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ESTIMATION OF A SMOOTHNESS PARAMETER BY SPLINE WAVELETS

Abstract. We consider the smoothness parameter of a function $f \in L^2(\mathbb{R})$ in terms of Besov spaces $B^s_{2,\infty}(\mathbb{R})$,

$$s^*(f) = \sup\{s > 0 : f \in B^s_{2,\infty}(\mathbb{R})\}.$$

The existing results on estimation of smoothness [K. Dziedziul, M. Kucharska and B. Wolnik, J. Nonparametric Statist. 23 (2011)] employ the Haar basis and are limited to the case $0 < s^*(f) < 1/2$. Using p-regular ($p \geq 1$) spline wavelets with exponential decay we extend them to density functions with $0 < s^*(f) < p + 1/2$. Applying the Franklin–Strömberg wavelet $p = 1$, we prove that the presented estimator of $s^*(f)$ is consistent for piecewise constant functions. Furthermore, we show that the results for the Franklin–Strömberg wavelet can be generalised to any spline wavelet ($p \geq 1$).

1. Introduction

Definition 1.1. Let $f \in L^2(\mathbb{R})$. Then

$$s^*(f) = \sup\{s > 0 : f \in B^s_{2,\infty}(\mathbb{R})\}$$

is called the *smoothness parameter* of f, where by convention $\sup\{\emptyset\} = 0$ and $\sup\{(0,\infty)\} = \infty$.

For the definition of $B^s_{2,\infty}(\mathbb{R})$ see [HW], [W]. From the continuous embedding

$$B^{s_1}_{2,\infty}(\mathbb{R}) \subset B^{s_2}_{2,\infty}(\mathbb{R}) \quad \text{for } s_1 > s_2,$$

it follows that for any $f \in L^2(\mathbb{R})$, either f belongs to all $B^s_{2,\infty}(\mathbb{R})$ spaces, or to none, or there exists $s^* = s^*(f)$ such that $f \in B^s_{2,\infty}(\mathbb{R})$ for all $0 < s < s^*$ and $f \notin B^s_{2,\infty}(\mathbb{R})$ for all $s > s^*$.

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Note that the smoothness parameter based on the Hölder–Zygmund space $B^s_{\infty,\infty}$ was considered in [GN], [HN], [J]. It is essential in adaptive inference [HN] considering an estimation of a density function f to test a nonparametric hypothesis: $H_0 : s^*(f) \le t$ versus $H_a : s^*(f) > t$. To achieve that, one needs a consistent estimator. In our discussion we show that there exists a consistent estimator for the class of piecewise-smooth density functions.

We fix a scaling function ϕ and a wavelet ψ associated with ϕ which form an r-regular multiresolution analysis (further denoted by r-RMA). For the definition see [M, Definitions 1 and 2, p. 21]. By [D, Proposition 5.5.2], ψ satisfies the zero oscillation condition, i.e. there exists $d \ge r$ such that

$$\begin{aligned} &\int_{\mathbb{R}} x^k \psi(x)\,dx = 0 \quad \text{for } 0 \le k \le d, \\ &\int_{\mathbb{R}} x^{d+1}\psi(x)\,dx \ne 0. \end{aligned} \tag{1.1}$$

In our paper we consider a special case of r-RMA, namely a spline multiresolution analysis of order p (p-SMA). For a construction see [HW, Chapter 4.2] or [W]. The multiresolution analysis, the wavelet and, finally, the scaling function are constructed using the spline space of order $p \ge 1$. For the convenience of the reader we recall the construction of the Franklin–Strömberg wavelet for $p = 1$, denoted by S (see [W]). Let us define the following subsets of $\mathbb{R}$:

$$\begin{aligned} &\mathbb{Z}_+ = \{1, 2, \dots\}, \quad \mathbb{Z}_- = -\mathbb{Z}_+, \\ &A_0 = \mathbb{Z}_+ \cup \{0\} \cup \tfrac{1}{2}\mathbb{Z}_-, \quad A_1 = \{1/2\} \cup A_0, \end{aligned}$$

where $aA = \{ax : x \in A\}$ and $a + A = \{a + x : x \in A\}$. Let V be a discrete subset of $\mathbb{R}$. Then we denote by $\mathbb{S}(V)$ the space of all functions $f \in L^2(\mathbb{R})$ continuous on $\mathbb{R}$ and linear on every interval $I \subset \mathbb{R}$ such that $I \cap V = \emptyset$. A function $S \in \mathbb{S}(A_1)$ such that $\|S\|_2 = 1$ and S is orthogonal to $\mathbb{S}(A_0)$ is called the *Franklin–Strömberg wavelet* (see Figure 1). One of the main properties of this spline wavelet is that, although it is supported on the whole $\mathbb{R}$, it decays exponentially at infinity, i.e. there are constants $\alpha > 0$ and $\beta > 0$ such

$$|S(x)| < \beta e^{-\alpha|x|} \quad \text{for all } x \in \mathbb{R}. \tag{1.2}$$

In the general case we denote by ϕ^p the scaling function and by ψ^p the spline wavelet, where $p \ge 2$, which both have exponential decay with first $p-1$ derivatives at infinity [HW, Theorem 2.18]:

$$\underset{C>0}{\exists}\ \underset{\gamma>0}{\exists}\ \underset{x\in\mathbb{R}}{\forall}\quad |D^m\phi(x)| \le Ce^{-\gamma|x|}, \quad m = 0, 1, \dots, p-1. \tag{1.3}$$

Note that, by (1.3), every p-SMA is a $(p-1)$-RMA. We treat p-SMA separately, because p-SMA has better approximation properties: we can characterise the Besov space $B^s_{2,\infty}(\mathbb{R})$ for $0 < s < p + 1/2$ [C, Theorem 9.3],

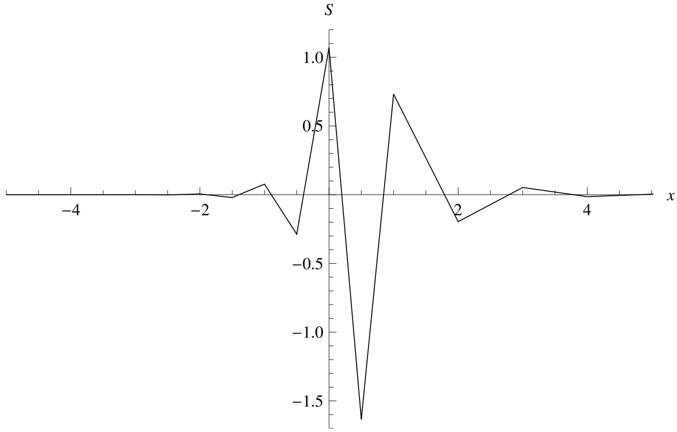

Fig. 1. The Franklin–Strömberg wavelet

instead of $0 < s < p-1$ (in the case of $(p-1)$-RMA). The characterisation with the use of p-SMA is done on the interval $[0,1]$, but it holds on $\mathbb{R}$ too.

Denote by $P_h f$, where $h > 0$, the orthogonal projection of $f \in L^2(\mathbb{R})$ given by

$$P_h f(x) = \int_{\mathbb{R}} K_h(x,y) f(y)\, dy$$

with the kernel K_h defined as follows:

$$K_h(x,y) = \frac{1}{h} \sum_{k\in\mathbb{Z}} \phi\Big(\frac{x}{h} - k\Big) \phi\Big(\frac{y}{h} - k\Big),$$

where ϕ is a scaling function. One can easily obtain the following proposition.

Proposition 1.2. *Let a p-SMA be given, where $p \ge 1$. Then*

$$\underset{C>0}{\exists}\ \underset{\gamma>0}{\exists}\ \underset{x,y\in\mathbb{R}}{\forall}\quad |K_1(x,y)| < Ce^{-\gamma|x-y|} \quad \textit{with } \phi = \phi^p.$$

Define

$$Q_h = P_{h/2} - P_h.$$

By [M, Proposition 4, Section 2.9] we have the following characterisation of Besov spaces with $h = 2^{-j}$, $j \in \mathbb{Z}$. Let an r-RMA be given. Then a function f belongs to $B^s_{2,\infty}(\mathbb{R})$ for $0 < s < r$ if and only if $f \in L^2(\mathbb{R})$ and

$$\sup_{j\ge 0} 2^{js} \|P_{2^{-(j+1)}} f - P_{2^{-j}} f\|_2 = \sup_{j\ge 0} 2^{js} \|Q_{2^{-j}} f\|_2 < \infty. \tag{1.4}$$

Similarly, in view of the result of Ciesielski [C, Theorem 9.2], we have the characterisation of Besov spaces for a p-SMA: a function f belongs to $B^s_{2,\infty}(\mathbb{R})$ for some $0 < s < p + 1/2$ if and only if $f \in L^2(\mathbb{R})$ and (1.4) holds.

One can observe that the above characterisations are also true for any $0 < h < 1$, i.e. a function f belongs to $B^s_{2,\infty}(\mathbb{R})$ for some $0 < s < r$, resp.

$0 < s < p + 1/2$, if and only if $f \in L^2(\mathbb{R})$ and

$$\sup_{0<h<1} h^{-s}\|P_{h/2}f - P_h f\|_2 = \sup_{0<h<1} h^{-s}\|Q_h f\|_2 < \infty. \tag{1.5}$$

This is a consequence of the simple observation that

$$\forall_{0<h<1} \ \exists!_{j\geq 0} \ \exists!_{1/2\leq c<1} \qquad h = c\cdot 2^{-j}$$

and

$$Q_{c2^{-j}} = \sigma_c \circ Q_{2^{-j}} \circ \sigma_{1/c}, \quad \text{where} \quad \sigma_c f(x) = f(x/c).$$

2. Main results. Using the above characterisations one can obtain the proposition given below. It is an extension of Theorem 1.1 from [DKW], where all results are obtained only in the case of the Haar basis and for the sequence $h = 2^{-j}$.

All proofs of our results are postponed to Section 5.

We set $\mathcal{P}_f := \{0 < h < 1 : \|Q_h f\|_2 \neq 0\}$; we will write $\{h_k \in \mathcal{P}_f\}_{k=1}^{\infty} \to 0$ to mean that $h_k \in \mathcal{P}_f$ for $k \geq 1$ and $\lim_{k\to\infty} h_k = 0$.

Proposition 2.1. *Let $f \in L^2(\mathbb{R})$ and an r-RMA be given such that $0 < s^*(f) < r$, or a p-SMA such that $0 < s^*(f) < p+1/2$. Then there exists a sequence $\{\tau_k \in \mathcal{P}_f\}_{k=1}^{\infty} \to 0$ such that*

$$s^*(f) = \lim_{\tau_k\to 0} \log_{\tau_k} \|Q_{\tau_k} f\|_2 \tag{2.1}$$

and whenever $\{h_k \in \mathcal{P}_f\}_{k=1}^{\infty} \to 0$ then

$$s^*(f) \leq \liminf_{h_k\to 0} \log_{h_k} \|Q_{h_k} f\|_2. \tag{2.2}$$

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with density function $f \in L^2(\mathbb{R})$. For every $h > 0$ and sample size $n(h)$ we define a density estimator by

$$f_{h,n(h)}(x) := \frac{1}{n(h)} \sum_{i=1}^{n(h)} K_h(x, X_i).$$

Let

$$\mathcal{P}_f^* := \Big\{\{h_l \in \mathcal{P}_f\}_{l=1}^{\infty} \to 0 : h_l \leq \lambda 2^{-l} \text{ for some } \lambda > 0, \\ \lim_{h_l\to 0} \log_{h_l} \|Q_{h_l} f\|_2 = s^*(f)\Big\}.$$

Note that by Proposition 2.1, $\mathcal{P}_f^*$ is not empty.

The following theorem is an extension of Theorem 2.1 from [DKW] and proposes an estimator of the smoothness parameter.

Theorem 2.2. *Let a p-SMA or an r-RMA be given where the scaling function ϕ has exponential decay. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with density function $f \in L^2(\mathbb{R})$ and $0 < s^*(f) < p + 1/2$,* *resp.* $0 < s^*(f) < r$. *Then for* $\{h_k\}_{k=1}^{\infty} \in \mathcal{P}_f^*$,

$$\lim_{h_k \to 0} \log_{h_k} \|f_{h_k/2, n(h_k/2)} - f_{h_k, n(h_k)}\|_2 = s^*(f) \quad a.s, \tag{2.3}$$

where $n(h_k) \asymp h_k^{-2(p+1)}$ *for the* p*-SMA, while* $n(h_k) \asymp h_k^{-2(r+1/2)}$ *for the* r*-RMA.*

In [CD], $Q_h f$ is estimated with the help of empirical wavelet coefficients with $h = 2^{-j}$.

Note that the conditions of Proposition 2.1 and Theorem 2.2 hold for the Franklin–Strömberg wavelet. We will prove that for that wavelet and any piecewise constant function f the formula (2.1) holds for every sequence $\{h_k \in \mathcal{P}_f\}_{k=1}^{\infty} \to 0$.

LEMMA 2.3. *Let* S *be the Franklin–Strömberg wavelet* ($p = 1$). *Then*

$$\forall_{z \in [0,1/2) \cup [3/2,2)} \quad \Big|\int_z^{\infty} S(x)\,dx\Big| > M, \tag{2.4}$$

where

$$M = \left|\frac{S(1)}{24}(3 - 2\sqrt{3})\right| \approx 0.01415608.$$

LEMMA 2.4. *With the same constants* α, β *as in the exponential decay property of the Franklin–Strömberg wavelet* (1.2),

$$\forall_{x \in \mathbb{R}} \quad \Big|\int_x^{\infty} S(u)\,du\Big| \le \frac{\beta}{\alpha} e^{-\alpha|x|}.$$

We can immediately obtain the following corollary from Lemma 2.4.

COROLLARY 2.5. *For any real numbers* $a_1 < \cdots < a_n$ *and* $v_1, \ldots, v_n \in \mathbb{R} \setminus \{0\}$ *and for each* $h \ge 0$ *and* $k \in \mathbb{Z}$,

$$\Big|\sum_{i=1}^{n} v_i \int_{a_i/h - k}^{\infty} S(u)\,du\Big| \le \tilde{\beta} e^{-\alpha\eta}, \tag{2.5}$$

where

$$\tilde{\beta} = \frac{v\beta}{\alpha}, \quad v = \sum_{i=1}^{n} |v_i|, \quad \eta = \eta(j, k, a_i) = \min_{1 \le i \le n} \left|\frac{a_i}{h} - k\right|.$$

A similar theorem for r-RMA with ϕ and ψ having compact support was proved in [CD] with $h = 2^{-j}$.

THEOREM 2.6. *Define the following functions on* $\mathbb{R}$:

$$g_a(x) = \begin{cases} 0 & \text{if } x \le a, \\ 1 & \text{otherwise,} \end{cases} \tag{2.6}$$

where $a \in \mathbb{R}$, and

$$H = v_1 g_{a_1} + v_2 g_{a_2} + \cdots + v_n g_{a_n}, \tag{2.7}$$

where $a_i \in \mathbb{R}$ and $v_i \in \mathbb{R} \setminus \{0\}$, $i = 1, \ldots, n$, satisfy

$$a_1 < \cdots < a_n \quad \text{and} \quad v_1 + \cdots + v_n = 0.$$

Then $H \in L^2(\mathbb{R})$ and $s^*(H) = 1/2$. Furthermore, if we consider the Franklin–Strömberg wavelet S, for the function H we have

$$\lim_{h_k \to 0} \log_{h_k} \|Q_{h_k}(H)\|_2 = 1/2 = s^*(H) \tag{2.8}$$

for any $\{h_k \in \mathcal{P}_H\}_{k=1}^{\infty} \to 0$.

Using the same techniques as in the proof of Theorems 2.2 and 2.6 we can obtain the following corollary.

COROLLARY 2.7. *Let an SMA of order 1 be given and let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with density function $f \in L^2(\mathbb{R})$, given by (2.7). Whenever $\{h_k \in \mathcal{P}_f\}_{k=1}^{\infty} \to 0$ is such that there exists $\lambda > 0$ with $h_k \le \lambda 2^{-k}$ for any k, then*

$$\lim_{h_k \to 0} \log_{h_k} \|f_{h_k/2, n(h_k/2)} - f_{h_k, n(h_k)}\|_2 = 1/2 = s^*(f) \qquad a.s, \tag{2.9}$$

where $n(h_k) \asymp h_k^{-4}$.

From Corollary 2.7 it follows that the above estimator of $s^*(f)$ is consistent.

3. Extensions. Having the analogue of (2.4) for spline wavelets ψ^p of order $p > 1$, we can obtain Theorem 2.6 and Corollary 2.7. We consider the Battle–Lemarié wavelet of order p as an example of ψ^p (for the definition see [D, Subsection 5.4]). Using MATHEMATICA for every $p \ge 1$ we find intervals I_{1p}, I_{2p} and a constant $M_p > 0$ such that

$$\exists_{k_{1p}, k_{2p} \in \mathbb{Z}} \quad (I_{1p} - k_{1p}) \cup (I_{2p} - k_{2p}) = [0, 1)$$

and

$$\forall_{z \in I_{1p} \cup I_{2p}} \quad \Big| \int_z^\infty \psi^p(x)\, dx \Big| > M_p.$$

Let $F_p(z) = \int_z^\infty \psi^p(x)\, dx$.

We choose, for odd $p = 1, 3, 5$,

$$I_{1p} = [-1, -0.5), \quad I_{2p} = [1.5, 2),$$

and for even $p = 2, 4, 6$,

$$I_{1p} = [-0.5, -1), \quad I_{2p} = [3, 3.5),$$

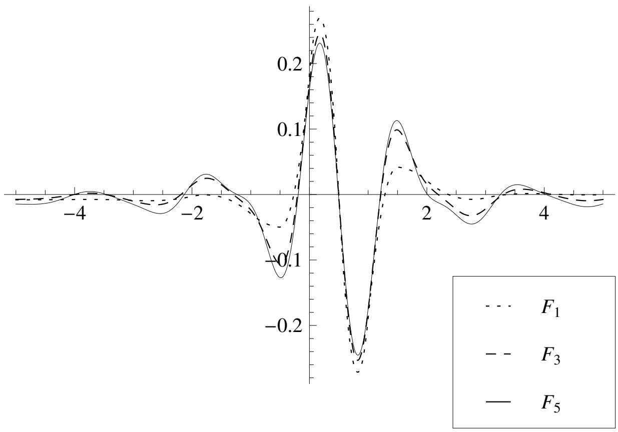

Fig. 2. The functions F_p, $p = 1, 3, 5$, obtained using MATHEMATICA

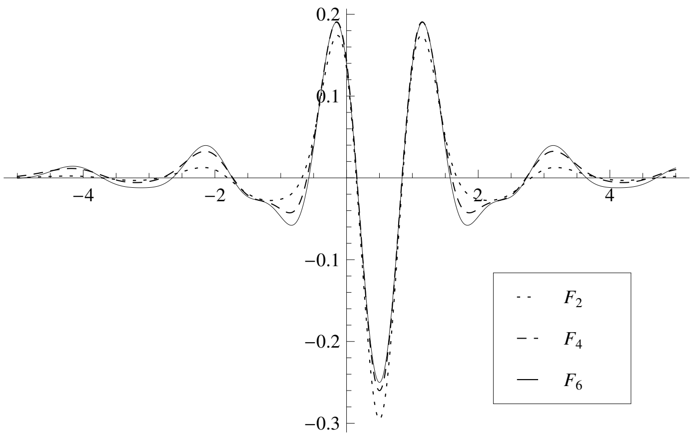

Fig. 3. The functions F_p, $p = 2, 4, 6$, obtained using MATHEMATICA

because the function F_p has nonzero values on I_{1p}, I_{2p}. Furthermore, we observe that $|F_p|$ is concave on those intervals. Thus, to find M_p, it is sufficient to consider the values of $|F_p|$ at the ends of I_{1p}, I_{2p} (see Table 1).

Moreover, we can replace the function (2.6) by a truncated power function of order $m < p$, i.e. $(x - a)_+^m$, and (2.7) by a linear combination of truncated power functions g such that $g \in L^2(\mathbb{R})$. Then, in the case of any spline wavelet, the conclusion of Theorem 2.6 holds with $m + 1/2$ instead of $1/2$. Analogously, we can convert Corollary 2.7 to the case of p-SMA and the density function f being a linear combination of truncated power functions of order m. Then in the conclusion we have $m + 1/2$ instead of $1/2$.

Table 1. Values of $|F_p|$ at the ends of $I_{1p}, I_{2p},\ p = 1, 2, \ldots, 6$

ends \ p	1	2	3	4	5	6
-1.5	–	0.01936	–	0.02120	–	0.02311
-1	0.02918	0.02608	0.02347	0.03811	0.01559	0.04741
-0.5	0.04976	–	0.10620	–	0.12692	–
1.5	0.04184	–	0.09862	–	0.11281	–
2	0.02111	–	0.01589	–	0.00148	–
3	–	0.00999	–	0.02782	–	0.03363
3.5	–	0.00743	–	0.01579	–	0.01285
M_p	0.02111	0.00743	0.01589	0.01579	0.00148	0.01285

4. Simulations. In this section we present the behaviour of the smoothness parameter estimator (2.9). Following the conclusions of the previous section, we use the scaling function ϕ^1 associated with the Battle–Lemarié wavelet of order 1 to construct the estimator. To obtain values of ϕ^1 we use linear interpolation between dyadic discretization points.

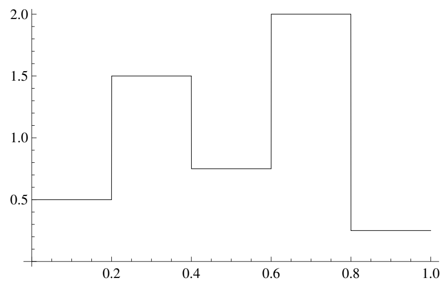

Fig. 4. The density function f (4.1)

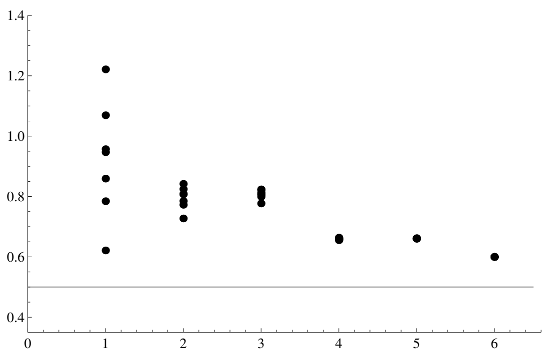

Fig. 5. Simulation results for the estimator of $s^*(f)$ for $k = 1, \ldots, 6$ (the experiment was repeated seven times)

We focus on the case where $h_k = 2^{-k}$ and $n(h_k) = 2^{4k}$, $k \geq 1$. Data samples are generated from the following piecewise constant density function:

$$f = 0.5\mathbb{1}_{[0,0.2]} + 1.5\mathbb{1}_{(0.2,0.4]} + 0.75\mathbb{1}_{(0.4,0.6]} + 2\mathbb{1}_{(0.6,0.8]} + 0.25\mathbb{1}_{(0.8,1]}, \tag{4.1}$$

where $\mathbb{1}_A$ is the characteristic function of the set A. The true value of the smoothness parameter for f is $s^*(f) = 1/2$.

To better illustrate the behaviour of the proposed estimator we repeated the simulation experiment seven times. The results are shown in Figure 5. The simulations were limited to $k \leq 6$, because of excessive time needed to perform computations for $k = 7$.

5. Proofs

5.1. Proof of Proposition 2.1. For all $0 < s < s^*(f)$ by (1.5) we have

$$\underset{D>0}{\exists} \ \underset{h>0}{\forall} \quad h^{-s}\|Q_h f\|_2 \leq D.$$

Hence

$$\log_h \|Q_h f\|_2 \geq \log_h D + s \quad \text{for } h \in \mathcal{P}_f. \tag{5.1}$$

Then for every $\{h_k \in \mathcal{P}_f\}_{k=1}^{\infty} \to 0$,

$$\liminf_{h_k \to 0} \log_{h_k} \|Q_{h_k} f\|_2 \geq s \quad \text{for } s < s^*(f).$$

So

$$\liminf_{h_k \to 0} \log_{h_k} \|Q_{h_k} f\|_2 \geq s^*(f).$$

For all $s^*(f) < s < r$ there exists $h = h(s) \in \mathcal{P}_f$ such that

$$h^{-s}\|Q_h f\|_2 \geq 1.$$

Then

$$\log_h \|Q_h f\|_2 \leq s. \tag{5.2}$$

Hence for $s_j \searrow s^*(f)$ we have

$$\liminf_{h(s_j) \to 0} \log_{h(s_j)} \|Q_{h(s_j)} f\|_2 \leq s^*(f). \ \blacksquare$$

5.2. Proof of Lemma 2.3. We can see that the function S is decreasing on $I_1 = [0, 1/2)$ and on $I_2 = [3/2, 2)$. For $F(z) = \int_z^\infty S(x)\,dx$ we have $F'(z) = -S(z)$. Since F' is increasing on I_1 and on I_2, F is convex on I_1 and on I_2. From the definition it follows that

$$\begin{aligned} \sup_{x \in I_1 \cup I_2} F(x) &= \max\{F(0), F(1/2), F(3/2), F(2)\} \\ &= F(1/2) = \frac{S(1)}{24}(3 - 2\sqrt{3}) < 0. \end{aligned}$$

Thus, $|F|$ is concave on I_1 and on I_2 and achieves its infimum at the point $1/2$. Moreover,

$$\forall_{z\in[0,1/2)\cup[3/2,2)} \quad \Big|\int_z^\infty S(x)\,dx\Big| > M, \tag{5.3}$$

where

$$M = \left|\frac{S(1)}{24}(3-2\sqrt{3})\right| \approx 0.01415608.$$

The constant M is calculated with the aid of a computer. ∎

5.3. Proof of Theorem 2.2. First, we need to estimate the quantity

$$\begin{aligned}
\|f_{h,n(h)} - P_h(f)\|_2^2 &= \int_{\mathbb{R}} \frac{1}{n^2}\Big(\sum_{i=1}^n [K_h(x,X_i) - EK_h(x,X_i)]\Big)^2\,dx \\
&= \frac{1}{n^2}\sum_{i=1}^n \int_{\mathbb{R}} [K_h(x,X_i) - EK_h(x,X_i)]^2\,dx \\
&\quad + \frac{2}{n^2}\sum_{m<l}\int_{\mathbb{R}} (K_h(x,X_l) - EK_h(x,X_l))(K_h(x,X_m) - EK_h(x,X_m))\,dx \\
&= I_{h,n,2} + I_{h,n,3}.
\end{aligned}$$

Lemma 5.1. *With the above notation:*

1. $EI_{h,n,2} \le \dfrac{C^2}{\gamma n h}$,
2. $EI_{h,n,3} = 0$,
3. $\operatorname{Var} I_{h,n,2} \le \dfrac{16C^4}{\gamma^2 n^3 h^2}$,
4. $\operatorname{Var} I_{h,n,3} \le \dfrac{32C^4}{\gamma^2 n^2 h^2}$,

where the constant C is from the exponential decay condition and $n = n(h)$.

Proof. Set $Y_{x,l} = K_h(x,X_l) - EK_h(x,X_l)$. We can see that $EY_{x,l} = 0$.

1. We have

$$\begin{aligned}
EI_{h,n,2} &= E\Big(\frac{1}{n^2}\sum_{i=1}^n \int_{\mathbb{R}} [K_h(x,X_i) - EK_h(x,X_i)]^2\,dx\Big) \\
&= \frac{1}{n}E\Big(\int_{\mathbb{R}} [K_h(x,X_1) - EK_h(x,X_1)]^2\,dx\Big)
\end{aligned}$$

$$\le \frac{1}{n} E\Big(\int_{\mathbb{R}} K_h^2(x, X_1)\, dx\Big)$$

$$= \frac{1}{n} \int_{\mathbb{R}} \int_{\mathbb{R}} K_h^2(x, u) f(u)\, du\, dx$$

$$\le \frac{C^2}{nh^2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\gamma|x/h - u/h|} f(u)\, du\, dx$$

$$= \frac{C^2}{nh} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\gamma|t - u/h|}\, dt f(u)\, du = \frac{C^2}{\gamma nh} \int_{\mathbb{R}} f(u)\, du = \frac{C^2}{\gamma nh}.$$

2. From the independence of X_m and X_l, $m \neq l$, one obtains

$$EI_{h,n,3} = \frac{2}{n^2} E\Big(\sum_{m<l} \int_{\mathbb{R}} Y_{x,l} Y_{x,m}\, dx\Big)$$
$$= \frac{2}{n^2} \sum_{m<l} \int_{\mathbb{R}} EY_{x,l} EY_{x,m}\, dx = 0.$$

3. Using $(a+b)^2 \le 2(a^2+b^2)$ and Jensen's inequality, we have

$$\mathrm{Var}(I_{h,n,2}) = \frac{1}{n^4} \sum_{i=1}^{n} \mathrm{Var}\Big(\int_{\mathbb{R}} Y_{x,i}^2\, dx\Big)$$

$$= \frac{1}{n^3} \mathrm{Var}\Big(\int_{\mathbb{R}} Y_{x,1}^2\, dx\Big)$$

$$\le \frac{1}{n^3} E\Big(\int_{\mathbb{R}} Y_{x,1}^2\, dx\Big)^2 = \frac{1}{n^3} E\Big(\int_{\mathbb{R}} \int_{\mathbb{R}} Y_{x,1}^2 Y_{y,1}^2\, dx\, dy\Big)$$

$$= \frac{1}{n^3} \int_{\mathbb{R}} \int_{\mathbb{R}} E[K_h(x, X_1) - EK_h(x, X_1)]^2 [K_h(x, X_1) - EK_h(x, X_1)]^2\, dx\, dy$$

$$\le \frac{4}{n^3} \int_{\mathbb{R}} \int_{\mathbb{R}} E[K_h^2(x, X_1) + (EK_h(x, X_1))^2][K_h^2(x, X_1) + (EK_h(x, X_1))^2]\, dx\, dy$$

$$= \frac{4}{n^3} \int_{\mathbb{R}} \int_{\mathbb{R}} E[K_h^2(x, X_1) K_h^2(y, X_1)] + EK_h^2(x, X_1)(EK_h(y, X_1))^2$$
$$+ (EK_h(x, X_1))^2 EK_h^2(y, X_1) + (EK_h(x, X_1))^2 (EK_h(y, X_1))^2\, dx\, dy$$

$$\le \frac{4}{n^3} \int_{\mathbb{R}} \int_{\mathbb{R}} E[K_h^2(x, X_1) K_h^2(y, X_1)]\, dx\, dy$$
$$+ \frac{12}{n^3} \int_{\mathbb{R}} \int_{\mathbb{R}} EK_h^2(x, X_1) EK_h^2(y, X_1)\, dx\, dy = A_1 + A_2.$$

Observe that A_2 can be evaluated using item 1:

$$A_2 = \frac{12}{n^3} \int_{\mathbb{R}}\int_{\mathbb{R}} EK_h^2(x, X_1) EK_h^2(y, X_1)\, dx\, dy$$
$$= \frac{12}{n^3} \int_{\mathbb{R}} EK_h^2(x, X_1)\, dx \int_{\mathbb{R}} EK_h^2(y, X_1)\, dy$$
$$\leq \frac{12}{n^3} \cdot \frac{C^2}{\gamma h} \cdot \frac{C^2}{\gamma h} = \frac{12C^4}{\gamma^2 n^3 h^2}.$$

Furthermore

$$A_1 = \frac{4}{n^3} \int_{\mathbb{R}}\int_{\mathbb{R}} E[K_h^2(x, X_1) K_h^2(y, X_1)]\, dx\, dy$$
$$= \frac{4}{n^3} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} K_h^2(x, u) K_h^2(y, u) f(u)\, du\, dx\, dy$$
$$\leq \frac{4C^4}{n^3 h^4} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\gamma(|x/h-u/h|+|y/h-u/h|)} f(u)\, du\, dx\, dy$$
$$= \frac{4C^4}{\gamma^2 n^3 h^2},$$

which leads to

$$\operatorname{Var} I_{h,n,2} \leq A_1 + A_2 \leq \frac{16C^4}{\gamma^2 n^3 h^2}.$$

4. Recall that

$$I_{h,n,3} = \frac{2}{n^2} \sum_{m<l} \int_{\mathbb{R}} Y_{x,l} Y_{x,m}\, dx,$$

where

$$Y_{x,l} = K_h(x, X_l) - EK_h(x, X_l).$$

Hence

$$\operatorname{Var} I_{h,n,3} = E(I_{h,n,3})^2 - (EI_{h,n,3})^2 = E(I_{h,n,3})^2$$
$$= E\Big(\frac{4}{n^4} \sum_{i<j} \sum_{m<l} \int_{\mathbb{R}}\int_{\mathbb{R}} Y_{x,i} Y_{x,j} Y_{y,m} Y_{y,l}\, dx\, dy\Big)$$
$$= \frac{4}{n^4} \sum_{i<j} \sum_{m<l} E\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} Y_{x,i} Y_{x,j} Y_{y,m} Y_{y,l}\, dx\, dy\Big).$$

Since the variables $X_1, \ldots, X_n$ are independent, it follows that if $i \neq m$ or $j \neq l$ then

$$E\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} Y_{x,i} Y_{x,j} Y_{y,m} Y_{y,l}\, dx\, dy\Big) = 0.$$

So it is sufficient to consider the case where $i = m$ and $j = l$.

Using Jensen's inequality and $(a-b)^2 \le 2(a^2+b^2)$, we obtain

$$
\begin{aligned}
&E\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} Y_{x,i}Y_{x,j}Y_{y,i}Y_{y,j}\,dx\,dy\Big)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} E((K_h(x,X_i)-EK_h(x,X_i))(K_h(y,X_i)-EK_h(y,X_i)))\\
&\quad\cdot E((K_h(x,X_j)-EK_h(x,X_j))(K_h(x,X_j)-EK_h(x,X_j)))\,dx\,dy\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}(K_h(x,X_i)-EK_h(x,X_i))(K_h(y,X_i)-EK_h(y,X_i))f(u)\,du\Big)^2 dx\,dy\\
&\le \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(K_h(x,u)-EK_h(x,X_i))^2(K_h(y,u)-EK_h(y,X_i))^2 f(u)\,du\,dx\,dy\\
&\le 4\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}(K_h^2(x,u)+(EK_h(x,X_i))^2)(K_h^2(y,u)+(EK_h(y,X_i))^2)f(u)\,du\,dx\,dy\\
&= 4\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}[K_h^2(x,u)K_h^2(y,u)+K_h^2(y,u)(EK_h(x,X_i))^2\\
&\quad+K_h^2(x,u)(EK_h(y,X_i))^2+(EK_h(x,X_i))^2(EK_h(y,X_i))^2]f(u)\,du\,dx\,dy\\
&= 4\int_{\mathbb{R}}\int_{\mathbb{R}}[EK_h^2(x,u)K_h^2(y,u)+EK_h^2(y,u)(EK_h(x,X_i))^2\\
&\quad+EK_h^2(x,u)(EK_h(y,X_i))^2+(EK_h(x,X_i))^2(EK_h(y,X_i))^2]\,dx\,dy\\
&\le 4\int_{\mathbb{R}}\int_{\mathbb{R}}[EK_h^2(x,u)K_h^2(y,u)+3EK_h^2(x,X_i)EK_h^2(y,X_i)]\,dx\,dy\\
&= 4\int_{\mathbb{R}}\int_{\mathbb{R}} K_h^2(x,u)K_h^2(y,u)\,dx\,dy+12\int_{\mathbb{R}}\int_{\mathbb{R}} EK_h^2(x,X_i)EK_h^2(y,X_i)\,dx\,dy.
\end{aligned}
$$

Using the results of items 1 and 3 we obtain

$$
\begin{aligned}
E\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} Y_{x,i}Y_{x,j}Y_{y,i}Y_{y,j}dxdy\Big)
&\le 4\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} K_h^2(x,u)K_h^2(y,u)\,dx\,dy\Big)f(u)\,du\\
&\quad+12\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} EK_h^2(x,X_i)EK_h^2(y,X_i)\,dx\,dy\Big)f(u)\,du\\
&\le \frac{16C^4}{\gamma^2h^2},
\end{aligned}
$$

which leads to

$$
\operatorname{Var} I_{h,n,3} \le \frac{4}{n^4}\cdot\frac{n^2-n}{2}\cdot\frac{16C^4}{\gamma^2h^2} \le \frac{32C^4}{\gamma^2n^2h^2}. \ \blacksquare
$$

Having obtained the inequalities from Lemma 5.1, we can finish the proof of Theorem 2.2. We present it in the case of r-RMA, because the proof for

p-SMA is similar. Note also that the proof is analogous to that of [DKW, Theorem 2.1].

To shorten notation, in the following we write f_h for $f_{h,n(h)}$. We show that there exists $L > 0$ such that for every $\varepsilon > 0$, there are a natural number N and subset $A_N \subset \Omega$ with $P(A_N) > 1 - \varepsilon$ such that

$$\forall_{k \geq N} \ \forall_{\omega \in A_N} \quad \|f_{h_k} - P_{h_k} f\|_2^2(\omega) < 3L h_k^{2s}. \tag{5.4}$$

We recall that

$$\begin{aligned}\|f_{h_k} - P_{h_k} f\|_2^2 &= I_{h_k,n(h_k),2} + I_{h_k,n(h_k),3} \\ &= (I_{h_k,n(h_k),2} - EI_{h_k,n(h_k),2}) + EI_{h_k,n(h_k),2} + I_{h_k,n(h_k),3}.\end{aligned}$$

We know that there exist constants $M_1, M_2 > 0$ such that

$$M_1 h_k^{-(2r+1)} \leq n(h_k) \leq M_2 h_k^{-(2r+1)}.$$

Using Lemma 5.1 we obtain

$$\begin{aligned}\operatorname{Var} I_{h_k,n(h_k),2} &\leq \frac{16C^4}{M_1^3\gamma^2} \frac{1}{h_k^2} \frac{1}{h_k^{-3(2r+1)}} \leq L h_k^{6r+1}, \\ \operatorname{Var} I_{h_k,n(h_k),3} &\leq \frac{32C^4}{M_1^2\gamma^2} \frac{1}{h_k^2} \frac{1}{h_k^{-2(2r+1)}} \leq L h_k^{4r}, \\ EI_{h_k,n(h_k),2} &\leq \frac{C^2}{M_1\gamma} h_k^{2r} \leq L h_k^{2r}.\end{aligned}$$

From Chebyshev's inequality, for every $0 < s < r$,

$$\begin{aligned}P(|I_{h_k,n(h_k),2} - EI_{h_k,n(h_k),2}| \geq L h_k^{2s}) &\leq L^{-1} h_k^{4(r-s)+2r+1} \\ P(|I_{h_k,n(h_k),3}| \geq L h_k^{2s}) &\leq L^{-1} h_k^{4(r-s)}.\end{aligned}$$

So,

$$\sum_{k=1}^{\infty} P(|I_{h_k,n(h_k),2} - EI_{h_k,n(h_k),2}| \geq L h_k^{2s}) < \infty,$$

$$\sum_{k=1}^{\infty} P(|I_{h_k,n(h_k),3}| \geq L h_k^{2s}) < \infty.$$

Thus by the Borel–Cantelli lemma, for N large enough, $P(A_N)$ is at least $1 - \varepsilon$, where

$$A_N = \Big\{\omega : \forall_{k \geq N} \ |I_{h_k,n(h_k),2} - EI_{h_k,n(h_k),2}| \leq L h_k^{2s},\ |I_{h_k,n(h_k),3}| \leq L h_k^{2s}\Big\}.$$

Therefore, the statement (5.4) is true.

For $s < s^*(f) < r$ take N large enough such that $\|Q_{h_k} f\|_2 \le h_k^s$ for $k \ge N$. Thus using the triangle inequality, we get, for $\omega \in A_N$,

$$\begin{aligned}\|f_{h_k/2} - f_{h_k}\|_2 &\le \|f_{h_k/2} - P_{h_k/2} f\|_2 + \|f_{h_k} - P_{h_k} f\|_2 + \|Q_{h_k} f\|_2\\ &\le \big(1 + \sqrt{3L}(1 + 2^{-s})\big) h_k^s.\end{aligned}$$

Therefore,

$$\liminf_{k\to\infty} \log_{h_k} \|f_{h_k/2} - f_{h_k}\|_2(\omega) \ge s.$$

For $s^* < s < r$ take N so large that $\|Q_{h_k} f\|_2 \ge h_k^s$ for $k \ge N$. Let $\delta > 0$ be such that $s^* < s + \delta < r$. Then, from the triangle inequality for $\omega \in A_N$,

$$\begin{aligned}\|f_{h_k/2} - f_{h_k}\|_2 &\ge -\|f_{h_k/2} - P_{h_k/2} f\|_2 - \|f_{h_k} - P_{h_k} f\|_2 + \|Q_{h_k} f\|_2\\ &\ge \big(1 - \sqrt{3L} h_k^{\delta} (1 + 2^{-(s+\delta)})\big) h_k^s,\end{aligned}$$

which means that

$$\limsup_{k\to\infty} \log_{h_k} \|f_{h_k/2} - f_{h_k}\|_2(\omega) \le s. \ \blacksquare$$

5.4. Proof of Lemma 2.4. Let $x > 0$. Using the exponential decay of the Franklin–Strömberg wavelet (1.2), we obtain

$$\Big|\int_x^\infty S(u)\,du\Big| \le \int_x^\infty |S(u)|\,du \le \beta \int_x^\infty e^{-\alpha|u|} du = \frac{\beta}{\alpha} e^{-\alpha x}.$$

If $x \le 0$, then by the zero oscillation condition (1.1),

$$\Big|\int_x^\infty S(u)\,du\Big| = \Big|\int_{-\infty}^x S(u)\,du\Big| \le \int_{-\infty}^x |S(u)|\,du \le \beta \int_{-\infty}^x e^{-\alpha|u|}\,du = \frac{\beta}{\alpha} e^{\alpha x}.$$

So finally,

$$\mathop{\forall}_{x\in\mathbb{R}} \quad \Big|\int_x^\infty S(u)\,du\Big| \le \frac{\beta}{\alpha} e^{-\alpha|x|}. \ \blacksquare$$

5.5. Proof of Corollary 2.5. Using Lemma 2.4 we have

$$\begin{aligned}\Big|\sum_{i=1}^n v_i \int_{a_i/h-k}^\infty S(u)\,du\Big| &\le \sum_{i=1}^n |v_i| \Big|\int_{a_i/h-k}^\infty S(u)\,du\Big| = \frac{\beta}{\alpha}\sum_{i=1}^n |v_i| e^{-\alpha(a_i/h-k)}\\ &\le \frac{\beta}{\alpha} v e^{-\alpha\eta},\end{aligned}$$

where $v = \sum_{i=1}^n |v_i|$ and $\eta = \min_{1\le i\le n} |a_i/h - k|$. $\blacksquare$

5.6. Proof of Theorem 2.6. Our aim is to show that

$$\mathop{\exists}_{A>0,B>0} \ \mathop{\exists}_{h_0>0} \ \mathop{\forall}_{h<h_0} \quad hA \le \|Q_h(H)\|_2^2 \le hB. \tag{5.5}$$

Let us choose an index l such that $v_l = \max_{1\le i\le n}|v_i|$. Then

$$\begin{aligned}\|Q_h(H)\|_2^2 &= \sum_{k\in\mathbb{Z}}\langle H, S_{h,k}\rangle^2 = \sum_{k\in\mathbb{Z}}\Big(\sum_{i=1}^n \langle v_i g_{a_i}, S_{h,k}\rangle\Big)^2\\ &\le n\sum_{k\in\mathbb{Z}}\sum_{i=1}^n \langle v_i g_{a_i}, S_{h,k}\rangle^2 \le n v_l^2 \sum_{k\in\mathbb{Z}}\sum_{i=1}^n\Big(\int_{a_i}^{\infty} S_{h,k}(x)\,dx\Big)^2\\ &= hnv_l^2\sum_{i=1}^n\sum_{k\in\mathbb{Z}}\Big(\int_{a_i/h-k}^{\infty} S(x)\,dx\Big)^2.\end{aligned}$$

Using Lemma 2.4, we get

$$\begin{aligned}\|Q_h(H)\|_2^2 &\le hnv_l^2\frac{C^2}{\alpha^2}\sum_{i=1}^n\sum_{k\in\mathbb{Z}} e^{-2\alpha|a_i/h-k|} \le 2hn^2v_l^2\frac{C^2}{\alpha^2}\sum_{k\ge 0}e^{-2\alpha k}\\ &= 2hn^2v_l^2\frac{C^2}{\alpha^2}\,\frac{1}{1-e^{-2\alpha}} = h\frac{2(nv_lC)^2}{(1-e^{-2\alpha})\alpha^2}.\end{aligned}$$

Let us calculate the lower bound of $\|Q_h(H)\|_2^2$. We have

$$\begin{aligned}\|Q_h(H)\|_2^2 &= \sum_{k\in\mathbb{Z}}\langle H, S_{h,k}\rangle^2\\ &= h\sum_{k\in\mathbb{Z}}\Big(\int_{a_1/h-k}^{\infty} v_1 S(u)\,du + \cdots + \int_{a_n/h-k}^{\infty} v_n S(u)\,du\Big)^2\\ &= h\sum_{k\in\mathbb{Z}}\Big(\int_{a_l/h-k}^{\infty} v_l S(u)\,du + \sum_{i\ne l} v_i \int_{a_i/h-k}^{\infty} S(u)\,du\Big)^2.\end{aligned}$$

Let us now define $\delta = a_l/h - [a_l/h]$. Clearly, $\delta\in[0,1)$. If $\delta\in[0,1/2)$, then for $k=[a_l/h]$ we have

$$\begin{aligned}\|Q_h(H)\|_2^2 &\ge h\Big(\int_{a_l/h-k}^{\infty} v_l S(u)\,du + \sum_{i\ne l} v_i \int_{a_i/h-k}^{\infty} S(u)\,du\Big)^2\\ &\ge h\Big(\Big|\int_{a_l/h-k}^{\infty} v_l S(u)\,du\Big| - \Big|\sum_{i\ne l} v_i \int_{a_i/h-k}^{\infty} S(u)\,du\Big|\Big)^2.\end{aligned}$$

By (2.4) and Corollary 2.5,

$$\|Q_h(H)\|_2^2 \ge h\Big(v_lM - \frac{\beta}{\alpha} v e^{-\alpha(\min_{i\ne l}|a_i/h-k|)}\Big)^2,$$

where $v = \sum_{i \neq l} |v_i|$. Note that

$$\left|\frac{a_i}{h} - k\right| = \left|\frac{a_i}{h} - \left[\frac{a_l}{h}\right]\right| = \left|\frac{a_i}{h} - \frac{a_l}{h} + \frac{a_l}{h} - \left[\frac{a_l}{h}\right]\right|$$
$$\geq \left|\frac{a_i}{h} - \frac{a_l}{h}\right| - \left|\frac{a_l}{h} - \left[\frac{a_l}{h}\right]\right| \geq \left|\frac{a_i}{h} - \frac{a_l}{h}\right| - 1.$$

So,

$$\begin{aligned}\|Q_h(H)\|_2^2 &\geq h\left(v_l M - \frac{\beta}{\alpha} v e^{-\alpha(\min_{i\neq l}|a_i/h - a_l/h| - 1)}\right)^2 \\ &= h\left(v_l M - \frac{\beta}{\alpha} v e^{\alpha} e^{-\alpha/h \min_{i\neq l}|a_i - a_l|}\right)^2 \\ &= h(v_l M - \beta_1 e^{-\alpha/h\theta_l})^2,\end{aligned}$$

where

(5.6) $$\beta_1 = \frac{\beta}{\alpha} v e^{\alpha}, \qquad \theta_l = \min_{i \neq l} |a_i - a_l|.$$

Similarly, for $\delta \in [1/2, 1)$ and $k = [a_l/h] - 1$ we obtain

$$\|Q_h(H)\|_2^2 \geq h\Big(\int_{a_l/h-k}^{\infty} v_l S(u)\,du + \sum_{i\neq l}^{n} v_i \int_{a_i/h-k}^{\infty} S(u)\,du\Big)^2,$$

and

$$\left|\frac{a_i}{h} - k\right| \geq \left|\frac{a_i}{h} - \frac{a_l}{h}\right| - 2.$$

Thus

$$\begin{aligned}\|Q_h(H)\|_2^2 &\geq h\left(v_l M - \frac{\beta}{\alpha} v e^{2\alpha} e^{-\alpha/h \min_{i\neq l}(|a_i - a_l|)}\right)^2 \\ &\geq h(v_l M - \beta_2 e^{-\frac{\alpha}{h}\theta_l})^2,\end{aligned}$$

where $\beta_2 = \frac{\beta}{\alpha} v e^{2\alpha}$. Finally,

$$\|Q_h(H)\|_2^2 \geq h(v_l M - \beta_2 e^{-\alpha/h\theta_l})^2.$$

So, by (5.6) there exists h_0 such that for $h < h_0$,

$$\|Q_h(H)\|_2^2 \geq h\frac{(v_l M)^2}{2}.$$

We take

$$A = \frac{(v_l M)^2}{2} \quad \text{and} \quad B = \frac{2(n v_l \beta)^2}{(1 - e^{-2\alpha})\alpha^2}.$$

Thus, for $h < h_0$ we get

$$\frac{1}{2} + \frac{1}{2}\log_h B \leq \log_h \|Q_h(H)\|_2 \leq \frac{1}{2} + \frac{1}{2}\log_h A. \blacksquare$$

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