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LOCAL EXISTENCE OF SOLUTIONS OF THE FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF A MAGNETOHYDRODYNAMIC COMPRESSIBLE FLUID

Abstract. Local existence of solutions for the equations describing the motion of a magnetohydrodynamic compressible fluid in a domain bounded by a free surface is proved. In the exterior domain we have an electromagnetic field which is generated by some currents located on a fixed boundary. First by the Galerkin method and regularization techniques the existence of solutions of the linearized equations is proved, next by the method of successive aproximations local existence to the nonlinear problem is shown.

1. Introduction. In this paper we prove the existence of a local solution to the equations describing the motion of a magnetohydrodynamic compressible fluid in a domain $\Omega_t \subset \mathbb{R}^3$ bounded by a free surface S_t. In a domain $D_t \subset \mathbb{R}^3$ which is exterior to Ω_t we have a gas under constant pressure p_0. Moreover in D_t we have an electromagnetic field generated by some currents located on a fixed boundary B of D_t.

In the domain Ω_t the motion is described by the following problem:

$$
(1.1)\quad
\begin{aligned}
&\varrho(v_t + v\cdot\nabla v) - \operatorname{div}\mathbb{T}(v,p) - \mu_1 \overset{1}{H}\cdot\nabla\overset{1}{H} + \mu_1\nabla\overset{1}{H}{}^2 = f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\mu_1 \overset{1}{H}{}^1 = -\operatorname{rot}\overset{1}{E} && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}) && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{div}(\mu_1\overset{1}{H}) = 0 && \text{in } \widetilde{\Omega}^T,
\end{aligned}
$$

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where: $\widetilde{\Omega}^T = \bigcup_{0\le t\le T} \Omega_t \times \{t\}$, $v = v(x,t)$ is the velocity of the fluid, $p = p(\varrho)$ the pressure, $\varrho = \varrho(x,t)$ the density, $\overset{1}{H} = \overset{1}{H}(x,t)$ the magnetic field, $f = f(x,t)$ the external force field per unit mass, μ_1 the constant magnetic permeability, σ_1 the constant electric conductivity, $\overset{1}{E} = \overset{1}{E}(x,t)$ the electric field,

$$\mathbb{T}(v,p) = \mathbb{D}(v) - pI \tag{1.2}$$

the stress tensor, where I is the unit matrix and

$$\mathbb{D}(v) = \{\mu(\partial_{x_i} v_j + \partial_{x_j} v_i) + (\nu - \mu)\delta_{ij} \operatorname{div} v\}_{i,j=1,2,3} \tag{1.3}$$

is the dilatation tensor, and ν, μ the viscosity coefficients of the fluid.

In the domain D_t which is a dielectric (gas) we assume that there is no fluid motion inside ($v = 0$). Therefore we have the electromagnetic field only described by the following system:

$$\begin{aligned} &\mu_2 \overset{2}{H}_t = -\operatorname{rot} \overset{2}{E} && \text{in } \widetilde{D}^T,\\ &\operatorname{rot} \overset{2}{H} = \sigma_2 \overset{2}{E} && \text{in } \widetilde{D}^T,\\ &\operatorname{div}(\mu_2 \overset{2}{H}) = 0 && \text{in } \widetilde{D}^T, \end{aligned} \tag{1.4}$$

where $\widetilde{D}^T = \bigcup_{0\le t\le T} D_t \times \{t\}$.

On $S_t = \partial\Omega_t \cap \partial D_t$ we assume the following transmission and boundary conditions:

$$\begin{aligned} &n \cdot \mathbb{T}(v,p) = -p_0 n && \text{on } \widetilde{S}^T,\\ &\frac{1}{\sigma_1} \overset{1}{H} = \frac{1}{\sigma_2} \overset{2}{H} && \text{on } \widetilde{S}^T,\\ &\overset{1}{E} \cdot \tau_\alpha = \overset{2}{E} \cdot \tau_\alpha, \quad \alpha = 1,2, && \text{on } \widetilde{S}^T,\\ &v \cdot n = -\frac{\phi_t}{|\nabla \phi_t|} && \text{on } \widetilde{S}^T, \end{aligned} \tag{1.5}$$

where $\widetilde{S}^T = \bigcup_{0\le t\le T} S_t \times \{t\}$, n is the unit vector outward to Ω_t and normal to S_t, τ_α, $\alpha = 1,2$, is a tangent vector to S_t, and $\phi(x,t) = 0$ describes S_t at least locally.

Next we assume the following boundary conditions on B:

$$\begin{aligned} &\overset{2}{H} = H_* && \text{on } B,\\ &\overset{2}{E} = E_* && \text{on } B, \end{aligned} \tag{1.6}$$

where H_* and E_* are connected by

$$\sigma_2 E_{*n} = \frac{1}{A_{\tau_1}A_{\tau_2}}\,(\partial_{\tau_1}(H_{*\tau_1}A_{\tau_2}) - \partial_{\tau_2}(H_{*\tau_1}A_{\tau_1})),$$

$$\mu_2\partial_t H_{*n} = \frac{1}{A_{\tau_1}A_{\tau_2}}\,(\partial_{\tau_2}(E_{*\tau_1}A_{\tau_1}) - \partial_{\tau_1}(E_{*\tau_2}A_{\tau_2})),$$

$$\begin{aligned}-\partial_t\partial_{\tau_1}(H_{*\tau_1}A_{\tau_2}A_n) - \partial_t\partial_{\tau_2}(H_{*\tau_2}A_{\tau_1}A_n) &= \partial_{\tau_1}\partial_{\tau_2}(E_{*n}A_n)\\ -\mu_2\partial_{\tau_1}(A_{\tau_2}A_{\tau_3}\partial_t H_{*\tau_1}) - \mu_2\partial_{\tau_2}(A_{\tau_1}A_{\tau_3}\partial_t H_{*\tau_2}) &- \partial_{\tau_2}\partial_{\tau_1}(E_{*n}A_n),\end{aligned}$$

where (τ_1,τ_2,n) are curvilinear coordinates and $A_{\tau_1}, A_{\tau_2}, A_n$ are the Lamé coefficients of the transformation $(\tau_1,\tau_2,n)\mapsto(x_1,x_2,x_3)$.

Finally, we assume the initial conditions

$$\begin{aligned}&\Omega_t|_{t=0} = \Omega, \quad S_t|_{t=0} = S, \quad D_t|_{t=0} = D,\\ &\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0, \quad \overset{1}{H}|_{t=0} = \overset{1}{H}_0 \quad \text{in } \Omega,\\ &\overset{2}{H}|_{t=0} = \overset{2}{H}_0 \quad \text{in } D.\end{aligned} \tag{1.7}$$

To prove the existence of solutions to the above problem we introduce the Lagrangian coordinates $\xi\in\Omega$. The Lagrangian coordinates connected with the velocity v are the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi\in\Omega. \tag{1.8}$$

Therefore $x_v(\xi,t) = \xi + \int_0^t \overline{v}(\xi,\tau)\,d\tau$, where

$$\overline{v}(\xi,t) = v(x_v(\xi,t),t).$$

To introduce the Lagrangian coordinates in D_t we extend v on D_t. Let us denote the extended function by v'. Then we define $\xi\in D$ to be the Cauchy data to the problem

$$\frac{dx}{dt} = v'(x,t), \quad x|_{t=0} = \xi\in D. \tag{1.9}$$

Therefore $x_{v'}(\xi,t) = \xi + \int_0^t \overline{v}'(\xi,\tau)\,d\tau$, where $\overline{v}'(\xi,t) = v'(x_{v'}(\xi,t),t)$. Then by $(1.1)_5$,

$$\begin{aligned}\Omega_t &= \{x\in\mathbb{R}^3 : x = x_v(\xi,t),\ \xi\in\Omega\},\\ S_t &= \{x\in\mathbb{R}^3 : x = x_v(\xi,t),\ \xi\in S\}.\end{aligned}$$

Since S_t is determined at least locally by $\phi(x,t) = 0$, S is described by $\phi(x_v(\xi,t),t)|_{t=0} = 0$. Moreover, we have

$$\overline{n}_v = n(x_v(\xi,t),t) = \frac{\nabla_x\phi(x,t)}{|\nabla_x\phi(x,t)|}\bigg|_{x=x_v(\xi,t)}.$$

We introduce the following notation:

$$\begin{aligned}
&\|u\|_{l,Q} = \|u\|_{H^l(Q)}, && Q \in \{\Omega, S, D, \Pi, B\},\ 0 \le l \in \mathbb{Z},\\
&\|u\|_{k,p,q,Q^T} = \|u\|_{L_q(0,T,W_p^k(Q))}, && Q \in \{\Omega, S, D, \Pi, B\},\\
&p, q \in [1,\infty], && 0 \le k \in \mathbb{Z},
\end{aligned}$$

where $Q^t = Q \times (0,t)$,

$$|u|_{p,Q} = \|u\|_{L_p(Q)}, \qquad Q \in \{\Omega, S, D, \Pi, B\},\ p \in [1,\infty].$$

2. Weak solutions. Weak solutions to problem (1.1)–(1.7) are defined in Lagrangian coordinates.

Definition 2.1. By a *weak solution* for problem (1.1)–(1.7) we mean functions $\overline{v}, \overline{H}$ which satisfy the integral identities

$$\begin{aligned}
\int_0^T\int_\Omega (\overline{\varrho}\,\overline{v}_t\overline{\varphi} + \mathbb{D}_v(\overline{v})\mathbb{D}_v(\overline{\varphi}))I_v\,d\xi\,dt - \int_0^T\int_\Omega (\mu_1 \overset{1}{\overline{H}}\nabla_v \overset{1}{\overline{H}}\overline{\varphi} - \mu_1\nabla_v \overset{1}{\overline{H}}{}^2\overline{\varphi})I_v\,d\xi\,dt \\
= \int_0^T\int_\Omega \overline{f}\overline{\varphi}I_v\,d\xi\,dt - \int_0^T\int_S (\overline{p}-p_0)\overline{n}_v\overline{\varphi}I_v\,d\xi_s\,dt + \int_0^T\int_\Omega \nabla_v\overline{p}I_v\,d\xi\,dt,
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
\int_0^T\int_\Pi \Big(-\mu\overline{H}_t\overline{\psi} - \mu\overline{v}\nabla_v\overline{H}\overline{\psi} + \frac{1}{\sigma}\operatorname{rot}_v \overline{H}\operatorname{rot}_v\overline{\psi}\Big)I_v\,d\xi\,dt \\
- \int_0^T\int_\Omega \mu_1(\overline{v}\times\overset{1}{\overline{H}})\operatorname{rot}_v\overline{\psi}I_v\,d\xi\,dt = \frac{1}{\sigma_2}\int_0^T\int_B (\overline{n}_v\times\overline{E}_*)\overline{\psi}I_v\,d\xi_B\,dt,
\end{aligned}\tag{2.2}$$

where φ, ψ are sufficiently regular, and $\overline{n}_v$ is the unit outward vector normal to S or B.

In (2.1), (2.2) we use the notation $\overline{A}(\xi,t) = A(x_v(\xi,t),t)$,

$$\begin{aligned}
&\overline{H}|_\Omega = \overset{1}{\overline{H}}, \quad \overline{H}|_D = \overset{2}{\overline{H}}, \quad \sigma|_\Omega = \sigma_1, \quad \sigma|_D = \sigma_2,\\
&\Pi = \Omega\cup D, \quad \mu|_\Omega = \mu_1, \quad \mu|_D = \mu_2.
\end{aligned}$$

In (2.2), v is the extension on Π,

$$\mathbb{D}_v(\overline{v}) = \{\mu(\partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_j + \partial_{x_j}\xi_k\nabla_{\xi_k}\overline{v}_i) + (\nu-\mu)\delta_{ij}\operatorname{div}_v\overline{v}\}_{i,j=1,2,3},$$

$$\operatorname{rot}_v\overline{v} = \nabla_v\times\overline{v},$$

$$\nabla_v = \partial_x\xi_i\nabla_{\xi_i}, \qquad \operatorname{div}_v\overline{v} = \nabla_v\cdot\overline{v} = \partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_i, \qquad \partial_{\xi_i} = \nabla_{\xi_i}.$$

Let A be the Jacobi matrix of the transformation $x = x_v(\xi,t)$. Then

$$\det A = \exp\Big(\int_0^t \operatorname{div}_v\overline{v}\,d\tau\Big) = I_v$$

and if

$$\sup_{\xi\in\Omega}\sup_{t\in[0,T]}|\nabla_\xi\overline{v}|<\mu$$

then

$$0<c_1(1-\mu t)^3\le\det\{\partial_\xi x\}\le c_2(1+\mu t)^3,\qquad t\in[0,T],$$

where c_1,c_2 are constants and T is sufficiently small.

Moreover $x^i_{\xi^j}=\delta_{ij}+\int_0^t\partial_{\xi^j}\overline{v}^i(\xi,\tau)\,d\tau$ and $\xi_x=x_\xi^{-1}$. Then we get

$$\begin{aligned}\sup_{\xi\in\Omega}|x_\xi|&\le 1+\sup_{\xi\in\Omega}\int_0^t|\overline{v}_\xi(\xi,\tau)|\,d\tau\le 1+c\int_0^t\|\overline{v}\|_{3,\Omega}\,d\tau\\&\le 1+c\sqrt{t}\sqrt{\int_0^t\|\overline{v}\|^2_{3,\Omega}\,d\tau}\le 1+c\sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t}.\end{aligned}$$

Hence $\sup_{x\in\Omega_t}|\xi_x|\le\varphi(a)$, where $a=\sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t}$ and φ is an increasing positive function.

To prove the existence of a solution to the above problem we introduce Lagrangian coordinates connected with a given divergence-free function u. Moreover we linearize the nonlinear terms with v in (2.1) writing them in the form $u\nabla v$ and $u\times\overset{1}{H}$. Then from (2.1), (2.2) we get

$$\begin{aligned}(2.3)\qquad&\int_0^T\int_\Omega(\overline{\varrho}\,\overline{v}_t\overline{\varphi}+\mathbb{D}_u(\overline{v})\mathbb{D}_u(\overline{\varphi}))I_u\,d\xi\,dt\\&\quad-\int_0^T\int_\Omega(u_1\overset{1}{\overline{H}}{}'\nabla_u\overset{1}{\overline{H}}{}'\cdot\overline{\varphi}-\mu_1\nabla_u\overset{1}{\overline{H}}{}'^2\overline{\varphi})I_u\,d\xi\,dt\\&=\int_0^T\int_\Omega\overline{f}\overline{\varphi}I_u\,d\xi\,dt-\int_0^T\int_S(\overline{p}-p_0)\overline{n}_u\overline{\varphi}I_u\,d\xi_s\,dt+\int_0^T\int_\Omega\nabla_u\overline{p}I_u\,d\xi\,dt,\end{aligned}$$

$$\begin{aligned}(2.4)\qquad&\int_0^T\int_\Pi\left(-\mu\overline{H}\overline{\psi}_t-\mu\overline{u}\nabla_u\overline{H}\overline{\psi}+\frac{1}{\sigma}\operatorname{rot}_u\overline{H}\operatorname{rot}_u\overline{\psi}\right)I_u\,d\xi\,dt\\&\quad-\int_0^T\int_\Omega\mu_1(\overline{u}\times\overset{1}{\overline{H}})\operatorname{rot}_u\overline{\psi}I_u\,d\xi\,dt=\frac{1}{\sigma_2}\int_0^T\int_B(\overline{n}_u\times\overline{E}_*)\overline{\psi}I_u\,d\xi_B\,dt,\end{aligned}$$

where $\overline{u}$, $\overset{1}{\overline{H}}{}'$ are given functions and moreover $\overline{\varrho}$ is such that

$$0<\varphi_*\le\overline{\varrho}\le\varphi^*<\infty,\qquad\text{where }\varphi_*,\varphi^*\in\mathbb{R}.$$

3. Existence and regularity of solutions of the linearized problem (2.3). To prove the existence of solutions to the problem (2.3), (2.4) we use the Galerkin method. Take a basis $\{\overline{\varphi}_k\}$ in $L_2(\Omega)$ and $\{\overline{\psi}_k\}$ in $L_2(\Pi)$.

Then we are looking for an approximate solution in the form

$$(3.1)\qquad \overline{v}_n = \sum_{k=1}^{n} c_{kn}(t)\overline{\varphi}_k(\xi), \qquad \overline{H}_n = \sum_{k=1}^{n} d_{kn}(t)\overline{\psi}_k(\xi),$$

where the functions c_{kn}, d_{kn}, $k = 1, \ldots, n$, are solutions of the following system of ordinary differential equations:

$$(3.2)\qquad \int_{\Omega} (\overline{\varrho}\,\overline{v}_{nt}\overline{\varphi}_k + \mathbb{D}_u(\overline{v}_n)\mathbb{D}_u(\overline{\varphi}_k))I_u\,d\xi$$
$$= \mu_1 \int_{\Omega} (\overset{1}{\overline{H}}{}'\nabla_u \overset{1}{\overline{H}}{}'\overline{\varphi}_k - \nabla_u \overset{1}{\overline{H}}{}'^2\overline{\varphi}_k)I_u\,d\xi$$
$$= \int_{\Omega} \overline{f}\overline{\varphi}_k I_u\,d\xi - \int_{S} (\overline{p} - p_0)\overline{n}_u\overline{\varphi} I_u\,d\xi_S + \int_{\Omega} \nabla_u\overline{p} I_u\,d\xi\,dt,$$

$$(3.3)\qquad \int_{\Pi} \left(\mu\overline{H}_{nt}\overline{\psi}_k - \mu\overline{u}\nabla_u\overline{H}_n\overline{\psi}_k + \frac{1}{\sigma}\operatorname{rot}_u \overline{H}_n \operatorname{rot}_u \overline{\psi}_k\right) I_u\,d\xi$$
$$- \int_{\Omega} \mu_1(\overline{u} \times \overset{1}{\overline{H}})\operatorname{rot}_u \overline{\psi}_k I_u\,d\xi = \frac{1}{\sigma_2}\int_{B} (\overline{n}_u \times \overline{E}_*)\overline{\psi} I_u\,d\xi_B,$$

for $k = 1, \ldots, n$. The equations (3.2), (3.3) can be written in the form

$$(3.2)_1\qquad \overline{\varrho}\,\frac{d}{dt}\,c_{kn} + a_{ki}(t)c_{in} = f_k(t),$$

$$(3.3)_1\qquad \frac{d}{dt}\,d_{nk} + b_{ki}(t)d_{in} = g_k(t),$$

where $k = 1, \ldots, n$, and summation over repeated indices is assumed. Then from (3.2), (3.3) we see that

$$\sum_{k,i}\int_0^T |a_{ki}(t)|\,dt \le \varphi(a)(\|\overset{1}{\overline{H}}{}'\|^2_{1,2,2,\Omega^T} + 1),$$
$$\sum_{k,i}\int_0^T |b_{ki}(t)|\,dt \le \varphi(a)(\|\overline{u}\|^2_{1,2,2,\Omega^T} + 1),$$

where $a = T^{1/2}\|\overline{u}\|_{3,2,2,\Omega^t}$ and φ is an increasing positive function.

Next we have to assume that

$$\sup_{t\in[0,T]}\sup_{\xi\in\Omega} |I - \xi_x| \le \delta,$$

where δ is sufficiently small and I is the unit matrix.

LEMMA 3.1. *Assume that* $\overset{1}{\overline{H}}{}' \in L_\infty(0, T, L_2(\Omega)) \cap L_2(0, T, H^3(\Omega))$; $\overline{f}, \overset{1}{\overline{H}}{}'_t \in L_2(0, T, L_2(\Omega))$; $\overline{u} \in L_2(0, T, H^3(\Omega))$; $\overline{\varrho}, \overline{\varrho}_t \in L_2(0, T, H^1(\Omega))$. *Then for solutions of* (3.2) *the following inequality holds*:

$$\begin{aligned}(3.4)\qquad \|\overline{v}_n\|^2_{0,2,\infty,\Omega^t} + \|\overline{v}_n\|^2_{1,2,2,\Omega^t} &\le \alpha(t,a,\|\overline{\varrho}_t\|_{1,2,2,\Omega^t})[\|\bar{\overset{1}{H}}{}'\|^2_{0,2,\infty,\Omega^t}\\ &\quad\cdot(\varepsilon\|\bar{\overset{1}{H}}{}'\|^2_{3,2,2,\Omega^t} + c(\varepsilon)t(\|\bar{\overset{1}{H}}{}'(0)\|^2_{0,\Omega} + \|\bar{\overset{1}{H}}{}'_t\|^2_{0,2,2,\Omega^t}))\\ &\quad + c\|\bar{f}\|^2_{0,2,2,\Omega^t} + \|\overline{\varrho}\|^2_{1,2,2,\Omega^t}] + \int_\Omega \overline{\varrho}(0)\overline{v}^2(0)\,d\xi,\end{aligned}$$

where α is an increasing positive function.

Proof. Multiplying (3.2) by c_{kn} and summing over k from 1 to n we get

$$\begin{aligned}(3.5)\qquad &\frac{1}{2}\int_\Omega\left(\overline{\varrho}\,\frac{d}{dt}\overline{v}_n + |\mathbb{D}_u(\overline{v}_n)|^2\right)I_u\,d\xi\\ &\quad = \mu_1\int_\Omega(\bar{\overset{1}{H}}{}'\nabla_u\bar{\overset{1}{H}}{}'\overline{v}_n - \nabla_u\bar{\overset{1}{H}}{}'^2\overline{v}_n)I_u\,d\xi\\ &\quad\quad + \int_\Omega \bar{f}\overline{v}_nI_u\,d\xi - \int_S(\overline{p}-p_0)\overline{n}_u\overline{v}_nI_u\,d\xi_S + \int_\Omega\nabla_u\overline{p}\,\overline{v}_nI_u\,d\xi.\end{aligned}$$

Using the Korn inequality we get

$$\begin{aligned}(3.6)\qquad \frac{d}{dt}\int_\Omega\overline{\varrho}\,\overline{v}_nI_u\,d\xi + c\|\overline{v}_n\|^2_{1,\Omega} &\le \varphi(a)\|\bar{\overset{1}{H}}{}'\|^2_{0,\Omega}\|\bar{\overset{1}{H}}{}'\|^2_{2,\Omega} + c\|(\overline{p}-p_0)\overline{n}_u\|^2_{0,S}\\ &\quad + \|\bar{f}\|^2_{0,\Omega} + c\|\overline{v}_n\|^2_{0,\Omega} + c\int_\Omega(|\overline{\varrho}_t| + \overline{\varrho}|\mathrm{div}_u\,\overline{u}|)|\overline{v}_n|I_u\,d\xi + c\|\overline{p}\|^2_{1,\Omega}.\end{aligned}$$

Integrating with respect to time and using the Gronwall inequality we get (3.4).

To obtain more regular solutions to (3.2) we show

LEMMA 3.2. *Let the assumptions of Lemma* 3.1 *be satisfied and* $\overline{v}(0) \in H^1(\Omega)$. *Then*

$$\begin{aligned}(3.7)\qquad \|\overline{v}_{nt}\|^2_{0,2,2,\Omega^t} + \|\overline{v}_n\|^2_{1,2,\infty,\Omega^t} &\le \alpha(t,a)[\|\bar{\overset{1}{H}}{}'\|^2_{0,2,\infty,\Omega^t}(\varepsilon\|\bar{\overset{1}{H}}{}'\|^2_{3,2,2,\Omega^t}\\ &\quad + c(\varepsilon)t(\|\bar{\overset{1}{H}}{}'(0)\|^2_{0,\Omega} + \|\bar{\overset{1}{H}}{}'_t\|^2_{0,2,2,\Omega^t})) + c\|\bar{f}\|^2_{0,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{1,\Omega}\\ &\quad + c\|\overline{v}_n\|^2_{0,2,2,\Omega^t} + \varepsilon\|\overline{v}_{nt}\|^2_{1,2,2,\Omega^t} + \|\overline{\varrho}\|^2_{1,2,2,\Omega^t}],\end{aligned}$$

where α is an increasing positive function.

Proof. Multiplying (3.2) by $\frac{d}{dt}c_{kn}$ and summing over k from 1 to n we get

$$\begin{aligned}\int_\Omega(\overline{\varrho}\,\overline{v}^2_{nt} + \mathbb{D}_u(\overline{v}_n)\mathbb{D}_u(\overline{v}_{nt}))I_u\,d\xi &= \mu_1\int_\Omega(\bar{\overset{1}{H}}{}'\nabla_u\bar{\overset{1}{H}}{}'\overline{v}_{nt} - \nabla_u\bar{\overset{1}{H}}{}^2\,\overline{v}_{nt})I_u\,d\xi\\ &\quad + \int_\Omega\bar{f}\,\overline{v}_{nt}I_u\,d\xi - \int_S(\overline{p}-p_0)\overline{n}_u\overline{v}_{nt}I_u\,d\xi_S + \int_\Omega\nabla_u\overline{p}\,\overline{v}_{nt}I_u\,d\xi.\end{aligned}$$

Using the Hölder and Young inequalities we get

$$
\begin{aligned}
(3.8)\qquad & \|\overline{v}_{nt}\|_{0,\Omega}^2 + \frac{d}{dt}\|\mathbb{D}_u(\overline{v}_n)\|_{0,\Omega}^2 \\
&\le \varphi(a)\Big(\|\overset{1}{\overline{H}}{}'\|_{0,\Omega}^2\|\overset{1}{\overline{H}}{}'\|_{2,\Omega}^2 + \int\limits_\Omega |\overline{u}_\xi|\,|\mathbb{D}_u(\overline{v}_n)|\,|\overline{v}_{n\xi}|I_u\,d\xi\Big) \\
&\quad + c\|(\overline{p}-p_0)\overline{n}_u\|_{0,S}^2 + c\|\overline{f}\|_{0,\Omega}^2 + \varepsilon\|\overline{v}_{nt}\|_{1,\Omega}^2 + \|\overline{p}\|_{1,\Omega}^2.
\end{aligned}
$$

Integrating (3.8) with respect to time and using the Korn and Gronwall inequalities we get (3.7).

To estimate $\|\overline{v}_{nt}\|_{1,2,2,\Omega^t}$ we need the following result.

LEMMA 3.3. *Let the assumptions of Lemma* 3.2 *be satisfied and*

$$
\overline{f}_t \in L_2(0,T,L_2(\Omega)),\qquad \overset{1}{\overline{H}}{}', \overset{1}{\overline{H}}{}'_t \in L_\infty(0,T,H^1(\Omega)),
$$
$$
\int\limits_\Omega \overline{\varrho}(0)\overline{v}_t^2(0)\,d\xi < \infty.
$$

Then

$$
\begin{aligned}
(3.9)\qquad & \|\overline{v}_{nt}\|_{0,2,\infty,\Omega^t}^2 + \|\overline{v}_{nt}\|_{1,2,2,\Omega^t}^2 \le \alpha(a,t\|\overline{\varrho}_t\|_{2,2,2,\Omega^t}^2)\Big[(\varepsilon\|\overline{u}\|_{3,2,2,\Omega^t}^2 \\
&+ c(\varepsilon)t(\|\overline{u}_t\|_{0,2,2,\Omega^t}^2 + \|\overline{u}(0)\|_{0,\Omega}^2))(\|\overline{v}_n\|_{1,2,\infty,\Omega^t}^2 + \|\overset{1}{\overline{H}}{}'\|_{1,2,\infty,\Omega^t}^4) \\
&+ \|\overset{1}{\overline{H}}{}'_t\|_{1,2,\infty,\Omega^t}^2(\varepsilon\|\overset{1}{\overline{H}}{}'\|_{3,2,2,\Omega^t}^2 + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'(0)\|_{0,\Omega}^2 + \|\overset{1}{\overline{H}}{}'_t\|_{0,2,2,\Omega^t}^2)) \\
&+ c\|\overline{v}_{nt}\|_{0,2,2,\Omega^t}^2 + c\|\overline{f}_t\|_{0,2,2,\Omega^t}^2 + \int\limits_\Omega \overline{\varrho}(0)\overline{v}_t^2(0)\,d\xi + \|\overline{\varrho}_t\|_{1,2,2,\Omega^t}^2 \\
&+ \|\overline{\varrho}\|_{1,2,2,\Omega^t}^2\Big].
\end{aligned}
$$

Proof. Differentiating (3.2) with respect to t, multiplying by $\frac{d}{dt}c_{kn}$, summing over k from 1 to n and using the Korn, Hölder and Young inequalities we get

$$
\begin{aligned}
(3.10)\qquad & \frac{d}{dt}\|\overline{\varrho}\,\overline{v}_{nt}\|_{0,\Omega}^2 + \|\overline{v}_{nt}\|_{1,\Omega}^2 \le \varphi(a)\Big[\|\overline{u}_\xi\|_{\infty,\Omega}\Big(\varepsilon\|\overline{v}_{nt}\|_{1,\Omega}^2 + \|\overline{p}\|_{1,\Omega}^2 \\
&+ \|\overline{v}_n\|_{1,\Omega}^2 + \|\overset{1}{\overline{H}}{}'\|_{1,\Omega}^4 + \int\limits_\Omega \overline{\varrho}\,\overline{v}_{nt}^2 I_u\,d\xi\Big) + \frac{1}{\varrho_*}\|\overline{\varrho}_t\|_{2,\Omega}\int\limits_\Omega \overline{\varrho}\,\overline{v}_{nt}^2 I_u\,d\xi \\
&+ \|\overset{1}{\overline{H}}{}'_t\|_{1,\Omega}^2\|\overset{1}{\overline{H}}{}'\|_{1,\Omega}^2 + \varepsilon\|\overline{v}_{nt}\|_{1,\Omega}^2\Big] + c\|\overline{f}_t\|_{0,\Omega}^2 + c\|(\overline{p}-p_0)\overline{n}_{ut}\|_{0,S}^2 \\
&+ \|\overline{v}_{nt}\|_{0,\Omega}^2 + \|\overline{p}\|_{1,\Omega}^2.
\end{aligned}
$$

Integrating (3.10) with respect to t and using the Gronwall inequality we get (3.9).

From Lemmas 3.1–3.3 we have

LEMMA 3.4. *Let the assumptions of Lemmas* 3.1–3.3 *be satisfied*, *and* $\overline{\varrho}, \overline{\varrho}_t \in L_2(0,T,H^2(\Omega))$, $\overline{\varrho}_{tt} \in L_2(0,T,L_2(\Omega))$. *Then*

$$
\begin{aligned}
&\|\overline{v}_n\|^2_{1,2,2,\Omega^t} + \|\overline{v}_{nt}\|^2_{1,2,2,\Omega^t} + \|\overline{v}_n\|^2_{2,2,\infty,\Omega^t} + \|\overline{v}_{nt}\|^2_{0,2,\infty,\Omega^t} \\
&\quad \le \alpha(t,a,\|\overline{\varrho}_t\|_{2,2,2,\Omega^t})[(\varepsilon\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t} \\
&\qquad + \|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega}))(\|\overset{1}{\overline{H}}{}'\|^2_{0,2,\infty,\Omega^t} + \|\overset{1}{\overline{H}}{}'\|^2_{1,2,\infty,\Omega^t}) + (\varepsilon\|\overline{u}\|^2_{3,2,2,\Omega^t} \\
&\qquad + c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Omega^t} + \|\overline{u}(0)\|^2_{0,\Omega}))\|\overset{1}{\overline{H}}{}'\|^4_{1,2,\infty,\Omega^t} + c\|\overline{f}\|^2_{0,2,2,\Omega^t} \\
&\qquad + c\|\overline{f}_t\|^2_{0,2,2,\Omega^t} + \int_\Omega \overline{\varrho}(0)\overline{v}^2(0)\,d\xi + \|\overline{v}_n(0)\|^2_{1,\Omega} + \int_\Omega \overline{\varrho}(0)\overline{v}_t^2(0)\,d\xi \\
&\qquad + \|\overline{v}_t(0)\|^2_{0,\Omega} + \varepsilon(\|\overline{\varrho}\|^2_{2,2,2,\Omega^t} + \|\overline{\varrho}_t\|^2_{2,2,2,\Omega^t}) \\
&\qquad + c(\varepsilon)t(\|\overline{\varrho}_t\|^2_{0,2,2,\Omega^t} + \|\overline{\varrho}_{tt}\|^2_{0,2,2,\Omega^t} + \|\overline{\varrho}(0)\|^2_{0,\Omega} + \|\overline{\varrho}_t(0)\|^2_{0,\Omega})] \equiv \overline{F},
\end{aligned}
$$

where α is an increasing positive function.

Now choosing a subsequence and letting $n \to \infty$ we get

LEMMA 3.5. *Let the assumptions of Lemmas* 3.1–3.4 *be satisfied. Then there exists a weak solution of problem* (3.2) *such that* $\overline{v} \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^1(\Omega))$; $\overline{v}_t \in L_2(0,T,H^1(\Omega)) \cap L_\infty(0,T,L_2(\Omega))$ *and*

$$\|\overline{v}\|^2_{1,2,2,\Omega^t} + \|\overline{v}_t\|^2_{1,2,2,\Omega^t} + \|\overline{v}\|^2_{1,2,\infty,\Omega^t} + \|\overline{v}_t\|^2_{0,2,\infty,\Omega^t} \le \overline{F}.$$

To show that $\overline{v} \in L_2(0,T,H^3(\Omega))$ we consider the following elliptic problem:

$$
\begin{aligned}
&-\operatorname{div}_u \mathbb{D}_u(\overline{v}) = \overline{\varrho}\,\overline{v}_t + \mu_1 \overset{1}{\overline{H}}{}'\nabla_u \overset{1}{\overline{H}}{}' + \mu_1 \overset{1}{\overline{H}}{}'^2 + \overline{f} + \nabla_u \overline{p} && \text{in } \Omega^T, \\
&\overline{n}_u \mathbb{D}_u(\overline{v}) = (\overline{p} - p_0)\overline{n}_u && \text{on } S^T.
\end{aligned}
\tag{3.17}
$$

Using in (3.17) the regularization technique for elliptic problems (see [7]) and Lemma 3.5 we get

LEMMA 3.6. *Let the assumptions of Lemma* 3.5 *be satisfied and* $\overset{1}{\overline{H}}{}' \in L_\infty(0,T,H^2(\Omega))$; $\overline{f}, \overline{\varrho} \in L_2(0,T,H^1(\Omega))$. *Then*

$$
\begin{aligned}
\|\overline{v}\|^2_{3,2,2,\Omega^t} &\le \alpha(a,t)[(\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t})(\varepsilon\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} \\
&\quad + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega^t} + \|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t})) + \|\overline{v}_t\|^2_{1,2,2,\Omega^t} \\
&\quad + \|\overline{f}\|^2_{1,2,2,\Omega^t} + \|\overline{\varrho}\|^2_{2,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{2,\Omega}].
\end{aligned}
\tag{3.18}
$$

To show that $v_t \in L_2(0,T,H^2(\Omega))$ we differentiate (3.17) with respect to t. Then we get the following parabolic problem:

$$
\begin{aligned}
&\overline{\varrho}\,\overline{v}_{tt} - \operatorname{div}_u \mathbb{D}_u(\overline{v}_t) = (\operatorname{div}_u)_t\mathbb{D}_u(\overline{v}) + \operatorname{div}_u(\mathbb{D}_u)_t(\overline{v}) \\
&\qquad - \overline{v}_t\overline{\varrho}_t + (\mu_1 \overset{1}{\overline{H}}{}'\nabla_u \overset{1}{\overline{H}}{}'^2)_t - (\mu_1\nabla_u \overset{1}{\overline{H}}{}'^2)_t + \overline{f}_t + (\nabla_u\overline{p})_t \quad && \text{in } \Omega^T \\
&\overline{n}_u\mathbb{D}_u(\overline{v}_t) = (\overline{p}-p_0)_t\overline{n}_u + (\overline{p}-p_0)(\overline{n}_u)_t + (\overline{n}_u\mathbb{D}_u)_t(\overline{v}) && \text{on } S^T. \\
&\overline{v}|_{t=0} = \overline{v}_0 && \text{in } \Omega.
\end{aligned}
\tag{3.19}
$$

Using the regularization technique for parabolic problems (see [7]) we get

$$
\begin{aligned}
\|\overline{v}_t\|^2_{2,\Omega} &\le \varphi(a)[\|\overline{u}\|^2_{2,\Omega}(\|\overline{v}\|^2_{3,\Omega} + \|\overline{\varrho}\|^2_{2,\Omega} + \|\overset{1}{\overline{H}}{}'\|^4_{2,\Omega}) \\
&\quad + \|\overset{1}{\overline{H}}{}'\|^2_{1,\Omega}\|\overset{1}{\overline{H}}{}'\|^2_{2,\Omega} + \|\overset{1}{\overline{H}}{}'_t\|^2_{2,\Omega}\|\overset{1}{\overline{H}}{}'\|^2_{1,\Omega}] + \|\overline{v}_{tt}\|^2_{0,\Omega} + \|\overline{f}_t\|^2_{0,\Omega} \\
&\quad + \varepsilon\|\overline{\varrho}\|^2_{0,\Omega} + \varepsilon\|\overline{\varrho}_t\|^2_{0,\Omega}.
\end{aligned}
\tag{3.20}
$$

Integrating (3.2) with respect to time and using Lemmas 3.5–3.6 we get

Lemma 3.7. *Let the assumptions of Lemmas* 3.5–3.6 *be satisfied and* $\overline{u} \in L_\infty(0,T,H^2(\Omega))$; $\overset{1}{\overline{H}}{}'_t \in L_2(0,T,H^2(\Omega))$; $\overset{1}{\overline{H}}{}'_{tt}, \overline{f}_t \in L_2(0,T,L_2(\Omega))$; $\overset{1}{\overline{H}}{}'(0), \overset{1}{\overline{H}}{}'_t(0) \in L_2(\Omega)$. *Then*

$$
\begin{aligned}
\|\overline{v}_t\|^2_{2,2,2,\Omega^t} &\le \alpha(a,t)[\|\overline{u}\|^2_{2,2,\infty,\Omega^t}(\|\overline{v}\|^2_{3,2,2,\Omega^t} + \|\overline{\varrho}\|^2_{1,2,2,\Omega^t}) \\
&\quad + (\|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t}(1 + \|\overline{u}\|^2_{2,2,\infty,\Omega^t}) + \|\overset{1}{\overline{H}}{}'_t\|^2_{2,2,2,\Omega^t}) \\
&\quad \cdot (\varepsilon(\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'_t\|^2_{2,2,2,\Omega^t}) + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t} \\
&\quad + \|\overset{1}{\overline{H}}{}'_{tt}\|^2_{0,2,2,\Omega^t} + \|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega} + \|\overset{1}{\overline{H}}{}'_t(0)\|^2_{0,\Omega})) + \|\overline{v}_{tt}\|^2_{0,2,2,\Omega^t} \\
&\quad + \|\overline{f}_t\|^2_{0,2,2,\Omega^t} + \|\overline{\varrho}\|^2_{1,2,2,\Omega^t} + \|\overline{\varrho}_t\|^2_{0,2,2,\Omega^t}].
\end{aligned}
\tag{3.21}
$$

where α is an increasing positive function.

Next we have to estimate $\|\overline{v}_{tt}\|_{1,2,2,\Omega^t}$. Then similarly to Lemma 3.8 of [3] we prove

Lemma 3.8. *Let the assumptions of Lemmas* 3.5–3.7 *be satisfied and* $\overline{u}_t \in L_2(0,T,H^2(\Omega) \cap L_\infty(0,T,H^1(\Omega))$; $\overset{1}{\overline{H}}_{tt} \in L_2(0,T,H^1(\Omega))$; $\overline{u}_{tt} \in L_2(0,T,H^1(\Omega))$; $\overline{\varrho}_t \in L_\infty(0,T,H^1(\Omega))$; $\overline{\varrho} \in L_\infty(0,T,H^2(\Omega))$. *Then*

$$(3.22)\quad \|\overline{v}_{tt}\|^2_{0,2,\infty,\Omega^t}+\|\overline{v}_{tt}\|^2_{1,2,2,\Omega^t}\le \alpha(a,t)[(\varepsilon\|\overline{v}\|^2_{3,2,2,\Omega^t}+c(\varepsilon)t(\|\overline{v}_t\|^2_{0,2,2,\Omega^t}$$
$$+\|\overline{v}(0)\|^2_{0,\Omega}))\|\overline{u}_t\|^2_{1,2,\infty,\Omega^t}+\varepsilon\|\overset{1}{\overline{H}}{}'_{tt}\|^4_{1,2,2,\Omega^t}+ct^2\|\overset{1}{\overline{H}}{}'_t\|^4_{1,2,2,\Omega^t}$$
$$+c\|\overset{1}{\overline{H}}{}'(0)\|^4_{2,\Omega}+(\varepsilon\|\overset{1}{\overline{H}}{}'\|^2_{2,2,2,\Omega^t}+c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'_{tt}\|^2_{0,2,2,\Omega^t}+\|\overset{1}{\overline{H}}{}'_t(0)\|^2_{0,\Omega}))$$
$$\cdot(\|\overline{u}\|^2_{2,2,\infty,\Omega^t}\|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t}+\|\overset{1}{\overline{H}}{}'_t\|^2_{1,2,2,\Omega^t})+(1+\|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t})$$
$$\cdot(\varepsilon\|\overline{u}_t\|^2_{2,2,2,\Omega^t}+c(\varepsilon)(\|\overline{u}_{tt}\|^2_{0,2,2,\Omega^t}+\|\overline{u}_t(0)\|^2_{0,\Omega}))+\|\bar{f}_{tt}\|^2_{0,2,2,\Omega^t}$$
$$+\|\overline{v}_{tt}(0)\|^2_{0,\Omega}+\|\overline{\varrho}\|^2_{2,2,\infty,\Omega^t}(\varepsilon\|\overline{u}\|^2_{2,2,2,\Omega^t}+\|\overline{u}_t\|^2_{0,2,\infty,\Omega^t}\|\overline{v}_t\|^2_{1,2,2,\Omega^t}$$
$$+\|\overline{u}\|^4_{2,2,\infty,\Omega^t}+\|\overline{v}_t\|^2_{1,2,2,\Omega^t}+c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Omega^t}+\|\overline{u}(0)\|^2_{0,\Omega}))$$
$$+\|\overline{\varrho}_{tt}\|^2_{1,2,2,\Omega^t}+\|\overline{v}_t\|^2_{0,2,\infty,\Omega^t}+\|\overline{\varrho}_t\|^2_{1,2,\infty,\Omega^t}\|\overline{u}\|^2_{2,2,\infty,\Omega^t}+\|\overline{v}_t\|^2_{1,2,2,\Omega^t}$$
$$+\|\overline{v}\|^2_{2,2,2,\Omega^t}+\|\overline{v}\|^4_{2,2,2,\Omega^t}+\|\overline{\varrho}_{tt}\|^2_{0,2,2,\Omega^t}+\|\overline{\varrho}_t\|^2_{1,2,\infty,\Omega^t}],$$

where α is an increasing positive function.

From Lemmas 3.5–3.8 we get

$$\overline{v}\in L_2(0,T,H^3(\Omega))\cap L_\infty(0,T,H^1(\Omega));$$
$$v_t\in L_\infty(0,T,H^1(\Omega))\cap L_2(0,T,H^2(\Omega)),$$
$$\overline{v}_{tt}\in L_\infty(0,T,L_2(\Omega))\cap L_2(0,T,H^1(\Omega)).\ \blacksquare$$

4. Existence and regularity of solutions of the linearized problem (2.4). In [3] we prove the following lemmas for problem (1.1)–(1.7):

Lemma 4.1. *Assume $\overline{u}\in L_2(0,T,H^3(\Pi))$; $\overline{u}_t\in L_2(0,T,H^2(\Pi))$; $\overline{u}_{tt}\in L_2(0,T,H^1(\Pi))$; $\overline{H}_*,\overline{H}_{*t}\in L_2(0,T,H^2(B))$; $\overline{H}_*,\bar{E}_*\in L_\infty(0,T,H^1(B))$; $\bar{E}_{*t}\in L_2(0,T,L_2(B))$; $\overline{H}(0)\in H^1(\Pi)$; $\overline{H}_t(0)\in L_2(\Pi)$. Then there exists a weak solution of problem* (3.3) *such that*

$$\overline{H}\in L_\infty(0,T,H^1(\Pi))\cap L_2(0,T,H^1(\Pi)),$$
$$\overline{H}_t\in L_2(0,T,H^1(\Pi))\cap L_\infty(0,T,L_2(\Pi))$$

and

$$(4.1)\quad \|\overline{H}\|^2_{1,2,2,\Pi^t}+\|\overline{H}_t\|^2_{1,2,2,\Pi^t}+\|\overline{H}_t\|^2_{0,2,\infty,\Pi^t}+\|\overline{H}\|^2_{1,2,\infty,\Pi^t}$$
$$\le\alpha(a,t)[(\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t}+c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Pi^t}$$
$$+\|\overline{u}(0)\|^2_{0,\Pi}))\|\bar{E}_*\|^2_{1,2,\infty,B^t}+\|\overline{H}(0)\|^2_{1,\Pi}+\|\overline{H}_t(0)\|^2_{0,\Pi}$$
$$+\|\bar{E}_{*t}\|^2_{0,2,2,B^t}+\|\overline{H}_*\|^2_{1,2,2,B^t}+\|\overline{H}_*\|^2_{1,2,\infty,B^t}+\|\overline{H}_{*t}\|^2_{1,2,2,B^t}],$$

where α is an increasing positive function.

LEMMA 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied and* $\overline{H}_* \in L_2(0,T,H^3(B))$; $\overline{H}_{*t} \in L_2(0,T,H^2(B))$; $\overline{H}_{*tt} \in L_2(0,T,L_2(B))$. *Then*

$$
\begin{aligned}
(4.2)\qquad \|\overline{H}\|^2_{3,2,2,\Pi^t} \le \alpha(a,t)[((1 &+ \|\overline{u}\|^2_{2,2,\infty,\Pi^t} + \|\overline{u}\|^2_{3,2,2,\Pi^t}\\
&+ \|\overline{u}\|^4_{2,2,\infty,\Pi^t} + \|\overline{u}_t\|^2_{2,2,2,\Pi^t})(\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t} + c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Pi^t}\\
&+ \|\overline{u}(0)\|^2_{0,\Pi}))(\|\overline{u}\|^2_{2,2,\infty,\Pi^t} + 1) + \|\overline{H}(0)\|^2_{0,\Pi} + \|\overline{H}_*\|^2_{3,2,2,B^t}\\
&+ \|\overline{H}_*(0)\|^2_{0,B})(t\|\overline{u}_t\|^2_{2,2,2,\Pi^t} + \|\overline{u}(0)\|^2_{2,\Pi} + 1)^2 + ct\|\overline{u}_t\|^2_{2,2,2,\Pi^t}\\
&+ \|\overline{u}(0)\|^2_{2,\Pi} + \varepsilon\|\overline{u}_t\|^2_{2,2,2,\Pi^t} + c(\varepsilon)t(\|\overline{u}_{tt}\|^2_{0,2,2,\Pi^t} + \|\overline{u}_t(0)\|^2_{0,\Pi})\\
&+ \|\overline{H}_{*t}\|^2_{2,2,2,B^t} + \|\overline{H}_{*tt}\|^2_{0,2,2,B^t} + \|\overline{H}(0)\|^2_{1,\Pi}],
\end{aligned}
$$

where α *is an increasing positive function.*

LEMMA 4.3. *Let the assumptions of Lemmas* 4.1–4.2 *be satisfied and* $\bar{E}^*, \bar{E}_{*t} \in L_2(0,T,L_2(B))$; $\overline{u}_{tt} \in L_2(0,T,H^1(\Pi))$; $\overline{u}_t \in L_\infty(0,T,H^1(\Pi))$; $\overline{u} \in L_\infty(0,T,H^2(\Pi))$. *Then*

$$
\begin{aligned}
(4.3)\qquad \|\overline{H}_{tt}\|^2_{0,2,\infty,\Pi^t} + \|\overline{H}_{tt}\|^2_{1,2,2,\Pi^t} &\le \alpha(a,t)[\varepsilon(\|\overline{u}_{tt}\|^2_{1,2,2,\Pi^t}\\
&+ \|\overline{u}_t\|^4_{1,2,\infty,\Pi^t}\|\overline{u}\|^4_{2,2,\infty,\Pi^t} + \|\overline{u}_t\|^4_{1,2,\infty,\Pi^t} + \|\overline{u}\|^4_{2,2,\infty,\Pi^t}\\
&+ \|\overline{u}_{tt}\|^4_{1,2,2,\Pi^t})(t^2\|\overline{H}_t\|^4_{1,2,2,\Pi^t} + \|\overline{H}(0)\|^4_{1,\Pi}) + (\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t}\\
&+ c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Pi^t} + \|\overline{u}(0)\|^2_{0,\Pi}))(\|\overline{u}_t\|^2_{1,2,\infty,\Pi^t} + 1)\|\overline{H}_{tt}(0)\|^2_{0,\Pi}\\
&+ \|\overline{u}_t\|^2_{1,2,\infty,\Pi^t}(\varepsilon\|\overline{H}\|^2_{3,2,2,\Pi^t} + c(\varepsilon)t(\|\overline{H}_t\|^2_{0,2,2,\Pi^t} + \|\overline{H}(0)\|^2_{0,\Pi}))\\
&+ \|\bar{E}_*\|^2_{0,2,2,B^t} + \|\bar{E}_{*tt}\|^2_{0,2,2,B^t}],
\end{aligned}
$$

where α *is an increasing positive function.*

LEMMA 4.4. *Let the assumptions of Lemmas* 4.1–4.3 *be satisfied. Then*

$$
\begin{aligned}
(4.4)\qquad \|\overline{H}_t\|^2_{2,2,2,\Pi^t} &\le \alpha(a,t)[(\varepsilon(\|\overline{H}\|^2_{3,2,2,\Pi^t} + \|\overline{H}_*\|^2_{3,2,2,B^t} + \|\overline{u}_t\|^2_{2,2,2,\Pi^t})\\
&+ c(\varepsilon)t(\|\overline{H}_t\|^2_{0,2,2,\Pi^t} + \|\overline{H}_{tt}\|^2_{0,2,2,\Pi^t} + \|\overline{H}_{*t}\|^2_{0,2,2,B^t} + \|\overline{H}_{*tt}\|^2_{0,2,2,B^t}\\
&+ \|\overline{u}_t\|^2_{0,2,2,\Pi^t} + \|\overline{u}_{tt}\|^2_{0,2,2,\Pi^t} + \|\overline{u}(0)\|^2_{0,\Pi} + \|\overline{u}_t(0)\|^2_{0,\Pi} + \|\overline{H}(0)\|^2_{0,\Pi}\\
&+ \|\overline{H}_t(0)\|^2_{0,\Pi}))(\|\overline{u}_t\|^2_{2,2,2,\Pi^t} + \|\overline{u}_t\|^2_{1,2,\infty,\Pi^t} + \|\overline{u}(0)\|^2_{2,\Pi} + \|\overline{H}(0)\|^2_{2,\Pi})\\
&\cdot \varepsilon\|\overline{u}_t\|^4_{2,2,2,\Pi^t} + \|\overline{H}_{*t}\|^2_{1,2,\infty,B^t} + \|\overline{H}_{*t}\|^4_{1,2,2,B^t} + \|\overline{H}_t\|^2_{1,2,2,\Pi^t}\\
&\cdot (t\|\overline{u}_t\|^2_{2,2,2,\Pi^t} + \|\overline{u}(0)\|^2_{2,\Pi})\|\overline{H}\|^2_{3,2,2,\Pi^t}],
\end{aligned}
$$

where α *is an increasing positive function.*

Then from Lemmas 4.1–4.4 we conclude that

$$\begin{aligned}
&\overline{H} \in L_2(0,T,H^3(\Pi)) \cap L_\infty(0,T,H^1(\Pi));\\
&\overline{H}_t \in L_\infty(0,T,H^1(\Pi)) \cap L_2(0,T,H^2(\Pi));\\
&\overline{H}_{tt} \in L_\infty(0,T,L_2(\Pi)) \cap L_2(0,T,H^1(\Pi)).
\end{aligned}$$

5. Existence of solution of (1.1)–(1.7). Now we prove the existence of solutions of (1.1) by the method of successive approximations:

$$\begin{aligned}
&\overline{\varrho}_m \overline{v}_{m+1,t} - \operatorname{div}_{\overline{v}_m} \mathbb{D}_{\overline{v}_n}(\overline{v}_{m+1}) = \mu_1 \overset{1}{\overline{H}}_m \nabla_{\overline{v}_m} \overset{1}{\overline{H}}_m \\
&\qquad + \nabla_{\overline{v}_m} \overset{1}{\overline{H}}{}^2_m + \nabla_{\overline{v}_m} \overline{p}_m + \overline{f} && \text{in } \Omega^T,\\
&\overline{\varrho}_{mt} + \overline{\varrho}_m \operatorname{div}_{\overline{v}_m} \overline{v}_m = 0 && \text{in } \Omega^T\\
&\mu_1 \overset{1}{\overline{H}}_{m+1,t} = -\operatorname{rot}_{\overline{v}_m} \overset{1}{\overline{E}}_m && \text{in } \Omega^T,\\
&\operatorname{rot}_{\overline{v}_m} \overset{1}{\overline{H}}_{m+1} = \sigma_1(\overset{1}{\overline{E}}_{m+1} + \mu_1 \overline{v}_m \times \overset{1}{\overline{H}}_{m+1}) && \text{in } \Omega^T,\\
&\operatorname{div}_{\overline{v}_m}(\mu_1 \overset{1}{\overline{H}}_{m+1}) = 0 && \text{in } \Omega^T,\\
&\overline{n}_{\overline{v}_m} \mathbb{D}_{\overline{v}_m}(\overline{v}_{m+1}) = (\overline{p}_m - p_0)\overline{n}_{\overline{v}_m} && \text{on } S^T,\\
&\frac{1}{\sigma_1} \overset{1}{\overline{H}}_{m+1} = \frac{1}{\sigma_2} \overset{2}{\overline{H}}_{m+1} && \text{on } S^T,\\
&\overset{1}{\overline{E}}_{m+1} \cdot \overline{\tau}_{\alpha \overline{v}_m} = \overset{2}{\overline{E}}_{m+1} \cdot \overline{\tau}_{\alpha \overline{v}_m}, \qquad \alpha = 1,2, && \text{on } S^T,\\
&\overline{v}_{m+1}|_{t=0} = \overline{v}_0, \qquad \overset{1}{\overline{H}}_{m+1}|_{t=0} = \overset{1}{\overline{H}}_0, \qquad \overline{\varrho}_m|_{t=0} = \varrho_0,
\end{aligned}\tag{5.1}$$

$$\begin{aligned}
&\mu_2 \overset{2}{\overline{H}}_t = -\operatorname{rot}_{\overline{v}_m} \overset{2}{\overline{E}} && \text{in } D^T,\\
&\sigma_2 \overset{2}{\overline{E}} = \operatorname{rot}_{\overline{v}_m} \overset{2}{\overline{H}} && \text{in } D^T,\\
&\operatorname{div}_{\overline{v}_m}(\mu_2 \overset{2}{\overline{H}}_{m+1}) = 0 && \text{in } D^T,\\
&\overset{2}{\overline{H}}_{m+1} = \overline{H}_*, \qquad \overset{2}{\overline{E}}_{m+1} = \overline{E}_* && \text{on } B,\\
&\overset{2}{\overline{H}}_{m+1}|_{t=0} = \overset{2}{\overline{H}}_0.
\end{aligned}\tag{5.2}$$

First we show the boundedness of the sequence described by (5.1)–(5.2) in the norm

$$\begin{aligned}
B_m(t) &= \|\overline{v}_{mt}\|^2_{0,\Omega} + \|\overline{v}_m\|^2_{1,\Omega} + \|\overline{v}_m\|^2_{3,2,2,\Omega^t} + \|\overline{v}_{mt}\|^2_{2,2,2,\Omega^t}\\
&\quad + \|\overline{v}_{mtt}\|^2_{1,2,2,\Omega^t} + \|\overline{H}_{mt}\|^2_{0,\Pi} + \|\overline{H}_m\|^2_{1,\Pi} + \|\overline{H}_m\|^2_{3,2,2,\Pi^t}\\
&\quad + \|\overline{H}_{mt}\|^2_{2,2,2,\Pi^t} + \|\overline{H}_{mtt}\|^2_{1,2,2,\Omega^t}.
\end{aligned}\tag{5.3}$$

Lemma 5.1. *There exists a positive function P and $A > 0$ such that $P(0,0,F_0) < A$, $B_m(0) < A$ and there exists T_* such that $B_m(t) \le A$ for $t \le T_*$, $m = 1,2,\dots$, where F_0 is a function depending on $\overline{H}_*$, $\overline{E}_*$, $\overline{H}(0)$, $\overline{v}(0)$, $\overline{\varrho}(0)$, $\overline{f}$ in the norms of Lemmas* 3.1–4.4.

Proof. Integrating $(5.1)_2$ we get

$$\overline{\varrho}_m(\xi,t) = \overline{\varrho}_0(\xi)e^{-\int_0^t \operatorname{div}_{\overline{v}_m} \overline{v}_m\,d\tau}. \tag{5.4}$$

From (5.4) we have

$$\begin{aligned}\|\overline{\varrho}_m\|^2_{2,2,\infty,\Omega^t} \le c\|\overline{\varrho}_0\|^2_{3,2,2,\Omega^t}\varphi_1(a)[(\varepsilon+t)\|\overline{v}_m\|^2_{3,2,2,\Omega^t}\\ + c(\varepsilon)t(\|\overline{v}_{mt}\|^2_{0,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{0,\Omega})],\end{aligned} \tag{5.5}$$

$$\begin{aligned}\|\overline{\varrho}_{mt}\|^2_{2,2,2,\Omega^t} \le c\|\overline{\varrho}_0\|^2_{3,2,2,\Omega^t}\varphi_2(a)[(\varepsilon+t)\|\overline{v}_m\|^2_{3,2,2,\Omega^t}\\ + c(\varepsilon)t(\|\overline{v}_{mt}\|^2_{0,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{0,\Omega})],\end{aligned} \tag{5.6}$$

$$\begin{aligned}\|\overline{\varrho}_{mtt}\|^2_{1,2,2,\Omega^t} \le c\|\overline{\varrho}_0\|^2_{3,2,2,\Omega^t}\varphi_3(a)[\varepsilon\|\overline{v}_m\|^2_{3,2,2,\Omega^t}\\ + c(\varepsilon)t(\|\overline{v}_{mt}\|^2_{0,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{0,\Omega})\\ + \varepsilon\|\overline{v}_{mt}\|^2_{2,2,2,\Omega^t}],\end{aligned} \tag{5.7}$$

$$\begin{aligned}\|\overline{\varrho}_m\|^2_{2,2,2,\Omega^t} \le c\|\overline{\varrho}_0\|^2_{3,2,2,\Omega^t}\varphi_4(a,t)[(\varepsilon+t)\|\overline{v}_m\|^2_{3,2,2,\Omega^t}\\ + c(\varepsilon)t(\|\overline{v}_{mt}\|^2_{0,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{0,\Omega})],\end{aligned} \tag{5.8}$$

where

$$a = \sqrt{t}\Big(\int_0^t \|\overline{v}_m\|^2_{3,\Omega}\,d\tau\Big)^{1/2}$$

and φ_i, $i = 1,2,3,4$ are increasing positive functions.

From Lemmas 3.5–4.4 assuming that $\overline{u} = \overline{v}_m$, $\overline{H}' = \overline{H}_m$, $\overline{\varrho} = \overline{\varrho}_m$ using the fact that

$$\begin{aligned}\|\overline{u}\|^2_{2,2,\infty,\Omega^t} &\le t\|\overline{u}_t\|^2_{2,2,2,\Omega^t} + \|\overline{u}(0)\|^2_{2,\Omega},\\ \|\overline{H}'\|^2_{2,2,\infty,\Pi^t} &\le t\|\overline{H}'_t\|^2_{2,2,2,\Pi^t} + \|\overline{H}(0)\|^2_{2,\Pi},\\ \|\overline{\varrho}_t\|^2_{1,2,\infty,\Omega^t} &\le t\|\overline{\varrho}_{tt}\|^2_{1,2,2,\Omega^t} + \|\overline{\varrho}_t(0)\|^2_{1,\Omega},\\ \|\overline{\varrho}\|^2_{2,2,\infty,\Omega^t} &\le t\|\overline{\varrho}_t\|^2_{2,2,2,\Omega^t} + \|\overline{\varrho}(0)\|^2_{2,\Omega},\end{aligned} \tag{5.9}$$

and from (5.5)–(5.8) we get

$$\begin{aligned}B_{m+1}(t) \le \alpha(a,t,\varepsilon B_m,F_0)[(\varepsilon+t+c(\varepsilon)+(\varepsilon+t+c(\varepsilon))^2)(X_m(t)\\ + X_m^2(t)+F_0+1)(Y_m(t)+Y_m^2(t)+F_0+1)+F_0+F_0^2] \equiv P,\end{aligned}$$

where

$$
\begin{aligned}
X_m(t) &= \|\overline{v}_{mt}\|^2_{0,\Omega} + \|\overline{v}_m\|^2_{1,\Omega} + \|\overline{v}\|^2_{3,2,2,\Omega^T} + \|\overline{v}_{mt}\|^2_{2,2,2,\Omega^t} \\
&\quad + \|\overline{v}_{mtt}\|^2_{1,2,2,\Omega^t}, \\
Y_m(t) &= B_m(t) - X_m(t), \\
F_0 &= \|\overline{H}(0)\|^2_{2,\Pi} + \|\overline{v}(0)\|^2_{3,\Pi} + \|\overline{\varrho}(0)\|^2_{3,\Omega} + \|\overline{H}_t(0)\|^2_{2,\Pi} + \|\overline{v}_t(0)\|^2_{2,\Pi} \\
&\quad + \|\overline{\varrho}(0)\|_{2,\Omega} + \|\overline{v}_{tt}(0)\|^2_{1,\Pi} + \|\overline{H}_{tt}(0)\|^2_{1,\Pi} + \|\bar{E}_*\|^2_{1,2,\infty,B^t} \\
&\quad + \|\bar{E}_*\|^2_{1,2,2,B^t} + \|\bar{E}_{*t}\|^2_{0,2,2,B^t} + \|\bar{E}_{*tt}\|^2_{0,2,2,B^t} + \|\overline{H}_*\|^2_{3,2,2,B^t} \\
&\quad + \|\overline{H}_{*t}\|^2_{1,2,2,B^t} + \|\overline{H}(0)\|^2_{0,B} + \|\overline{H}_*\|^2_{1,2,\infty,B^t} + \int_\Omega \overline{\varrho}(0)\overline{v}^2(0)\,d\xi \\
&\quad + \int_\Omega \overline{\varrho}(0)\overline{v}_t(0)\,d\xi + \|\bar{f}\|^2_{1,2,2,\Omega^t} + \|\bar{f}_t\|^2_{0,2,2,\Omega^t} + \|\bar{f}_{tt}\|^2_{0,2,2,\Omega^t},
\end{aligned}
$$

and α is an increasing positive function.

Now assuming that $\varepsilon = \sqrt{t}$ we obtain

$$
B_{m+1}(t) \le P(t, t^\gamma B_m(t), F_0), \tag{5.10}
$$

where $\gamma > 0$. Let A be such that $P(0,0,F_0) < A$. Since P is a continuous function there exists $T_* > 0$ such that for $t \le T_*$,

$$
P(t, t^\gamma A, F_0) \le A. \tag{5.11}
$$

From (5.11) we see that $B_m(t) \le A$. Then $B_{m+1}(t) \le A$ for $t \le T_*$.

Now we prove the convergence of the sequence $\{\overline{v}_m, \overline{H}_m\}$. To show this we obtain from (5.1)–(5.2) the following system of problems for the differences $\overline{\mathcal{V}}_m = \overline{v}_m - \overline{v}_{m-1}$, $\overline{\mathcal{H}}_m = \overline{H}_m - \overline{H}_{m-1}$, $\overline{\mathcal{N}}_m = \overline{\varrho}_m - \overline{\varrho}_{m-1}$:

$$
\begin{aligned}
&\overline{\varrho}_m \overline{\mathcal{V}}_{m+1,t} - \operatorname{div}_{\overline{v}_m} \mathbb{D}_{\overline{v}_m}(\overline{\mathcal{V}}_{m+1}) = \operatorname{div}_{\overline{v}_m}[\mathbb{D}_{\overline{v}_m} - \mathbb{D}_{\overline{v}_{m-1}}](\overline{v}_m) \\
&\quad + [\operatorname{div}_{\overline{v}_m} - \operatorname{div}_{\overline{v}_{m-1}}]\mathbb{D}_{\overline{v}_m}(\overline{v}_m) + [\nabla_{\overline{v}_m} - \nabla_{\overline{v}_{m-1}}]\overline{p}_m \\
&\quad + \mu_1 \overset{1}{\overline{H}}_m \cdot [\nabla_{\overline{v}_m} - \nabla_{\overline{v}_{m-1}}]\overset{1}{\overline{H}}_m + \nabla_{\overline{v}_{m-1}}(\overline{p}_m - \overline{p}_{m-1}) \\
&\quad + \mu_1 \overset{1}{\overline{H}}_{m-1} \nabla_{\overline{v}_{m-1}}(\overset{1}{\overline{\mathcal{H}}}_m) + \mu_1 \overset{1}{\overline{\mathcal{H}}}_m \nabla_{\overline{v}_{m-1}} \overset{1}{\overline{H}}_m - \overline{\mathcal{N}}_m \overline{v}_{mt} \equiv F^*,
\end{aligned} \tag{5.12}
$$

$$
\begin{aligned}
&\mu \overline{\mathcal{H}}_{m+1} - \frac{1}{\sigma} \operatorname{rot}_{\overline{v}_m} \operatorname{rot}_{\overline{v}_m} \overline{\mathcal{H}}_{m+1} = [\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}] \operatorname{rot}_{\overline{v}_m} \overline{H}_m \\
&\quad + \operatorname{rot}_{\overline{v}_{m-1}}[\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}]\overline{H}_m + \operatorname{rot}_{\overline{v}_m}(\overline{v}_m \times \overline{\mathcal{H}}_{m+1}) \\
&\quad + \operatorname{rot}_{\overline{v}_m}(\overline{\mathcal{V}}_m \times \overline{H}_m) + [\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}](\overline{v}_{m-1} \times \overline{H}_m) \\
&\quad + [\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}] \operatorname{rot}_{\overline{v}_m} \overline{H}_m + \operatorname{rot}_{\overline{v}_{m-1}}[\operatorname{rot}_{\overline{v}_m} - \operatorname{rot}_{\overline{v}_{m-1}}]\overline{H}_m \\
&\quad + \nabla_{\overline{v}_m} \overline{\mathcal{H}}_{m+1} \overline{v}_m + (\nabla_{\overline{v}_m} - \nabla_{\overline{v}_{m-1}})\overline{H}_m = K^*,
\end{aligned} \tag{5.13}
$$

$$
(5.14)\qquad
\begin{aligned}
&\bar{\mathcal{N}}_{mt} + \bar{\mathcal{N}}_m \operatorname{div}_{\bar{v}_m} \overline{v}_m = -\overline{\varrho}_{m-1}(\operatorname{div}_{\bar{v}_m} \overline{v}_m - \operatorname{div}_{\bar{v}_{m-1}} \overline{v}_{m-1}),\\
&\bar{\mathcal{N}}_m|_{t=0} = 0,\\
&\overline{\mathcal{H}}|_{t=0},\\
&\bar{\mathcal{V}}|_{t=0},
\end{aligned}
$$

and

$$
(5.15)\qquad
\begin{aligned}
\overline{n}_{\bar{v}_m}\mathbb{D}_{\bar{v}_m}(\bar{\mathcal{V}}_{m+1}) = {}&[\overline{n}_{\bar{v}_m} - \overline{n}_{\bar{v}_{m-1}}]\mathbb{D}_{\bar{v}_m}(\overline{v}_m)\\
&+ \overline{n}_{\bar{v}_{m-1}}[\mathbb{D}_{\bar{v}_m} - \mathbb{D}_{\bar{v}_{m-1}}]\overline{v}_m + [\overline{n}_{\bar{v}_m} - \overline{n}_{\bar{v}_{m-1}}]\overline{p}_m\\
&+ \overline{n}_{\bar{v}_{m-1}}[\overline{p}_m - \overline{p}_{m-1}] + p_0[\overline{n}_{\bar{v}_m} - \overline{n}_{\bar{v}_{m-1}}] \equiv G^*.
\end{aligned}
$$

LEMMA 5.2. *Let the assumptions of Lemma* 5.1 *be satisfied. Then there exists* T_{**} *sufficiently small such that*

$$
(5.16)\qquad
\begin{aligned}
&\|\bar{\mathcal{V}}_{m+1}\|^2_{0,\Pi} + \|\bar{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Pi^t} + \|\bar{\mathcal{V}}_{m+1}\|^2_{0,2,2,\Pi^t} + \|\bar{\mathcal{V}}_{m+1}\|^2_{1,\Pi}\\
&\qquad + \|\overline{\mathcal{H}}_{m+1}\|^2_{0,\Pi} + \|\overline{\mathcal{H}}_{m+1}\|^2_{1,2,2,\Pi^t}\\
&\qquad \le \alpha(A)t(t+1)(\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Pi^t} + \|\overline{\mathcal{H}}_m\|^2_{1,2,2,\Pi^t}),
\end{aligned}
$$

where $\alpha(A)t(t+1) < 1$ *for* $t < T_{**}$, *and* α *is an increasing positive function.*

Proof. Multiplying (5.12) by $\bar{\mathcal{V}}_{m+1}I_{\bar{v}_m}$ and integrating over Ω we get

$$
(5.17)\qquad
\frac{1}{2}\int \overline{\varrho}_m \frac{d}{dt}\bar{\mathcal{V}}^2_{m+1}I_{\bar{v}_m}\,d\xi + \int_\Omega |\mathbb{D}_{\bar{v}_m}(\bar{\mathcal{V}}_{m+1})|^2 I_{\bar{v}_m}\,d\xi
= \int_\Omega F^*\bar{\mathcal{V}}_{m+1}I_{\bar{v}_m}\,d\xi + \int_S G^*\bar{\mathcal{V}}_{m+1}I_{\bar{v}_m}\,d\xi_S.
$$

We have

$$
(5.18)\qquad
\int_{\Omega^t} |F^*\bar{\mathcal{V}}_{m+1}|I_{\bar{v}_m}\,d\xi\,dt
\le \alpha(A)t(t+1)(\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}) + \varepsilon\|\bar{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t},
$$

$$
(5.19)\qquad
\int_{S^t} |G^*\bar{\mathcal{V}}_{m+1}|I_{\bar{v}_m}\,d\xi_S\,dt
\le \alpha(A)t(t+1)\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \varepsilon\|\bar{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t},
$$

where α is an increasing positive function.

Using in (5.17) the Korn inequality, integrating with respect to time and using (5.18), (5.19) we get

$$
(5.20)\qquad
\|\bar{\mathcal{V}}_{m+1}\|^2_{0,\Omega} + \|\bar{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t}
\le \alpha(A)t(t+1)(\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}).
$$

In [3] we proved the inequality

$$\|\overline{\mathcal{H}}_{m+1}\|^2_{0,2,\infty,\Pi^t} + \|\overline{\mathcal{H}}_{m+1}\|^2_{1,2,2,\Pi^t} \le \alpha(A)t(t+1)\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Pi^t}. \tag{5.21}$$

Multiplying (5.12) by $\bar{\mathcal{V}}_{m+1,t}$, integrating over Ω, and using the Young and Hölder inequalities we get

$$\begin{aligned}\int_\Omega \bar{\varrho}_m|\bar{\mathcal{V}}_{m+1,t}|^2 I_{\bar{v}_m}\,d\xi + \frac{d}{dt}\int_\Omega |\mathbb{D}_{\bar{v}_m}(\bar{\mathcal{V}}_{m+1})|^2 I_{\bar{v}_m}\,d\xi \\ \le c\Big(\|\bar{v}_m\|^2_{3,\Omega}\int_\Omega |\mathbb{D}_{\bar{v}_m}(\bar{\mathcal{V}}_{m+1})|^2 I_{\bar{v}_m}\,d\xi + \frac{d}{dt}\int_S G^*\bar{\mathcal{V}}_{m+1}I_{\bar{v}_m}\,d\xi_S \\ + \|G^*_t\|^2_{0,S} + \|\bar{\mathcal{V}}_{m+1}\|^2_{1,\Omega} + \|F^*\|^2_{0,\Omega} + \|\bar{v}_m\|^2_{3,\Omega}\|G^*\|^2_{0,S}\Big).\end{aligned} \tag{5.22}$$

Integrating (5.22) with respect to time, and using the Gronwall and Korn inequalities we get

$$\begin{aligned}\|\bar{\mathcal{V}}_{m+1,t}\|^2_{0,2,2,\Omega^t} + \|\bar{\mathcal{V}}_{m+1}\|^2_{1,\Omega} \le \alpha(A)(\|G^*_t\|^2_{0,2,2,S^t} + \|\bar{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t} \\ + \|F^*\|^2_{0,2,2,\Omega^t} + c\|G^*\|^2_{0,2,\infty,S^t} + \varepsilon\|\bar{\mathcal{V}}_{m+1}\|^2_{1,2,\infty,\Omega^t}),\end{aligned} \tag{5.23}$$

where α is an increasing positive function.

We have

$$\|F^*\|^2_{0,2,2,\Omega^t} \le \alpha(A)t(t+1)(\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}), \tag{5.24}$$

$$\|G^*\|^2_{0,S} \le \alpha(A)t(t+1)\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t}, \tag{5.25}$$

$$\|G^*_t\|^2_{0,2,2,S^t} \le \alpha(A)t(t+1)\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t}, \tag{5.26}$$

where α is an increasing positive function.

Using (5.24)–(5.26) in (5.23) we get

$$\begin{aligned}\|\bar{\mathcal{V}}_{m+1,t}\|^2_{0,2,2,\Omega^t} + \|\bar{\mathcal{V}}_{m+1}\|^2_{1,\Omega} \le \alpha(A)[t(t+1)(\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} \\ + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}) + \|\bar{\mathcal{V}}_{m+1}\|^2_{1,2,2,\Omega^t} + \varepsilon\|\bar{\mathcal{V}}_{m+1}\|^2_{1,2,\infty,\Omega^t}],\end{aligned} \tag{5.27}$$

where α is an increasing positive function.

Now from the regularization technique for the parabolic problem (see [7])

$$\begin{aligned}&\bar{\varrho}_m\bar{\mathcal{V}}_{m+1,t} - \mathrm{div}_{\bar{v}_m}\mathbb{D}_{\bar{v}_{m+1}}(\bar{\mathcal{V}}_{m+1}) = F^* && \text{in } \Omega^T,\\ &\mathbb{D}_{\bar{v}_m}(\bar{\mathcal{V}}_{m+1}) = G^* && \text{on } S^T,\\ &\bar{\mathcal{V}}_{m+1}|_{t=0} = 0 && \text{in } \Omega,\end{aligned}$$

we get

$$\|\bar{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Omega^t} \le c(\|F^*\|^2_{0,2,2,\Omega^t} + \|G^*\|^2_{1/2,2,2,S^t} + c\|\bar{\mathcal{V}}_{m+1}\|^2_{0,\Omega}). \tag{5.28}$$

We have

$$\|G^*\|^2_{2,2,S^t} \le \alpha(A)t(t+1)\|\bar{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t}, \tag{5.29}$$

Now using in (5.28) inequality (5.24), (5.29) we get

$$\|\overline{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Omega^t} \le \alpha(A)t(t+1)(\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}) + c\|\overline{\mathcal{V}}_{m+1}\|^2_{0,2,2,\Omega^t}, \tag{5.30}$$

where α is an increasing positive function.

Summing inequalities (5.20), (5.27), (5.30) we get

$$\|\overline{\mathcal{V}}_{m+1,t}\|^2_{0,2,2,\Omega^t} + \|\overline{\mathcal{V}}_{m+1}\|^2_{1,\Omega} + \|\overline{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Omega^t} \le \alpha(A)t(t+1)[\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Omega^t} + \|\overset{1}{\overline{\mathcal{H}}}_m\|^2_{1,2,2,\Omega^t}], \tag{5.31}$$

where α is an icreasing positive function.

Summarizing, from (5.21) and (5.31) we obtain (5.16).

From Lemmas 5.1 and 5.2 we have

THEOREM 5.1. *Let the assumptions of Lemmas* 5.1 *and* 5.2 *be satisfied. Then there exists T^{**} sufficiently small such that for $T \le T^{**}$ there exists a solution to problem* (1.1)–(1.7) *such that*

$$\begin{aligned}
\overline{v} &\in L_2(0,T,H^3(\Omega)) \cap L_\infty(0,T,H^1(\Omega));\\
\overline{v}_t &\in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^2(\Omega));\\
\overline{v}_{tt} &\in L_2(T,0,H^1(\Omega)) \cap L_\infty(0,T,L_2(\Omega));\\
\overline{H} &\in L_2(0,T,H^3(\Pi)) \cap L_\infty(0,T,H^1(\Pi));\\
\overline{H}_t &\in L_\infty(0,T,H^1(\Pi)) \cap L_2(0,T,H^2(\Pi));\\
\overline{H}_{tt} &\in L_\infty(0,T,L_2(\Pi)) \cap L_2(0,T,H^1(\Pi))
\end{aligned}$$

and

$$\begin{aligned}
&\|\overline{v}_t\|^2_{1,2,2,\infty,\Omega^T} + \|\overline{v}\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{3,2,2,\Omega^T} + \|\overline{v}_t\|^2_{2,2,2,\Omega^T} + \|\overline{v}_{tt}\|^2_{1,2,2,\Omega^T}\\
&\quad + \|v_{tt}\|^2_{0,2,\infty,\Omega^T} + \|\overline{H}_t\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{3,2,2,\Pi^T}\\
&\quad + \|\overline{H}_t\|^2_{2,2,2,\Pi^T} + \|\overline{H}_{tt}\|^2_{2,2,\infty,\Pi^T} + \|\overline{H}_{tt}\|^2_{1,2,2,\Pi^T} \le A,
\end{aligned}$$

where A is defined in Lemma 5.1.

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