathbf{u} \in BD(\Omega_1)$ the distribution $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})$ is a regular measure on Ω_1. Thus there exist sequences $\{\Omega^k_c\}_{k\in\mathbb{N}}$ and $\{\Omega^k_0\}_{k\in\mathbb{N}}$ of subsets of Ω_1 such that

$$\mathrm{cl}\,\Omega^k_c = \Omega^k_c \subset \mathrm{Fr}\,\Omega \subset \Omega^k_0 = \mathrm{int}\,\Omega^k_0, \qquad \forall k \in \mathbb{N}, \tag{3.13}$$

$$\text{if } k_1 < k_2 \text{ then } \Omega^{k_1}_c \subset \Omega^{k_2}_c \subset \Omega^{k_2}_0 \subset \Omega^{k_1}_0, \tag{3.14}$$

$$|\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})|(\Omega^k_0 - \Omega^k_c) < 1/k, \qquad \forall k \in \mathbb{N}. \tag{3.15}$$

Moreover, by Urysohn's Lemma [19, Theorem 1.5.10], for every $k \in \mathbb{N}$, there exists a continuous function $\psi_k : \Omega_1 \to [0,1]$ such that $\psi_k(x) = 1$ for $x \in \Omega^k_c$ and $\psi_k(x) = 0$ for $x \in \Omega_1 - \Omega^k_0$. Then for every $\varphi \in C_c(\Omega_1)$ we have $\int_{\mathrm{Fr}\,\Omega} \varphi\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) = \lim_{k\to\infty} \int_{\Omega_1} \psi_k\varphi\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})$ (cf. [4, Theorem 3.1]).

Lemma 5 (see [9, Lemma 5]). *If there exists a closed Caccioppoli set* $\mathcal{C} \subset \Omega_1$ *with* $\mathcal{C} = \mathrm{cl}\,\mathrm{int}\,\mathcal{C}$ *such that* $\Gamma_2 = \mathrm{Fr}\,\Omega \cap \mathcal{C}$, *with* $ds(\mathrm{Fr}\,\Omega \cap \mathrm{Fr}\,\mathcal{C}) = 0$, *then for all* $\mathbf{u} \in BD(\Omega_1)$ *and all* $\boldsymbol{\sigma} \in W^n(\Omega_1, \mathrm{div})$,

$$\int_{\Gamma_2} \boldsymbol{\beta}_B(\boldsymbol{\sigma}_{|\Omega}) \cdot (\boldsymbol{\gamma}_B^O(\mathbf{u}) - \boldsymbol{\gamma}_B^I(\mathbf{u}))\, ds = \int_{\Gamma_2} \boldsymbol{\sigma} : [(\boldsymbol{\gamma}_B^O(\mathbf{u}) - \boldsymbol{\gamma}_B^I(\mathbf{u})) \otimes_s \boldsymbol{\nu}]\, ds, \tag{3.16}$$

where we denote $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})_{|\operatorname{Fr}\Omega}$ *by* $\boldsymbol{\sigma} : [(\boldsymbol{\gamma}_B^O(\mathbf{u}) - \boldsymbol{\gamma}_B^I(\mathbf{u})) \otimes_s \boldsymbol{\nu}]\, ds$. ■

ASSUMPTION 6. Let $\Gamma_1 = \operatorname{Fr}\Omega \cap \mathcal{C}$, where $\mathcal{C} = \operatorname{cl}\operatorname{int}\mathcal{C} \subset \Omega_1$ is a closed Caccioppoli set and $ds(\operatorname{Fr}\Omega \cap \operatorname{Fr}\mathcal{C}) = 0$. ■

Let $\boldsymbol{\mu} \in \mathbb{M}_b(\Omega, \mathbf{E}_s^n)$. We recall that $|\boldsymbol{\mu}|$ is the total variation measure associated with $\boldsymbol{\mu}$, i.e. for every $\boldsymbol{\mu}$-measurable subset $\widetilde{\Omega}$ of Ω we have $|\boldsymbol{\mu}|(\widetilde{\Omega}) = \sup\{\int_{\widetilde{\Omega}} \boldsymbol{\varphi} : \boldsymbol{\mu} \mid \boldsymbol{\varphi} \in C(\overline{\Omega}, \mathbf{E}_s^n),\ \max_{i,j}(\|\varphi_{ij}\|_{C(\Omega)}) \le 1\}$. Then $\|\boldsymbol{\mu}\|_{\mathbb{M}_b(\Omega)} = \int_\Omega |\boldsymbol{\mu}|$. The density of $\boldsymbol{\mu}$ with respect to $|\boldsymbol{\mu}|$ will be denoted by $d\boldsymbol{\mu}/d|\boldsymbol{\mu}|$. Let $\boldsymbol{\mu} = \boldsymbol{\mu}_a(x)\,dx + \boldsymbol{\mu}_s$ be the Lebesgue decomposition of $\boldsymbol{\mu}$ into the absolutely continuous and singular parts with respect to dx.

We consider the spaces $\mathbf{Y}^1(\overline{\Omega})$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ given by

$$\begin{aligned} \mathbf{Y}^1(\overline{\Omega}) \equiv \{\mathbf{M} \in \mathbb{M}_b(\overline{\Omega}, \mathbf{E}_s^n) \mid \exists \mathbf{u}_1 \in BD(\Omega_1),& \\ \boldsymbol{\varepsilon}(\mathbf{u}_1)_{|\overline{\Omega}} = \mathbf{M},\ \mathbf{u}_{1|\Omega_1 - \overline{\Omega}} = \mathbf{0}\},& \end{aligned} \tag{3.17}$$

$$C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \equiv \{\boldsymbol{\sigma} \in C(\overline{\Omega}, \mathbf{E}_s^n) \mid \boldsymbol{\sigma}_{|\Omega} \in W^n(\Omega, \mathrm{div})\}. \tag{3.18}$$

These are topological vector spaces put in duality by the bilinear pairing

$$\langle \mathbf{M}, \boldsymbol{\sigma} \rangle_{\mathbf{Y}^1 \times C} = \int_{\overline{\Omega}} \boldsymbol{\sigma} : \mathbf{M} = \sum_{i,j=1}^n \int_{\overline{\Omega}} \sigma_{ij} M^{ij}. \tag{3.19}$$

REMARK 2. The definition of spaces in duality requires that for every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$, $\boldsymbol{\sigma} \ne \mathbf{0}$, there exists $\mathbf{M} = \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathbf{Y}^1(\overline{\Omega})$ such that

$$\int_{\overline{\Omega}} \boldsymbol{\sigma} : \mathbf{M} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) - \int_{\operatorname{Fr}\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds \ne 0 \tag{3.20}$$

(cf. (3.12), (3.16)). But for every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ such that $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$ in Ω, and for every $\mathbf{M} = \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathbf{Y}^1(\overline{\Omega})$,

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) - \int_{\operatorname{Fr}\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds = -\int_\Omega (\operatorname{div}\boldsymbol{\sigma}) \cdot \mathbf{u}\, dx = 0 \tag{3.21}$$

(see (2.11) and (3.16)). Therefore the duality should be defined between the spaces $\mathbf{Y}^1(\overline{\Omega})$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)/\{\boldsymbol{\sigma} \in C(\overline{\Omega}, \mathbf{E}_s^n) \mid \operatorname{div}\boldsymbol{\sigma} = \mathbf{0}\}$. To simplify the proofs, the previous definition, given by (3.18) and (3.19), is considered here. We do not get a contradiction, since we do not use the Hausdorff property of the topology $\sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n), \mathbf{Y}^1(\overline{\Omega}))$.

This remark relates to the spaces $\mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1)$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$, put in duality in part II (by formulae (2.1), (2.4)). To simplify the proofs, we do not replace the above spaces by $\mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1)$ and

$$(3.22)\qquad [C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)/\{\boldsymbol{\sigma}\in C(\overline{\Omega},\mathbf{E}^n_s)\mid \operatorname{div}\boldsymbol{\sigma}=\mathbf{0}\}]$$
$$\times\Big[C(\Gamma_1,\mathbf{E}^n_s)/\Big\{\boldsymbol{\kappa}\in C(\Gamma_1,\mathbf{E}^n_s)\;\Big|\;\forall\boldsymbol{\mu}\in\mathbb{M}_b(\Gamma_1,\mathbb{R}^n),\ \int_{\Gamma_1}\boldsymbol{\kappa}:[\boldsymbol{\mu}\otimes_s\boldsymbol{\nu}]\,ds=0\Big\}\Big].$$

We say that a net $\{\mathbf{M}_k\}_{k\in K}\subset\mathbf{Y}^1(\overline{\Omega})$ converges to $\mathbf{M}_0$ in the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}),C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s))$ if $\langle(\mathbf{M}_k-\mathbf{M}_0),\boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C}\to 0$ for every $\boldsymbol{\sigma}\in C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$. Let $\mathbf{Y}^1(\overline{\Omega})$ be endowed with this topology. Then $C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$ is the dual space to $\mathbf{Y}^1(\overline{\Omega})$, i.e.

$$(3.23)\qquad [\mathbf{Y}^1(\overline{\Omega}),\sigma(\mathbf{Y}^1(\overline{\Omega}),C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s))]^*=C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$$

(cf. [17, Theorem V.3.9]). Similarly,

$$(3.24)\qquad [C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s),\sigma(C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s),\mathbf{Y}^1(\overline{\Omega}))]^*=\mathbf{Y}^1(\overline{\Omega}).$$

The space $BD(\Omega)$ is isomorphic to $\mathcal{A}\equiv\{\mathbf{u}\in BD(\Omega_1)\mid \mathbf{u}_{|\Omega_1-\overline{\Omega}}=\mathbf{0}\}$ (cf. Assumption 1). Moreover, $\mathcal{A}$ is isomorphic to $\mathbf{Y}^1(\overline{\Omega})$, and the isomorphism is given by $\mathcal{A}\ni\mathbf{u}\mapsto\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}\in\mathbf{Y}^1(\overline{\Omega})$. The Banach spaces $[BD(\Omega),\|\cdot\|_{BD}]$ and $[\mathbf{Y}^1(\overline{\Omega}),\|\cdot\|_{\mathbb{M}_b(\overline{\Omega})}]$ are isomorphic (cf. [8, Proposition 4.24]). Each closed ball $\mathrm{cl}_{\|\cdot\|}(B(0,r))$ (in $\mathbf{Y}^1$) is compact in the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}),C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s))$, where $\mathrm{cl}_{\|\cdot\|}$ denotes the closure in the norm of $BD(\Omega)$ (see [8, Proposition 4.23]). The space $[\mathrm{cl}_{\|\cdot\|_{BD}}(B_{BD}(0,r_2))$, weak* $BD(\Omega)$ topology] is isomorphic to $[\mathrm{cl}_{\|\cdot\|_{BD}}(B_{BD}(0,r_2)),\sigma(\mathbf{Y}^1(\overline{\Omega}),C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s))]$ (cf. [8, Proposition 4.25]).

The functional $\mathbb{B}^j_\lambda:\mathbf{Y}^1(\overline{\Omega})\to\mathbb{R}\cup\{\infty\}$ is defined by

$$(3.25)\qquad \mathbb{B}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}})\equiv-\int_{\Gamma_1}\boldsymbol{\sigma}_0:(\boldsymbol{\gamma}^I_B(\mathbf{u})\otimes_s\boldsymbol{\nu})\,ds+\int_{\Omega}j(x,\boldsymbol{\varepsilon}(\mathbf{u}))\,dx$$
$$+\int_{\Gamma_0}I_{\{[\mathbf{u}^0-\boldsymbol{\gamma}^I_B(\mathbf{u})]\otimes_s\boldsymbol{\nu}=0\}}([\mathbf{u}^0-\boldsymbol{\gamma}^I_B(\mathbf{u})]\otimes_s\boldsymbol{\nu})\,ds$$

if $\mathbf{u}_{|\Omega}\in LD(\Omega)$ and $\mathbf{u}_{|\Omega_1-\overline{\Omega}}=\mathbf{0}$, where $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0)=\lambda\mathbf{g}$ on Γ_1, and $\mathbb{B}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}})\equiv\infty$ otherwise. We assume that there exists $\widetilde{\mathbf{u}}\in BD(\Omega_1)$ such that $\widetilde{\mathbf{u}}_{|\Omega}\in LD(\Omega)$ and $\mathbb{B}^j_\lambda(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}})<\infty$.

4. Lower semicontinuous regularization. In this section the lower semicontinuous (l.s.c.) regularization of the functional $\mathbb{B}^j_\lambda$ is found, where the space $BD(\Omega)$ is endowed with the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}),C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s))$.

Because of the duality between $\mathbf{Y}^1(\overline{\Omega})$ and $C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$, we define a functional $(\mathbb{B}^j_\lambda)^*:C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)\to\mathbb{R}\cup\{\infty\}$ by

$$(4.1)\qquad (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma})=\sup\{\langle\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}},\boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C}-\mathbb{B}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}})\mid \mathbf{u}\in BD(\Omega_1),$$
$$\mathbf{u}_{|\Omega}\in LD(\Omega)\text{ and }\mathbf{u}_{|\Omega_1-\overline{\Omega}}=\mathbf{0}\}.$$

We say that $(\mathbb{B}^j_\lambda)^*$ is the dual functional to $\mathbb{B}^j_\lambda$ with respect to the duality between $\mathbf{Y}^1(\overline{\Omega})$ and $C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$ (see [18, pp. 16–18]). The bidual functional $(\mathbb{B}^j_\lambda)^{**}:\mathbf{Y}^1(\overline{\Omega})\to\mathbb{R}\cup\{\infty\}$ is defined by

$$(4.2)\quad (\mathbb{B}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = \sup\{\langle\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}},\boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} - (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma}\in C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)\}.$$

Because of (3.12), the space $\mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ is isomorphic to $\{-\boldsymbol{\gamma}_B(\mathbf{u})\otimes_s\boldsymbol{\nu}\in L^1(\mathrm{Fr}\,\Omega,\mathbf{E}^n_s)\mid \mathbf{u}\in BD(\Omega)\}$. Thus, the bilinear form between $\mathbb{M}_b(\Omega,\mathbf{E}^n_s)\times\mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ and $C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$ is given by

$$(4.3)\quad \langle(\mathbf{w},-\boldsymbol{\gamma}^I_B(\mathbf{u})\otimes_s\boldsymbol{\nu}),\boldsymbol{\sigma}\rangle_1 \equiv \int_\Omega \boldsymbol{\sigma}:\mathbf{w} + \int_{\mathrm{Fr}\,\Omega}\boldsymbol{\sigma}:(-\boldsymbol{\gamma}^I_B(\mathbf{u})\otimes_s\boldsymbol{\nu})\,ds$$

for every $\mathbf{w}\in\mathbb{M}_b(\Omega,\mathbf{E}^n_s)$, $-\boldsymbol{\gamma}^I_B(\mathbf{u})ds\otimes_s\boldsymbol{\nu}\in\mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ and $\boldsymbol{\sigma}\in C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$. Therefore a net $\{\boldsymbol{\sigma}_\delta\}_{\delta\in D}\subset C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$ is convergent to $\boldsymbol{\sigma}_0\in C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$ in the topology

$$(4.4)\quad \sigma(C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s), L^1(\Omega,\mathbf{E}^n_s)\times\mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega})$$

if $\langle(\mathbf{w},-\boldsymbol{\gamma}^I_B(\mathbf{u})\otimes_s\boldsymbol{\nu}),(\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_\delta)\rangle_1\to 0$ for every $\mathbf{w}\in L^1(\Omega,\mathbf{E}^n_s)$ and every $-\boldsymbol{\gamma}^I_B(\mathbf{u})ds\otimes_s\boldsymbol{\nu}\in\mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$. The extension $\widetilde{\mathbb{B}}^j_\lambda$ of $\mathbb{B}^j_\lambda$ onto the space $\mathbb{M}_b(\Omega,\mathbf{E}^n_s)\times\mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ is given by

$$(4.5)\quad \widetilde{\mathbb{B}}^j_\lambda(\mathbf{w},-\boldsymbol{\gamma}_B(\mathbf{u})ds\otimes_s\boldsymbol{\nu}) \equiv -\int_{\Gamma_1}\boldsymbol{\sigma}_0:(\boldsymbol{\gamma}_B(\mathbf{u})\otimes_s\boldsymbol{\nu})\,ds + \int_\Omega j(x,\mathbf{w})\,dx + \int_{\Gamma_0} I_{\{[\mathbf{u}^0-\boldsymbol{\gamma}^I_B(\mathbf{u})]\otimes_s\boldsymbol{\nu}=0\}}([\mathbf{u}^0-\boldsymbol{\gamma}^I_B(\mathbf{u})]\otimes_s\boldsymbol{\nu})\,ds$$

if $\mathbf{w}\in L^1(\Omega,\mathbf{E}^n_s)$ and $\mathbf{u}\in BD(\Omega)$, where $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0)=\lambda\mathbf{g}$ on Γ_1, and $\widetilde{\mathbb{B}}^j_\lambda(\mathbf{w},-\boldsymbol{\gamma}_B(\mathbf{u})ds\otimes_s\boldsymbol{\nu})\equiv\infty$ otherwise.

By duality between $\mathbb{M}_b(\Omega,\mathbf{E}^n_s)\times\mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$ and $C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)$, we define a functional $(\widetilde{\mathbb{B}}^j_\lambda)^*: C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)\to\mathbb{R}\cup\{\infty\}$ (cf. (4.3)). This functional is given by

$$(4.6)\quad (\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}) = \sup\Big\{\int_\Omega\boldsymbol{\sigma}:\mathbf{w}\,dx - \int_{\mathrm{Fr}\,\Omega}\boldsymbol{\beta}_B(\boldsymbol{\sigma})\cdot\boldsymbol{\gamma}_B(\mathbf{u})\,ds - \widetilde{\mathbb{B}}^j_\lambda(\mathbf{w},-\boldsymbol{\gamma}_B(\mathbf{u})ds\otimes_s\boldsymbol{\nu}) \;\Big|\; \mathbf{w}\in L^1(\Omega,\mathbf{E}^n_s),\ \mathbf{u}\in BD(\Omega)\Big\}.$$

The bidual functional $(\widetilde{\mathbb{B}}^j_\lambda)^{**}:\mathbf{Y}^1(\overline{\Omega})\to\mathbb{R}\cup\{\infty\}$ is defined by

$$(4.7)\quad (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\mathbf{w},-\boldsymbol{\gamma}^I_B(\mathbf{u})ds\otimes_s\boldsymbol{\nu}) = \sup\Big\{\int_\Omega\boldsymbol{\sigma}:\mathbf{w} - \int_{\mathrm{Fr}\,\Omega}\boldsymbol{\beta}_B(\boldsymbol{\sigma})\cdot\boldsymbol{\gamma}^I_B(\mathbf{u})\,ds - (\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}) \;\Big|\; \boldsymbol{\sigma}\in C_{\mathrm{div}}(\overline{\Omega},\mathbf{E}^n_s)\Big\}$$

for $(\mathbf{w},-\boldsymbol{\gamma}^I_B(\mathbf{u})ds\otimes_s\boldsymbol{\nu})\in\mathbf{Y}^1(\overline{\Omega})$ (cf. (3.16)).

Lemma 6 (see [8]). *For every* $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *we have* $(\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}) \geq (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma})$. *Moreover*, $(\widetilde{\mathbb{B}}^j_\lambda)^{**}(\mathbf{M}) \leq (\mathbb{B}^j_\lambda)^{**}(\mathbf{M})$ *for every* $\mathbf{M} \in \mathbf{Y}^1(\overline{\Omega})$.

Proof. Indeed, in the definition of $(\widetilde{\mathbb{B}}^j_\lambda)^*$ we take the supremum over a larger domain. The second inequality follows from the first. ∎

Definition 4 (cf. [12]). A subset H_0 of $\mathcal{L}^0(\overline{\Omega}, \mathbb{R}^m)_\mu$ is said to be *PCU-stable* if for any continuous partition of unity $(\alpha_0, \ldots, \alpha_d)$ such that $\alpha_0, \ldots, \alpha_d \in C^\infty(\overline{\Omega}, \mathbb{R})$, and any $\mathbf{z}_0, \ldots, \mathbf{z}_d \in H_0$, the sum $\sum_{i=0}^d \alpha_i \mathbf{z}_i$ is in H_0.

Proposition 7. *The functional* $(\widetilde{\mathbb{B}}^j_\lambda)^{**}$ *defined by* (4.7), (4.6) *and* (4.5) *is given by the expression*

$$
\begin{aligned}
(4.8)\qquad (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = &- \int_{\Gamma_1} \boldsymbol{\sigma}_0 : (\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds \\
&+ \int_{\Gamma_0} j_\infty(x, (\mathbf{u}^0 - \boldsymbol{\gamma}^I_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds \\
&+ \int_{\Omega} j(x, \boldsymbol{\varepsilon}(\mathbf{u})_a)\, dx + \int_{\Omega} j_\infty(x, d\boldsymbol{\varepsilon}(\mathbf{u})_s / d|\boldsymbol{\varepsilon}(\mathbf{u})_s|)\, d|\boldsymbol{\varepsilon}(\mathbf{u})_s|
\end{aligned}
$$

for every $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$, *where* $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) = \lambda \mathbf{g}$ *on* Γ_1.

Proof. Indeed, by [25, Theorem 3A and Proposition 2M], for every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ we have

$$
\begin{aligned}
(4.9)\qquad (\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}) \equiv \sup\Big\{ &\int_{\Omega} \boldsymbol{\sigma} : \mathbf{w}\, dx + \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (-\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds \\
&+ \int_{\Gamma_1} \boldsymbol{\sigma}_0 : (\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds - \int_{\Gamma_0} I_{\{\mathbf{u}^0 - \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0}\}}((\mathbf{u}^0 - \boldsymbol{\gamma}^I_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds \\
&- \int_{\Omega} j(x, \mathbf{w})\, dx \;\Big|\; \mathbf{u} \in BD(\Omega_1),\ \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0} \text{ and } \mathbf{w} \in L^1(\Omega, \mathbf{E}^n_s) \Big\} \\
= \sup\Big\{ &\int_{\Omega} \boldsymbol{\sigma} : \mathbf{w}\, dx - \int_{\Omega} j(x, \mathbf{w})\, dx \;\Big|\; \mathbf{w} \in L^1(\Omega, \mathbf{E}^n_s) \Big\} \\
+ \sup\Big\{ &\int_{\Gamma_1} \boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) \cdot \boldsymbol{\zeta}\, ds - \int_{\Gamma_1} \boldsymbol{\beta}_B(\boldsymbol{\sigma}) \cdot \boldsymbol{\zeta}\, ds \\
&+ \int_{\Gamma_0} \boldsymbol{\sigma} : ((\mathbf{u}^0 - \boldsymbol{\gamma}^I_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds \\
&- \int_{\Gamma_0} I_{\{\mathbf{u}^0 = \boldsymbol{\gamma}^I_B(\mathbf{u})\}}((\mathbf{u}^0 - \boldsymbol{\gamma}^I_B(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds \;\Big|\; \boldsymbol{\zeta} \in L^1(\Gamma_1, \mathbb{R}^n), \mathbf{u} \in BD(\Omega) \Big\}
\end{aligned}
$$

$$- \int_{\Gamma_0} \boldsymbol{\sigma} : (\mathbf{u}^0 \otimes_s \boldsymbol{\nu})\, ds$$

$$= \int_{\Omega} j^*(x, \boldsymbol{\sigma})\, dx + \int_{\Gamma_1} I_{\{\boldsymbol{\sigma} | \boldsymbol{\beta}_B(\boldsymbol{\sigma}) = \lambda \mathbf{g}\}}(\boldsymbol{\sigma})\, ds - \int_{\Gamma_0} \boldsymbol{\sigma} : (\mathbf{u}^0 \otimes_s \boldsymbol{\nu})\, ds$$

(cf. (3.16)). Since $\boldsymbol{\gamma}_B$ is a surjection from $BD(\Omega)$ onto $L^1(\mathrm{Fr}\,\Omega, \mathbb{R}^n)$ (cf. Theorem 2.1 of [27, Chapter 2]) and by (3.5) we deduce that $\inf \widetilde{\mathbb{B}}^j_\lambda < \infty$. Moreover, we replace $\boldsymbol{\gamma}^I_B(\mathbf{u})$ by $\boldsymbol{\zeta} \in L^1(\Gamma_1, \mathbb{R}^n)$.

By the duality between $\mathbf{Y}^1(\overline{\Omega})$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ we obtain $(\widetilde{\mathbb{B}}^j_\lambda)^{**}$. The space $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ is PCU-stable, so by the proofs of Theorem 1 and 4 of [12] we get

$$\begin{aligned}
(4.10)\quad & (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \\
&= \sup\Big\{ \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})_{|\mathrm{int}\,\Omega} + \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (-\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds \\
&\quad - \int_{\Omega} j^*(x, \boldsymbol{\sigma})\, dx - \int_{\Gamma_1} I_{\{\boldsymbol{\sigma} \,|\, \boldsymbol{\sigma}\cdot\boldsymbol{\nu} = \lambda\mathbf{g}\}}(\boldsymbol{\sigma})\, ds + \int_{\Gamma_0} \boldsymbol{\sigma} : (\mathbf{u}^0 \otimes_s \boldsymbol{\nu})\, ds \,\Big|\, \\
&\qquad\qquad \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \text{ and } \forall x \in \overline{\Omega},\ \boldsymbol{\sigma}(x) \in \mathcal{K}(x) \Big\} \\
&= \sup\Big\{ \int_{\Omega} [\boldsymbol{\sigma} : (\boldsymbol{\varepsilon}(\mathbf{u})_a) - j^*(x, \boldsymbol{\sigma})]\, dx + \int_{\Omega} [\boldsymbol{\sigma} : (d(\boldsymbol{\varepsilon}(\mathbf{u})_s)/d|\boldsymbol{\varepsilon}(\mathbf{u})_s|) \\
&\quad - j^*_\infty(x, \boldsymbol{\sigma})]\, d|\boldsymbol{\varepsilon}(\mathbf{u})_s| - \int_{\Gamma_1} (\boldsymbol{\sigma}\cdot\boldsymbol{\nu}) \cdot \boldsymbol{\gamma}^I_B(\mathbf{u})\, ds - \int_{\Gamma_1} I_{\{\boldsymbol{\sigma}\cdot\boldsymbol{\nu} \,|\, \boldsymbol{\sigma}\cdot\boldsymbol{\nu} = \lambda\mathbf{g}\}}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\, ds \\
&\quad + \int_{\Gamma_0} [\boldsymbol{\sigma} : ((\mathbf{u}^0 - \boldsymbol{\gamma}^I_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}) - j^*_\infty(x, \boldsymbol{\sigma})]\, ds \,\Big|\, \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \Big\}
\end{aligned}$$

for every $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$, which is (4.8) (cf. (3.4)). In the above calculations we use the equality $j^*_\infty(x, \boldsymbol{\sigma}) = I_{\mathcal{K}(x)}(\boldsymbol{\sigma})$, which holds for every $\boldsymbol{\sigma} \in \mathbf{E}^n_s$ and $x \in \overline{\Omega}$. Moreover, by (3.2) and (3.3), $\boldsymbol{\sigma}(x) \in \mathcal{K}(x)$ for every $x \in \overline{\Omega}$. Since $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) = \lambda\mathbf{g}$ on Γ_1, we have $\int_{\Gamma_1} \lambda\mathbf{g} \cdot \boldsymbol{\gamma}^I_B(\mathbf{u})\, ds = \int_{\Gamma_1} \boldsymbol{\sigma}_0 : (\boldsymbol{\gamma}^I_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds$. By Assumptions 5 and 3 we get $(\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}_0) < \infty$. ■

LEMMA 8. *For every $\mathbf{u} \in BD(\Omega_1)$ such that $\mathbf{u}_{|\Omega} \in LD(\Omega)$, $\mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}$ and $\boldsymbol{\gamma}^I_B(\mathbf{u})_{|\Gamma_0} = \mathbf{u}^0$, we have $(\mathbb{B}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) = \mathbb{B}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}})$.*

Proof. By Lemma 6, we have $(\widetilde{\mathbb{B}}^j_\lambda)^{**}(\mathbf{M}) \le (\mathbb{B}^j_\lambda)^{**}(\mathbf{M}) \le \mathbb{B}^j_\lambda(\mathbf{M})$ for every $\mathbf{M} \in \mathbf{Y}^1(\overline{\Omega})$ (see [18, pp. 16–18]). Therefore, by (4.8), we get the assertion. ■

LEMMA 9. *For every* $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *and every* $\boldsymbol{\sigma}_s \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *such that* $\operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0}$, *we have* $(\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}) = (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s)$.

Proof. By definition (4.1) and by Green's formula (2.11) we get

$$(\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}) = \sup\Big\{ -\int_\Omega (\operatorname{div} \boldsymbol{\sigma}) \cdot \mathbf{u}\, dx - \mathbb{B}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \;\Big|\; \mathbf{u}_{|\Omega} \in LD(\Omega) \text{ and } \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}\Big\}$$
$$= \sup\Big\{ -\int_\Omega [\operatorname{div}(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s)] \cdot \mathbf{u}\, dx - \mathbb{B}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \;\Big|\; \mathbf{u}_{|\Omega} \in LD(\Omega) \text{ and } \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}\Big\} = (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s). \quad \blacksquare \tag{4.11}$$

We say that a net $\{\boldsymbol{\sigma}_k\}_{k\in K} \subset C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ converges to $\widehat{\boldsymbol{\sigma}} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ in the topology

$$\sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), L^1(\Omega, \mathbf{E}^n_s) \times \{\boldsymbol{\varphi} \in \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega} \mid \boldsymbol{\varphi}_{|\Gamma_0} = \mathbf{0}\}) \tag{4.12}$$

if

$$\int_\Omega (\boldsymbol{\sigma}_k - \widehat{\boldsymbol{\sigma}}) : \mathbf{w}\, dx + \int_{\Gamma_1} (\boldsymbol{\sigma}_k - \widehat{\boldsymbol{\sigma}}) : (\mathbf{p} \otimes_s \boldsymbol{\nu})\, ds \to 0 \tag{4.13}$$

for every $\mathbf{w} \in L^1(\Omega, \mathbf{E}^n_s)$ and $\mathbf{p} \in L^1(\Gamma_1, \mathbb{R}^n)$.

LEMMA 10. *Let* $\widehat{f} : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \to \mathbb{R}$ *be a linear functional, continuous in the topology* (4.12), *such that* $\widehat{f}(\boldsymbol{\sigma}_s) = 0$ *for every* $\boldsymbol{\sigma}_s \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ *with* $\operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0}$ *in* Ω. *Then there exists* $\widetilde{\mathbf{u}} \in LD(\Omega)$ *such that for every* $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$,

$$\widehat{f}(\boldsymbol{\sigma}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\, ds, \tag{4.14}$$

and $\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) = \mathbf{0}$ *on* Γ_0.

Proof. Since $\widehat{f}$ is continuous in the topology (4.12), by Theorem V.3.9 of [17] there exist $\mathbf{m} \in L^1(\Omega, \mathbf{E}^n_s)$ and $\widehat{\mathbf{u}} \in BD(\Omega)$ such that $\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) = \mathbf{0}$ on Γ_0, and $\widehat{f}(\boldsymbol{\sigma}) = \int_\Omega \boldsymbol{\sigma} : \mathbf{m}\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\, ds$ for all $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$. For every $\boldsymbol{\sigma}_1 \in W^n(\Omega_1, \mathrm{div})$ with $\operatorname{div} \boldsymbol{\sigma}_1 = \mathbf{0}$ in Ω_1 and $\boldsymbol{\sigma}_{1|\overline{\Omega}} \in C(\overline{\Omega}, \mathbf{E}^n_s)$, we have

$$\widehat{f}(\boldsymbol{\sigma}_{1|\overline{\Omega}}) = \int_\Omega \boldsymbol{\sigma}_1 : \mathbf{m}\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma}_1 : (\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\, ds = 0. \tag{4.15}$$

Then by Proposition 1.1 and Theorem 1.3 of [27, Chapter II] there exists $\widetilde{\mathbf{u}} \in LD(\Omega)$ such that equality (4.14) holds.

Indeed, for all $\boldsymbol{\sigma}_2 \in C^1_c(\Omega_1, \mathbf{E}^n_s)$ such that $\operatorname{div} \boldsymbol{\sigma}_2 = \mathbf{0}$ in Ω_1, we have $\int_\Omega \boldsymbol{\sigma}_2 : \mathbf{m}\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma}_2 : (\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\, ds = 0$. Then, by Proposition 1.1 of [27, Chapter II], there exists $\widetilde{\mathbf{u}} \in D'(\Omega_1, \mathbb{R}^n)$ such that for every $\boldsymbol{\sigma} \in C_c^1(\Omega_1, \mathbf{E}_s^n)$,

$$\int_{\Omega_1} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\widetilde{\mathbf{u}}) = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{m}\,dx - \int_{\operatorname{Fr}\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\,ds = \widehat{f}(\boldsymbol{\sigma}_{|\overline{\Omega}}), \tag{4.16}$$

and

$$\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}}) = \begin{cases} \mathbf{m}\,dx & \text{in } \Omega, \\ -(\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\,ds & \text{on } \operatorname{Fr}\Omega, \\ \mathbf{0} & \text{in } \Omega_1 - \overline{\Omega} \end{cases} \tag{4.17}$$

(see [24]). For every $\boldsymbol{\sigma}_3 \in C_c^1(\Omega_1, \mathbf{E}_s^n)$ such that $\boldsymbol{\sigma}_3 = \mathbf{0}$ in $\overline{\Omega}$, we have

$$\int_{\Omega_1} \boldsymbol{\sigma}_3 : \boldsymbol{\varepsilon}(\widetilde{\mathbf{u}}) = \int_{\Omega} \boldsymbol{\sigma}_3 : \mathbf{m}\,dx - \int_{\operatorname{Fr}\Omega} \boldsymbol{\sigma}_3 : (\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\,ds = 0, \tag{4.18}$$

therefore we can assume that $\widetilde{\mathbf{u}}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}$. Moreover, by Theorem 1.3 of [27, Chapter II], $\widetilde{\mathbf{u}}_{|\Omega} \in LD(\Omega)$, because $\mathbf{m} \in L^1(\Omega, \mathbf{E}_s^n)$. ■

Let $Q : C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \to \mathbb{R} \cup \{\infty\}$ be defined by

$$Q(\boldsymbol{\sigma}) = \inf\{(\widetilde{\mathbb{B}}_\lambda^j)^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s) \mid \boldsymbol{\sigma}_s \in C(\overline{\Omega}, \mathbf{E}_s^n) \text{ and } \operatorname{div}\boldsymbol{\sigma}_s = \mathbf{0}\}. \tag{4.19}$$

Proposition 11. *Let $u^0 = \mathbf{0}$ on Γ_0. For every $\boldsymbol{\sigma} \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ we have*

$$(\mathbb{B}_\lambda^j)^*(\boldsymbol{\sigma}) = \operatorname{cl}_{(4.12)} Q(\boldsymbol{\sigma}), \tag{4.20}$$

where $\operatorname{cl}_{(4.12)} Q$ denotes the largest minorant which is less than Q and l.s.c. in the topology (4.12) (*i.e.* $\operatorname{cl}_{(4.12)} Q$ *is the l.s.c. regularization of Q in* (4.12)).

Proof. Step 1. Suppose there exist $\boldsymbol{\sigma}_1 \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ and a constant $\delta_0 > 0$ such that

$$(\mathbb{B}_\lambda^j)^*(\boldsymbol{\sigma}_1) + \delta_0 < \operatorname{cl}_{(4.12)} Q(\boldsymbol{\sigma}_1). \tag{4.21}$$

On account of Lemmas 6 and 9, to prove the proposition, it suffices to show that this assumption leads to a contradiction.

The linear space

$$\mathcal{M}_0 \equiv \{\boldsymbol{\sigma}_s \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \mid \operatorname{div}\boldsymbol{\sigma}_s = \mathbf{0}\} \tag{4.22}$$

is a closed subspace of $C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ endowed with the topology (4.12). Indeed, by the Green formula (2.11),

$$\mathcal{M}_0 = \bigcap_{\mathbf{u} \in LD(\Omega),\, \boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0} \Big\{\boldsymbol{\sigma} \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \;\Big|\; \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})\,dx - \int_{\operatorname{Fr}\Omega} \boldsymbol{\beta}_B(\boldsymbol{\sigma}) \cdot \boldsymbol{\gamma}_B(\mathbf{u})\,ds = 0 = \int_{\Omega} (\operatorname{div}\boldsymbol{\sigma}) \cdot \mathbf{u}\,dx\Big\}. \tag{4.23}$$

Step 2. Let

$$\Phi : [C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n),\ \text{topology (4.12)}] \to C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n)/\mathcal{M}_0 \tag{4.24}$$

be a linear function (canonical homomorphism) such that $\mathcal{M}_0 = \ker \Phi \equiv \{\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \mid \Phi(\boldsymbol{\sigma}) = 0\}$. Moreover, let $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0$ be endowed with the strongest topology for which Φ is continuous. Since $\mathcal{M}_0$ is closed in (4.12), $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0$ is a Hausdorff topological space (cf. [13, Chapter I]). Therefore the point $(\Phi(\boldsymbol{\sigma}_1), (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_1) + \delta_0)$ is a closed subspace of $[C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0] \times \mathbb{R}$. The epigraph of $\boldsymbol{\sigma} \mapsto \mathrm{cl}_{(4.12)}\, Q(\boldsymbol{\sigma})$, defined by $\mathrm{epi}\,\mathrm{cl}_{(4.12)}\, Q = \{(\boldsymbol{\sigma}, a) \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \times \mathbb{R} \mid \mathrm{cl}_{(4.12)}\, Q(\boldsymbol{\sigma}) \le a\}$, is convex. Then the set

$$\begin{aligned}(4.25)\qquad \widehat{A} \equiv \{(\widetilde{\boldsymbol{\sigma}}, a) \in [C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0] \times \mathbb{R} \mid \\ \exists \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), \mathrm{cl}_{(4.12)}\, Q(\boldsymbol{\sigma}) \le a \text{ and } \Phi(\boldsymbol{\sigma}) = \widetilde{\boldsymbol{\sigma}}\}\end{aligned}$$

is convex (cf. [13, Chapter I]). Moreover $\widehat{A}$ is closed in $[C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0] \times \mathbb{R}$, since $\mathrm{cl}_{(4.12)}\, Q(\boldsymbol{\sigma}) = \mathrm{cl}_{(4.12)}\, Q(\boldsymbol{\sigma} + \boldsymbol{\sigma}_b)$ for all $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ and $\boldsymbol{\sigma}_b \in \mathcal{M}_0$. By the Hahn–Banach theorem, there exists a closed affine hyperplane $\mathcal{H}$ which strictly separates $\widehat{A}$ and $(\Phi(\boldsymbol{\sigma}_1), (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_1) + \delta_0)$. Let

$$(4.26)\qquad \mathcal{H} = \{(\widetilde{\boldsymbol{\sigma}}, a) \in [C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0] \times \mathbb{R} \mid f_2(\widetilde{\boldsymbol{\sigma}}) + ba + c_2 = 0\},$$

where $b, c_2 \in \mathbb{R}$ and $f_2 : [C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0] \to \mathbb{R}$ is a continuous linear functional such that for every $(\widetilde{\boldsymbol{\sigma}}, a) \in \widehat{A}$,

$$(4.27)\qquad f_2(\widetilde{\boldsymbol{\sigma}}) + ba + c_2 > 0, \qquad f_2(\Phi(\boldsymbol{\sigma}_1)) + b((\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_1) + \delta_0) + c_2 < 0.$$

Step 3. Now we consider the case when $b = 0$. From (4.9) and Assumption 4, we deduce that $\inf\{(\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}) - \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}^e) : \boldsymbol{\sigma}\, dx \mid \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)\}$ is finite, since $\mathbf{u}^0 = \mathbf{0}$. Moreover, by the Green formula we obtain

$$\inf\Big\{\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}^e) : (\boldsymbol{\sigma}_s - \boldsymbol{\sigma})\, dx \,\Big|\, \operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0}\Big\} = -\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}^e) : \boldsymbol{\sigma}\, dx,$$

(see Assumption 4). Let

$$(4.28)\qquad h \equiv |f_2(\Phi(\boldsymbol{\sigma}_1)) + c_2| > 0,$$

$$\begin{aligned}(4.29)\qquad \widehat{d} \equiv \max\Big[1; (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_1) + \delta_0 - \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}^e) : \boldsymbol{\sigma}_1\, dx \\ - \inf\Big\{(\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}) - \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}^e) : \boldsymbol{\sigma}\, dx \,\Big|\, \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)\Big\}\Big]\end{aligned}$$

and $d \equiv h/(2\widehat{d})$. Then the functional

$$\begin{aligned}(4.30)\qquad [C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0] \times \mathbb{R} \ni (\widetilde{\boldsymbol{\sigma}}, a) \\ \mapsto f_2(\widetilde{\boldsymbol{\sigma}}) + d\Big(a + \widehat{d} - (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_1) - \delta_0 + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}^e) : \boldsymbol{\sigma}_1\, dx\Big) + c_2\end{aligned}$$

strictly separates $\{(\widetilde{\boldsymbol\sigma}, a) \in [C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0] \times \mathbb{R} \mid \exists \boldsymbol\sigma \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$, $\mathrm{cl}_{(4.12)}\, Q(\boldsymbol\sigma) - \int_\Omega \boldsymbol\varepsilon(\mathbf{u}^e) : \boldsymbol\sigma\, dx \le a$ and $\Phi(\boldsymbol\sigma) = \widetilde{\boldsymbol\sigma}\}$ and the point $(\Phi(\boldsymbol\sigma_1)$, $(\mathbb{B}^j_\lambda)^*(\boldsymbol\sigma_1) + \delta_0 - \int_\Omega \boldsymbol\varepsilon(\mathbf{u}^e) : \boldsymbol\sigma_1\, dx)$ (cf. (4.25)).

Step 4. By (4.27) and (4.30) there exists a continuous linear functional $f_3 : [C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0] \to \mathbb{R}$ and $c_3 \in \mathbb{R}$ such that

$$f_3(\Phi(\boldsymbol\sigma_1)) + c_3 > (\mathbb{B}^j_\lambda)^*(\boldsymbol\sigma_1) + \delta_0 \quad \text{and} \quad f_3(\widetilde{\boldsymbol\sigma}) + c_3 < a \tag{4.31}$$

for every $(\widetilde{\boldsymbol\sigma}, a) \in \widehat{A}$. Therefore the functional $\boldsymbol\sigma \mapsto f_4(\boldsymbol\sigma) + c_3$, defined by

$$C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \ni \boldsymbol\sigma \mapsto f_4(\boldsymbol\sigma) + c_3 = f_3(\Phi(\boldsymbol\sigma)) + c_3, \tag{4.32}$$

strictly separates $\operatorname{epi} \mathrm{cl}_{(4.12)}\, Q$ and

$$\{(\boldsymbol\sigma, a) \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \times \mathbb{R} \mid \boldsymbol\sigma \in \mathcal{M}_0 + \{\boldsymbol\sigma_1\},\ a = (\mathbb{B}^j_\lambda)^*(\boldsymbol\sigma_1) + \delta_0\}. \tag{4.33}$$

Moreover $\mathcal{M}_0 \subset \ker f_4$. Since Φ is continuous in the topology (4.12) and f_3 is continuous on $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)/\mathcal{M}_0$, it follows that $f_4 = f_3 \circ \Phi$ is continuous in the topology (4.12) over the space $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$.

Step 5. By Lemma 10, there exists $\widetilde{\mathbf{u}} \in LD(\Omega)$ such that $\boldsymbol\gamma_B(\widetilde{\mathbf{u}}) = \mathbf{0}$ on Γ_0 and

$$f_4(\boldsymbol\sigma) = \int_\Omega \boldsymbol\sigma : \boldsymbol\varepsilon(\widetilde{\mathbf{u}})\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol\sigma : (\boldsymbol\gamma_B(\widetilde{\mathbf{u}}) \otimes_s \boldsymbol\nu)\, ds \tag{4.34}$$

for every $\boldsymbol\sigma \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$, because $\mathcal{M}_0 \subset \ker f_4$.

Step 6. We say that a net $\{\boldsymbol\sigma_k\}_{k\in K} \subset C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ converges to $\widehat{\boldsymbol\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ in the topology

$$\begin{aligned} \sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), \{\boldsymbol\varphi \in \mathbf{Y}^1(\overline{\Omega}) \mid \exists \mathbf{u} \in BD(\Omega_1),& \\ \boldsymbol\varepsilon(\mathbf{u}) = \boldsymbol\varphi,\ \mathbf{u}_{|\Omega} \in LD(\Omega),\ \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0},\ \boldsymbol\gamma^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\})& \end{aligned} \tag{4.35}$$

if

$$\int_\Omega (\boldsymbol\sigma_k - \widehat{\boldsymbol\sigma}) : \boldsymbol\varepsilon(\mathbf{u})\, dx - \int_{\mathrm{Fr}\,\Omega} (\boldsymbol\sigma_k - \widehat{\boldsymbol\sigma}) : (\boldsymbol\gamma_B(\mathbf{u}) \otimes_s \boldsymbol\nu)\, ds \to 0 \tag{4.36}$$

for every $\mathbf{u} \in LD(\Omega)$ such that $\boldsymbol\gamma_B(\mathbf{u}) = \mathbf{0}$ on Γ_0. The l.s.c. regularization of $(\widetilde{\mathbb{B}}^j_\lambda)^*$ in the topology (4.35) (denoted by $\mathrm{cl}_{(4.35)}(\widetilde{\mathbb{B}}^j_\lambda)^*$) is given by

$$\begin{aligned} \mathrm{cl}_{(4.35)}\, (\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol\sigma) = \sup\Big\{ &\int_\Omega \boldsymbol\sigma : \boldsymbol\varepsilon(\mathbf{u})_{|\Omega}\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol\sigma : (\boldsymbol\gamma^I_B(\mathbf{u}) \otimes_s \boldsymbol\nu)\, ds \\ &- (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol\varepsilon(\mathbf{u})_{|\overline{\Omega}}) \,\Big|\, \mathbf{u} \in BD(\Omega_1),\ \mathbf{u}_{|\Omega} \in LD(\Omega),\ \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}, \\ &\boldsymbol\gamma^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0 \Big\} \\ = \sup\Big\{ &\int_\Omega \boldsymbol\sigma : \boldsymbol\varepsilon(\mathbf{u})_{|\Omega}\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol\sigma : (\boldsymbol\gamma^I_B(\mathbf{u}) \otimes_s \boldsymbol\nu)\, ds \end{aligned} \tag{4.37}$$

$$- \mathbb{B}^j_\lambda(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \,\Big|\, \mathbf{u} \in BD(\Omega_1),\, \mathbf{u}_{|\Omega} \in LD(\Omega),$$
$$\mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0},\, \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0 \Big\} = (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma})$$

for $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ (cf. Lemma 8 and [18, p. 15]). From (4.31), (4.32), (4.34) and (4.37) we obtain a contradiction. ■

LEMMA 12. *For every $\widehat{r} > 0$, the topology* (4.12) *is stronger than* $\sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), \mathbf{Y}^1(\overline{\Omega}))$ *over the set* $\{\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \mid \|\mathrm{div}\, \boldsymbol{\sigma}\|_{L^n} \le \widehat{r}\}$.

Proof. Let $\{\boldsymbol{\sigma}_\tau\}_{\tau\in T} \subset \{\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \mid \|\mathrm{div}\boldsymbol{\sigma}\|_{L^n(\Omega,\mathbb{R}^n)} \le \widehat{r}\}$ be a net convergent to $\widehat{\boldsymbol{\sigma}}$ in the topology (4.12). Then for every $\mathbf{u} \in LD(\Omega)$ with $\boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0}$ on Γ_0,

$$\int_\Omega (\boldsymbol{\sigma}_\tau - \widehat{\boldsymbol{\sigma}}) : \boldsymbol{\varepsilon}(\mathbf{u})\, dx + \int_{\mathrm{Fr}\,\Omega} (\boldsymbol{\sigma}_\tau - \widehat{\boldsymbol{\sigma}}) : (-\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds \to 0.$$

By the Green formula (2.11) we obtain $\int_\Omega \mathrm{div}(\boldsymbol{\sigma}_\tau - \widehat{\boldsymbol{\sigma}}) \cdot \mathbf{u}\, dx \to 0$ for every $\mathbf{u} \in LD(\Omega)$ such that $\boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0}$ on Γ_0. The set $\{\mathbf{u} \in LD(\Omega) \mid \boldsymbol{\gamma}_B(\mathbf{u})_{|\Gamma_0} = \mathbf{0}\}$ is dense in $[L^{n/(n-1)}(\Omega, \mathbb{R}^n), \|\cdot\|_{L^{n/(n-1)}}]$, since $C^1_c(\Omega, \mathbb{R}^n)$ is dense in $L^{n/(n-1)}(\Omega, \mathbb{R}^n)$ (see [1, Theorems 2.19 and 3.18], [27, Chapter II, Theorem 1.2]). Then, by [17, Theorem II.1.18],

$$\int_\Omega \mathrm{div}(\boldsymbol{\sigma}_\tau - \widehat{\boldsymbol{\sigma}}) \cdot \mathbf{w}\, dx \to 0 \quad \forall \mathbf{w} \in L^{n/(n-1)}(\Omega, \mathbb{R}^n), \tag{4.38}$$

since $\{\boldsymbol{\sigma}_\tau - \widehat{\boldsymbol{\sigma}}\}_{\tau\in T} \subset \{\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \mid \|\mathrm{div}\, \boldsymbol{\sigma}\|_{L^n(\Omega,\mathbb{R}^n)} \le \widehat{r} + \|\mathrm{div}\, \widehat{\boldsymbol{\sigma}}\|_{L^n}\}$. Therefore, $\int_\Omega \mathrm{div}(\boldsymbol{\sigma}_\tau - \widehat{\boldsymbol{\sigma}}) \cdot \mathbf{u}\, dx \to 0$ for every $\mathbf{u} \in BD(\Omega)$, because $BD(\Omega) \subset L^{n/(n-1)}(\Omega, \mathbb{R}^n)$ (cf. [27, Chapter II, Theorem 2.2]). By (2.11) the net $\{\boldsymbol{\sigma}_\tau\}_{\tau\in T}$ converges to $\widehat{\boldsymbol{\sigma}}$ in $\sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s), \mathbf{Y}^1(\overline{\Omega}))$. ■

PROPOSITION 13. *Let $\mathbf{u}^0 = \mathbf{0}$ on Γ_0 and let $A_k \equiv \{\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \mid \|\mathrm{div}\, \boldsymbol{\sigma}\|_{L^n} \le k\}$. For every $\widehat{\boldsymbol{\sigma}} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ and every $k > \|\mathrm{div}\, \widehat{\boldsymbol{\sigma}}\|_{L^n}$,*

$$(\mathbb{B}^j_\lambda)^*(\widehat{\boldsymbol{\sigma}}) = \mathrm{cl}_{A_k} Q(\widehat{\boldsymbol{\sigma}}), \tag{4.39}$$

where $\mathrm{cl}_{A_k} Q(\cdot)$ is the l.s.c. regularization of the function $\boldsymbol{\sigma} \mapsto Q(\boldsymbol{\sigma}) + I_{A_k}(\boldsymbol{\sigma})$ in the topology (4.12) *and $I_{A_k}(\cdot)$ is the indicator function of A_k.*

Proof. Step 1. Suppose there exist $\boldsymbol{\sigma}_1 \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ and constants δ_0, $k > 0$ such that $k > \|\mathrm{div}\, \boldsymbol{\sigma}_1\|_{L^n}$ and $(\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_1) + \delta_0 < \mathrm{cl}_{A_k} Q(\boldsymbol{\sigma}_1)$. On account of Lemmas 6 and 9, it suffices to show that this assumption leads to a contradiction.

For every $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$ let

$$(\widetilde{\mathbb{B}}^j_\lambda)^{*k}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \equiv \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} - (\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in A_k\}, \tag{4.40}$$

$$(\widetilde{\mathbb{B}}^j_\lambda)^*_{\|A_k}(\boldsymbol{\sigma}) \equiv (\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}) + I_{A_k}(\boldsymbol{\sigma}) \quad \forall \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s). \tag{4.41}$$

For every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ let

$$\operatorname{cl}_{(4.35)}(\widetilde{\mathbb{B}}^j_\lambda)^*_{\|A_k}(\boldsymbol{\sigma}) = \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} - (\widetilde{\mathbb{B}}^j_\lambda)^{*k}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \mid \mathbf{u} \in BD(\Omega_1),\ \mathbf{u}_{|\Omega} \in LD(\Omega),\ \mathbf{u}_{|\Omega_1-\overline{\Omega}} = \mathbf{0},\ \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\}. \tag{4.42}$$

Then for every $\widehat{k} > 0$ such that $\|\operatorname{div}\boldsymbol{\sigma}_1\|_{L^n} < \widehat{k}$ we have

$$\operatorname{cl}_{(4.35)}(\widetilde{\mathbb{B}}^j_\lambda)^*_{\|A_{\widehat{k}}}(\boldsymbol{\sigma}_1) = (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_1) \tag{4.43}$$

(cf. (4.37)). Indeed,

$$\begin{aligned} &\sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}_1\rangle_{\mathbf{Y}^1\times C} - (\widetilde{\mathbb{B}}^j_\lambda)^{*k}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \mid \mathbf{u} \in BD(\Omega_1),\\ &\qquad \mathbf{u}_{|\Omega} \in LD(\Omega),\ \mathbf{u}_{|\Omega_1-\overline{\Omega}} = \mathbf{0},\ \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\}\\ &= \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}_1\rangle_{\mathbf{Y}^1\times C} - (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}) \mid \mathbf{u} \in BD(\Omega_1),\\ &\qquad \mathbf{u}_{|\Omega} \in LD(\Omega),\ \mathbf{u}_{|\Omega_1-\overline{\Omega}} = \mathbf{0},\ \boldsymbol{\gamma}^I_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\} \end{aligned} \tag{4.44}$$

if $k > \|\operatorname{div}\boldsymbol{\sigma}_1\|_{L^n}$, since $(\widetilde{\mathbb{B}}^j_\lambda)^{*k}$ is the supremum over all affine mappings $\mathbf{Y}^1(\overline{\Omega}) \ni \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \mapsto \langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\sigma}\rangle_{\mathbf{Y}^1\times C} + \text{const}$ which are less than $(\widetilde{\mathbb{B}}^j_\lambda)$, and $\boldsymbol{\sigma} \in A_k$.

Step 2. Similarly to the proof of Proposition 11, for every $k > 0$, there exists a linear functional $f_k : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s) \to \mathbb{R}$ given by

$$f_k(\boldsymbol{\sigma}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}_k)\,dx - \int_{\operatorname{Fr}\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\mathbf{u}_k) \otimes_s \boldsymbol{\nu})\,ds, \tag{4.45}$$

where $\mathbf{u}_k \in LD(\Omega)$ and $\boldsymbol{\gamma}_B(\mathbf{u}_k) = \mathbf{0}$ on Γ_0 for every $k > 0$. Moreover, for all $k > 0$ there exists $c_k \in \mathbb{R}$ such that

$$(\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_1) + \delta_0 < f_k(\boldsymbol{\sigma}_1) + c_k \quad\text{and}\quad f_k(\widetilde{\boldsymbol{\sigma}}) + c_k < \operatorname{cl}_{A_k} Q(\widetilde{\boldsymbol{\sigma}}) \tag{4.46}$$

for every $\widetilde{\boldsymbol{\sigma}} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$. From (4.42), (4.43), (4.45) and (4.46) we obtain a contradiction. ∎

THEOREM 14. *Let $\mathbf{u}^0 = \mathbf{0}$ on Γ_0. For every $\boldsymbol{\varphi} \in \mathbf{Y}^1(\overline{\Omega})$ we have $(\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varphi}) = (\mathbb{B}^j_\lambda)^{**}(\boldsymbol{\varphi})$.*

Proof. Suppose that there exist $\mathbf{u}_1 \in BD(\Omega_1)$ with $\mathbf{u}_{1|\Omega_1-\overline{\Omega}} = \mathbf{0}$ and $\delta_1 > 0$ such that

$$(\mathbb{B}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u}_1)_{|\overline{\Omega}}) > (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u}_1)_{|\overline{\Omega}}) + 4\delta_1. \tag{4.47}$$

On account of Lemma 6, it suffices to show that this assumption leads to a contradiction. There exists $\boldsymbol{\sigma}_2 \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}^n_s)$ such that

$$(\mathbb{B}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u}_1)_{|\overline{\Omega}}) < \{\langle \boldsymbol{\varepsilon}(\mathbf{u}_1)_{|\overline{\Omega}}, \boldsymbol{\sigma}_2\rangle_{\mathbf{Y}^1\times C} - (\mathbb{B}^j_\lambda)^*(\boldsymbol{\sigma}_2)\} + \delta_1 \tag{4.48}$$

(cf. (3.19), (3.20), (4.2)). Therefore, by Proposition 11, Lemma 12, Proposition 13, Green's formula (2.11) and (4.48) there exists $k_0 > 0$ such that

$$
\begin{aligned}
(4.49)\quad (\mathbb{B}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u}_1)_{|\overline{\Omega}}) &< \Big\{ -\int_\Omega (\operatorname{div}\boldsymbol{\sigma}_2)\cdot \mathbf{u}_1\,dx - \operatorname{cl}_{A_{k_0}} Q(\boldsymbol{\sigma}_2)\Big\} + \delta_1 \\
&\le \sup_{\boldsymbol{\sigma}} \Big\{ -\int_\Omega (\operatorname{div}\boldsymbol{\sigma})\cdot \mathbf{u}_1\,dx - \operatorname{cl}_{A_{k_0}} Q(\boldsymbol{\sigma}) \,\Big|\, \boldsymbol{\sigma} \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}^n_s)\Big\} + \delta_1 \\
&= \sup_{\boldsymbol{\sigma}} \Big\{ -\int_\Omega (\operatorname{div}\boldsymbol{\sigma})\cdot \mathbf{u}_1\,dx - Q(\boldsymbol{\sigma}) \,\Big|\, \boldsymbol{\sigma} \in A_{k_0},\ \text{cf. Lemma 12}\Big\} + \delta_1 \\
&\le \sup_{\boldsymbol{\sigma}} \Big\{ -\int_\Omega (\operatorname{div}\boldsymbol{\sigma})\cdot \mathbf{u}_1\,dx - \inf\{(\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}+\boldsymbol{\sigma}_s) \mid \\
&\qquad \boldsymbol{\sigma}_s \in C(\overline{\Omega}, \mathbf{E}^n_s) \text{ and } \operatorname{div}\boldsymbol{\sigma}_s = 0 \text{ in } \Omega\} \,\Big|\, \boldsymbol{\sigma} \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}^n_s)\Big\} + \delta_1 \\
&= \sup_{\boldsymbol{\sigma}} \sup_{\boldsymbol{\sigma}_s} \Big\{ -\int_\Omega (\operatorname{div}(\boldsymbol{\sigma}+\boldsymbol{\sigma}_s))\cdot \mathbf{u}_1\,dx - (\widetilde{\mathbb{B}}^j_\lambda)^*(\boldsymbol{\sigma}+\boldsymbol{\sigma}_s) \,\Big| \\
&\qquad \boldsymbol{\sigma}, \boldsymbol{\sigma}_s \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}^n_s),\ \operatorname{div}\boldsymbol{\sigma}_s = 0\Big\} + \delta_1 \\
&= (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\mathbf{u}_1)_{|\overline{\Omega}}) + \delta_1.
\end{aligned}
$$

By (4.47) we have a contradiction. ∎

Remark 3. The space

$$
(4.50)\quad \{\boldsymbol{\varphi} \in \mathbf{Y}^1(\overline{\Omega}) \mid \exists \mathbf{u} \in BD(\Omega_1),\ \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} = \boldsymbol{\varphi},\ \mathbf{u}_{|\Omega} \in LD(\Omega),\ \mathbf{u}_{|\Omega_1-\Omega} = \mathbf{0}\}
$$

(included in $L^1(\Omega, \mathbf{E}^n_s) \times L^1(\operatorname{Fr}\Omega, \mathbf{E}^n_s)$) is not PCU-stable.

Proof. If the space (4.50) were PCU-stable, then $(\mathbb{B}^j_\lambda)^* = (\widetilde{\mathbb{B}}^j_\lambda)^*$. Hence we get a contradiction, since there exists $\boldsymbol{\sigma}_s \in C(\overline{\Omega}, \mathbf{E}^n_s)$ with $\operatorname{div}\boldsymbol{\sigma}_s = \mathbf{0}$ in Ω and $\|\boldsymbol{\sigma}_s\|_{L^\infty} > 0$ (cf. [21, formula (2.7)]). ∎

5. Basic conclusions. Now we pass to the mechanical conclusions. The displacement formulation of the equilibrium problem (studied in [8]) for the elastic-perfectly plastic body made of a Hencky material reads:

$$
(5.1)\qquad (P_{\lambda,j})\quad \text{Find } \inf\{\lambda F(\mathbf{u}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u})) \mid \mathbf{u} \in LD(\Omega)\},
$$

where the functionals F and G_j are defined by (3.9)–(3.11).

Moreover, in [8] the bidual relaxed problem

$$
(5.2)\qquad (RP^{**}_{\lambda,j})\quad \text{Find } \inf\{(\lambda F_R)^{**}(\mathbf{u}) + G^{**}_j(\boldsymbol{\varepsilon}(\mathbf{u})) \mid \mathbf{u} \in BD(\Omega)\}
$$

is studied, where for every $\mathbf{u} \in BD(\Omega)$,

$$(5.3)\qquad (\lambda F_R)^{**}(\mathbf{u}) \equiv -\lambda L(\mathbf{u}) + \int_{\Gamma_0} j_\infty(x, ((\mathbf{u}^0 - \boldsymbol{\gamma}_B(\mathbf{u})) \otimes_s \boldsymbol{\nu}))\, ds$$

(see (3.9)) and

$$(5.4)\qquad G_j^{**}(\boldsymbol{\varepsilon}(\mathbf{u})) = \int_\Omega j(x, \boldsymbol{\varepsilon}(\mathbf{u})_a)\, dx + \int_\Omega j_\infty(x, d\boldsymbol{\varepsilon}(\mathbf{u})_s/d|\boldsymbol{\varepsilon}(\mathbf{u})_s|)\, d|\boldsymbol{\varepsilon}(\mathbf{u})_s|.$$

LEMMA 15. *If* $\mathbf{f} \in L^{n+\delta}(\Omega, \mathbb{R}^n)$, *where* $\delta \ge 0$, *then the functional* $BD(\Omega) \ni \mathbf{u} \mapsto \int_\Omega \mathbf{f} \cdot \mathbf{u}\, dx \in \mathbb{R}$ *is continuous in the weak** $BD(\Omega)$ *and in* $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))$ *topologies on bounded subsets of* $BD(\Omega)$.

Proof. Indeed, by Proposition 8, the set $[\mathrm{cl}_{\|\cdot\|_{BD}}(B_{BD}(0,r)), \text{weak}^*\ BD(\Omega)]$ is homeomorphic to $\mathrm{cl}_{\|\cdot\|_{BD}}(B_{BD}(0,r))$ endowed with the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))$, for every $r > 0$. Moreover, by Proposition 2, the injection of $[BD(\Omega), \text{weak}^* \text{ topology}]$ into $[L^q(\Omega, \mathbb{R}^n), \text{weak topology}]$ is continuous on bounded subsets of $BD(\Omega)$, where $q = \frac{n+\delta}{n+\delta-1}$ ($q = \infty$ if $n + \delta = 1$). ∎

ASSUMPTION 7. There exist $k_b > 0$ and $r_1 > 0$ such that $j^*(x, \mathbf{w}^*) \le k_b$ for every $\mathbf{w}^* \in B_{\mathbf{E}_s^n}(0, r_1)$ and dx-a.e. $x \in \Omega$. ∎

Suppose the function (3.8) is coercive over $BD(\Omega)$, i.e.

$$(5.5)\qquad \text{if } \|\mathbf{u}_m\|_{BD} \to \infty \text{ then } \lambda F(\mathbf{u}_m) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}_m)) \to \infty$$

for every sequence $\{\mathbf{u}_m\}_{m\in\mathbb{N}} \subset BD(\Omega)$. Moreover, let $0 \le \lambda_1 < \lambda$. Then the function (3.8) (where we replace λ by λ_1) is coercive on $BD(\Omega)$. Similarly, if the function

$$(5.6)\qquad BD(\Omega) \ni \mathbf{u} \mapsto \lceil RP_{\lambda,j}^{**}\rceil(\mathbf{u}) = (\lambda F_R)^{**}(\mathbf{u}) + G_j^{**}(\boldsymbol{\varepsilon}(\mathbf{u})) \in \mathbb{R} \cup \{\infty\}$$

is coercive and $0 \le \lambda_1 < \lambda$, then $\lceil RP_{\lambda_1,j}^{**}\rceil$ is coercive over $BD(\Omega)$. Moreover, we obtain

$$(5.7)\qquad \lambda F(\mathbf{u}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u})) \ge \lceil RP_{\lambda,j}^{**}\rceil(\mathbf{u}) \quad \forall \mathbf{u} \in BD(\Omega).$$

LEMMA 16. *Let* $\widetilde{\mathbf{u}} \in BD(\Omega)$ *and let* $\{\mathbf{u}_p\}_{p\in P} \subset BD(\Omega)$ *be a net convergent to* $\widetilde{\mathbf{u}}$ *in the topology* $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))$ (cf. [8, Propositions 4.24 and 4.25]). *Moreover, for every* $p \in P$, *let* $\mathbf{u}_p = \mathbf{u}_p^1 + \mathbf{u}_p^2$, *where* $\mathbf{u}_p^2 \in \mathcal{R}_0$ *and the net* $\{\mathbf{u}_p^1\}_{p\in P}$ *is bounded in* $\|\cdot\|_{BD}$. *Then the net* $\{\mathbf{u}_p\}_{p\in P}$ *is bounded in* $\|\cdot\|_{BD}$ *and* $\mathbf{u}_p$ *is convergent to* $\widetilde{\mathbf{u}}$ *in the weak** BD *topology.*

Proof. For every $\Psi \in [BD(\Omega), \sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))]^*$ (see Proposition 6), $\Psi(\mathbf{u}_p - \widetilde{\mathbf{u}}) = \Psi(\mathbf{u}_p^2) + \Psi(\mathbf{u}_p^1 - \widetilde{\mathbf{u}})$ converges to 0. Therefore the set $\{\Psi(\mathbf{u}_p^2) \mid p \in P\}$ is bounded. Indeed, the set $\{\Psi(\mathbf{u}_p^1) \mid p \in P\}$ is bounded, because $\{\mathbf{u}_p^1\}_{p\in P}$ is bounded in $\|\cdot\|_{BD}$ and $\Psi \in [BD(\Omega), \|\cdot\|_{BD}]^*$.

The space $\mathcal{R}_0$ of rigid motions is finite-dimensional, so $\{\mathbf{u}_p^2\}_{p\in P}$ is bounded in $\|\cdot\|_{\mathbf{Y}^1}$, because for every $\Psi \in [BD(\Omega), \sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))]^*$

the set $\{\Psi(\mathbf{u}_p^2) \mid p \in P\}$ is bounded. Thus $\{\mathbf{u}_p\}_{p\in P}$ is bounded in $\|\cdot\|_{BD}$. Therefore $\{\mathbf{u}_p\}_{p\in P} \rightharpoonup \widetilde{\mathbf{u}}$ in the weak* BD topology. ■

LEMMA 17. *Let $\widetilde{\mathbf{u}} \in BD(\Omega)$ and let $\{\mathbf{u}_p\}_{p\in P} \subset BD(\Omega)$ be a net convergent to $\widetilde{\mathbf{u}}$ in the weak* $BD(\Omega)$ topology. Moreover, for every $p \in P$, let $\mathbf{u}_p = \mathbf{u}_p^1 + \mathbf{u}_p^2$, where $\mathbf{u}_p^2 \in \mathcal{R}_0$ and the net $\{\mathbf{u}_p^1\}_{p\in P}$ is bounded in $\|\cdot\|_{BD}$. Then the net $\{\mathbf{u}_p\}_{p\in P}$ is bounded in $\|\cdot\|_{BD}$ and $\mathbf{u}_p$ is convergent to $\widetilde{\mathbf{u}}$ in the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))$.*

Proof. The proof is similar to that of Lemma 16, with $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))$ replaced by the weak* BD topology. ■

The main conclusion of this section is the following.

THEOREM 18. *Let $\mathbf{u}^0 = \mathbf{0}$ on Γ_0, $ds(\Gamma_0) \neq 0$ and $\mathbf{f} \in L^{n+\delta}(\Omega, \mathbb{R}^n)$, where $\delta \geq 0$. If the function* (3.8) *is coercive over $BD(\Omega)$, then the l.s.c. regularization of* (3.8) *in the weak* $BD(\Omega)$ topology is the functional*

$$BD(\Omega) \ni \mathbf{u} \mapsto \lceil RP_{\lambda,j}^{**}\rceil(\mathbf{u}) = (\lambda F_R)^{**}(\mathbf{u}) + G_j^{**}(\boldsymbol{\varepsilon}(\mathbf{u})), \tag{5.8}$$

where $(\lambda F_R)^{**}$ and G_j^{**} are defined by (5.3) *and* (5.4).

Proof. Step 1. Let $\widetilde{\mathbf{u}} \in BD(\Omega)$ and $\{\mathbf{u}_p\}_{p\in P}$ be a net such that $\mathbf{u}_p \rightharpoonup \widetilde{\mathbf{u}}$ in the weak* $BD(\Omega)$ topology. Suppose the set

$$\{\lambda F(\mathbf{u}_p) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}_p)) \mid p \in P\} \tag{5.9}$$

is bounded. Then, by coercivity of (3.8) (or by Assumption 7 and Lemma 17), the net $\{\mathbf{u}_p\}_{p\in P}$ is bounded in $\|\cdot\|_{BD(\Omega)}$. Therefore by Theorem 14 and Lemma 15, $\liminf_{p\in P}(\lambda F(\mathbf{u}_p) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}_p))) \geq \lceil RP_{\lambda,j}^{**}\rceil(\widetilde{\mathbf{u}})$.

Step 2. Let $\mathbf{u}_p \rightharpoonup \widetilde{\mathbf{u}}$ in the weak* $BD(\Omega)$ topology, and suppose the set (5.9) is not bounded. Then either there exists a finer net $\{\mathbf{u}_p\}_{p\in P_1}$ $(P_1 \subset P)$ such that the set $\{\lambda F(\mathbf{u}_p) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}_p)) \mid p \in P_1\}$ is bounded, or, for every finer net $\{\mathbf{u}_p\}_{p\in P_2}$ $(P_2 \subset P)$, the set $\{\lambda F(\mathbf{u}_p) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}_p)) \mid p \in P_2\}$ is unbounded. The first case has been considered in Step 1. In the second case, by Assumption 7 and coercivity of (3.8), we get

$$\liminf(\lambda F(\mathbf{u}_p) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}_p))) = \infty \geq \lceil RP_{\lambda,j}^{**}\rceil(\widetilde{\mathbf{u}}) \quad \text{for } p \in P_2. \tag{5.10}$$

Step 3. Let $(\widetilde{\mathbb{B}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}}) < \infty$. By Theorem 14 there is a net $\{\mathbf{u}_t\}_{t\in\Sigma} \subset BD(\Omega)$ such that $\mathbf{u}_t \rightharpoonup \widetilde{\mathbf{u}}$ in $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n))$ and

$$\lim_{t\in\Sigma}\Big(\lambda F(\mathbf{u}_t) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}_t)) + \lambda\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,dx\Big) = (\widetilde{\mathbb{B}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}}). \tag{5.11}$$

The assertion of Theorem 14 holds in the special case when $\mathbf{g} = \mathbf{0}$ on Γ_1. Then

$$\lim_{t\in\Sigma}\Big[\lambda F(\mathbf{u}_t)+G_j(\boldsymbol{\varepsilon}(\mathbf{u}_t))+\lambda\Big(\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,dx+\int_{\Gamma_1}\mathbf{g}\cdot\boldsymbol{\gamma}_B(\mathbf{u}_t)\,ds\Big)\Big] = (\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}})+\lambda\int_{\Gamma_1}\mathbf{g}\cdot\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}})\,ds\in\mathbb{R}, \tag{5.12}$$

because $|\lambda\int_{\Gamma_1}\mathbf{g}\cdot\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}})\,ds|<\infty$. By Assumption 7, Lemma 16 and (5.12), the net $\{\mathbf{u}_t\}_{t\in\Sigma}$ is bounded in $\|\cdot\|_{BD(\Omega)}$ and $\mathbf{u}_t\rightharpoonup\widetilde{\mathbf{u}}$ in the weak* $BD(\Omega)$ topology. Therefore, by Theorem 14 and Lemma 15, we conclude that $\lim_{t\in\Sigma}(\lambda F(\mathbf{u}_t)+G_j(\boldsymbol{\varepsilon}(\mathbf{u}_t)))=\lceil RP^{**}_{\lambda,j}\rceil(\widetilde{\mathbf{u}})$.

Step 4. Let $(\widetilde{\mathbb{B}}^j_\lambda)^{**}(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}})=\infty$. Then $\lceil RP^{**}_{\lambda,j}\rceil(\widetilde{\mathbf{u}})=\infty$, as $\lambda\int_\Omega\mathbf{f}\cdot\widetilde{\mathbf{u}}\,dx$ is finite. If there exists a net $\{\widehat{\mathbf{u}}_p\}_{p\in P_3}$ such that $\widehat{\mathbf{u}}_p\rightharpoonup\widetilde{\mathbf{u}}$ in the weak* $BD(\Omega)$ topology and $\liminf_{p\in P_3}(\lambda F(\widehat{\mathbf{u}}_p)+G_j(\boldsymbol{\varepsilon}(\widehat{\mathbf{u}}_p)))<\infty$, then we have a contradiction with Steps 1 and 2 of this proof. Therefore, for every net $\{\mathbf{u}_p\}_{p\in P}$ such that $\mathbf{u}_p\rightharpoonup\widetilde{\mathbf{u}}$ in the weak* $BD(\Omega)$ topology, we have $\liminf_{p\in P}(\lambda F(\mathbf{u}_p)+G_j(\boldsymbol{\varepsilon}(\mathbf{u}_p)))=\infty=\lceil RP^{**}_{\lambda,j}\rceil(\widetilde{\mathbf{u}})$.

Step 5. For every $\widetilde{\mathbf{u}}\in BD(\Omega)$ we get

$$\inf\{\liminf(\lambda F(\mathbf{u}_p)+G_j(\boldsymbol{\varepsilon}(\mathbf{u}_p)))\mid\{\mathbf{u}_p\}_{p\in P}\text{ converges to }\widetilde{\mathbf{u}}\text{ in the weak* }BD(\Omega)\text{ topology}\}=\lceil RP^{**}_{\lambda,j}\rceil(\widetilde{\mathbf{u}}). \tag{5.13}$$

By [18, Chapter 1, Corollary 2.1] the proof is complete. ■

In Theorem 6.1 of [27, Chapter 2] and in Theorem 6.1 of [27, Chapter 1] only the equality of the infima of the relaxed and original problems has been shown. But it has not been proved that for every solution $\widehat{\mathbf{u}}$ of $(RP^{**}_{\lambda,j})$, there exists a net $\{\mathbf{u}_m\}_{m\in\mathbb{N}}$ which minimizes $P_{\lambda,j}$ and $\mathbf{u}_m\rightharpoonup\widehat{\mathbf{u}}$ weak* $BD(\Omega)$.

COROLLARY 19. *The function* (3.8) *is coercive over* $BD(\Omega)$ *if and only if* $\lceil RP^{**}_{\lambda,j}\rceil$ *is coercive.*

Proof. Suppose (3.8) is coercive. Then, by Theorem 18, so is $\lceil RP^{**}_{\lambda,j}\rceil$. Indeed, we have

$$\|\mathbf{u}\|_{BD(\Omega)}=\sup_{\mathbf{g},\mathbf{h}}\Big\{\int_\Omega\mathbf{g}\cdot\mathbf{u}\,dx+\int_\Omega\mathbf{h}:\boldsymbol{\varepsilon}(\mathbf{u})\ \Big|\ \mathbf{g}\in C(\Omega,\mathbb{R}^n),\ \mathbf{h}\in C(\Omega,\mathbf{E}^n_s),\ \|g_k\|_{C(\Omega,\mathbb{R})}\le 1,\ \|h_{ij}\|_{C(\Omega,\mathbb{R})}\le 1,\ \forall i,j,k=1,\dots,n\Big\} \tag{5.14}$$

for every $\mathbf{u}\in BD(\Omega)$ (cf. (2.3) and (2.4)–(2.6)). Then, for every $r>0$, $\mathrm{cl}_{\|\cdot\|_{BD}}B_{BD}(\mathbf{0},r)$ is the intersection of closed subsets in the weak* BD topology. Since (3.8) is coercive, for every k_s there exists r_s such that $\lambda F(\mathbf{u})+G_j(\boldsymbol{\varepsilon}(\mathbf{u}))>k_s$ for every $\mathbf{u}\in LD(\Omega)-B_{LD}(\mathbf{0},r_s)$. By (5.7) the proof is complete. ■

If $ds(\Gamma_0)=0$ and $L(\widehat{\mathbf{u}})=0$ for every $\widehat{\mathbf{u}}\in\mathcal{R}_0$, then the conclusions of Theorem 18 and Corollary 19 hold, where the functionals (3.8) and $\lceil RP^{**}_{\lambda,j}\rceil$ are defined over $BD(\Omega)/\mathcal{R}_0$.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] L. Ambrosio and G. Dal Maso, *On the relaxation in* $BV(\Omega, R^m)$ *of quasi-convex integrals*, J. Funct. Anal. 109 (1992), 76–97.

[3] L. Ambrosio and D. Pallara, *Integral representation of real functionals on* $BV(R^n;R^k)$ *and polyhedral approximation*, Indiana Univ. Math. J. 42 (1993), 295–321.

[4] G. Anzellotti, *On the extremal stress and displacement in Hencky plasticity*, Duke Math. J. 51 (1984), 133–147.

[5] P. Aviles and Y. Giga, *Variational integrals on mappings of bounded variation and their lower semicontinuity*, Arch. Rat. Mech. Anal. 115 (1991), 201–255.

[6] A. C. Barroso, G. Bouchitté, G. Buttazzo and I. Fonseca, *Relaxation of bulk and interfacial energies*, ibid. 135 (1996), 107–173.

[7] A. C. Barroso, I. Fonseca and R. Toader, *A relaxation theorem in the space of functions of bounded deformation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 29 (2000), 19–49.

[8] J. L. Bojarski, *The relaxation of Signorini problems in Hencky plasticity*, *I: three-dimensional solid*, Nonlinear Anal. 29 (1997), 1091–1116.

[9] —, *Regularity of solutions in plasticity. I: Continuum*, Appl. Math. (Warsaw) 30 (2003), 337–364.

[10] G. Bouchitté, I. Fonseca and L. Mascarenhas, *A global method for relaxation*, Arch. Rat. Mech. Anal. 145 (1998), 51–98.

[11] —, —, —, *Relaxation of variational problems under trace constraints*, Nonlinear Anal. 49 (2002), 221–246.

[12] G. Bouchitté and M. Valadier, *Integral representation of convex functionals on a space of measures*, J. Funct. Anal. 80 (1988), 398–420.

[13] N. Bourbaki, *Espaces Vectoriels Topologiques*, *Éléments de mathématique*, *Livre V*, Hermann, Paris, 1966.

[14] A. Braides, A. Defranceschi and E. Vitali, *A relaxation approach to Hencky's plasticity*, Appl. Math. Optim. 35 (1997), 45–68.

[15] E. Christiansen, *On the collapse solution in limit analysis*, Arch. Rat. Mech. Anal. 91 (1986), 119–135.

[16] —, *Limit analysis in plasticity as a mathematical programming problem*, Calcolo 17 (1980), 41–65.

[17] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.

[18] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.

[19] R. Engelking, *General Topology*, PWN–Polish Sci. Publ., Warszawa, 1977.

[20] I. Fonseca and S. Müller, *Relaxation of quasiconvex functionals in* $BV(\Omega, R^p)$ *for integrands* $f(x, u, \nabla u)$, Arch. Rat. Mech. Anal. 123 (1993), 1–49.

[21] A. Friedman and Y. Liu, *Propagation of cracks in elastic media*, ibid. 136 (1996), 235–290.

[22] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, lecture notes written by G. H. Williams, Dept. Math., Australian National Univ. 10, Canberra, 1977.

[23] R. Kohn and R. Temam, *Dual spaces of stresses and strains with applications to Hencky plasticity*, Appl. Math. Optim. 10 (1983), 1–35.

[24] J. J. Moreau, *Champs et distributions de tenseurs déformation sur un ouvert de connexité quelconque*, Séminaire d'Analyse Convexe, Université de Montpellier, 6 (1976).
[25] R. T. Rockafellar, *Integral functionals, normal integrands and measurable selections*, in: Nonlinear Operators and the Calculus of Variations, Lecture Notes in Math. 543, Springer, Berlin, 1975, 157–207.
[26] P. M. Suquet, *Discontinuities and plasticity*, in: Non-smooth Mechanics and Applications, J. J. Moreau and P. D. Panagiotopoulos (eds.), Springer, Wien, 1988, 278–340.
[27] R. Temam, *Mathematical Problems in Plasticity*, Gauthier-Villars, Paris, 1985.
[28] R. Temam and G. Strang, *Functions of bounded deformation*, Arch. Rat. Mech. Anal. 75 (1980), 7–21.
[29] F. Tomarelli, *Signorini problem in Hencky plasticity*, Ann. Univ. Ferrara-Sez. VII 36 (1990), 73–84.

Institute of Fundamental Technological Research
Polish Academy of Sciences
Świętokrzyska 21
00-049 Warszawa, Poland
E-mail: jbojar@ippt.gov.pl

Received on 28.1.2003;
revised version on 4.12.2003 (1671)