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LONG TIME EXISTENCE OF REGULAR SOLUTIONS TO 3D NAVIER–STOKES EQUATIONS COUPLED WITH HEAT CONVECTION

Abstract. We prove long time existence of regular solutions to the Navier–Stokes equations coupled with the heat equation. We consider the system in a non-axially symmetric cylinder, with the slip boundary conditions for the Navier–Stokes equations, and the Neumann condition for the heat equation. The long time existence is possible because the derivatives, with respect to the variable along the axis of the cylinder, of the initial velocity, initial temperature and external force are assumed to be sufficiently small in the L_2 norms. We prove the existence of solutions such that the velocity and temperature belong to $W^{2,1}_\sigma(\Omega\times(0,T))$, where $\sigma>5/3$. The existence is proved by using the Leray–Schauder fixed point theorem.

1. Introduction. We consider the following problem:

$$
\begin{aligned}
&v_{,t}+v\cdot\nabla v-\operatorname{div}\mathbb{T}(v,p)=\alpha(\theta)f && \text{in } \Omega^T=\Omega\times(0,T),\\
&\operatorname{div} v=0 && \text{in } \Omega^T,\\
&\theta_{,t}+v\cdot\nabla\theta-\varkappa\Delta\theta=0 && \text{in } \Omega^T,\\
&\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha=0,\quad \alpha=1,2 && \text{on } S^T=S\times(0,T),\\
&\bar n\cdot\bar v=0 && \text{on } S^T,\\
&\bar n\cdot\nabla\theta=0 && \text{on } S^T,\\
&v|_{t=0}=v_0,\quad \theta|_{t=0}=\theta_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

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where $\Omega \subset \mathbb{R}^3$ is a cylindrical domain, $S = \partial\Omega$, $v = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid motion, $p = p(x,t) \in \mathbb{R}^1$ the pressure, $\theta = \theta(x,t) \in \mathbb{R}_+$ the temperature, $f = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ the external force field, $\bar{n}$ is the unit outward normal vector to the boundary S, $\bar{\tau}_\alpha$, $\alpha = 1,2$, are tangent vectors to S and the dot denotes the scalar product in $\mathbb{R}^3$. We define the stress tensor by

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - p\mathbb{I},$$

where ν is the constant viscosity coefficient, $\mathbb{I}$ is the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor of the form

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

Finally $\varkappa$ is a positive heat conductivity coefficient.

By $x = (x_1, x_2, x_3)$ we denote the Cartesian coordinates, $\Omega \subset \mathbb{R}^3$ is a cylindrical type domain parallel to the x_3 axis with arbitrary cross section.

We assume that $S = S_1 \cup S_2$, where S_1 is the part of the boundary which is parallel to the x_3 axis and S_2 is perpendicular to that axis. More precisely,

$$S_1 = \{x \in \mathbb{R}^3 : \varphi_0(x_1,x_2) = c_*,\ -b < x_3 < b\},$$
$$S_2 = \{x \in \mathbb{R}^3 : \varphi_0(x_1,x_2) < c_*,\ x_3 \text{ equals either } -b \text{ or } b\},$$

where b, c_* are given positive numbers and $\varphi_0(x_1,x_2)$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$. We can assume $\bar{\tau}_1 = (\tau_{11}, \tau_{12}, 0)$ $\tau_2 = (0,0,1)$ and $\bar{n} = (\tau_{12}, -\tau_{11}, 0)$ on S_1.

We assume $\alpha \in C^2(\mathbb{R}_+)$ and Ω^T satisfies the weak l-horn condition, where $l = (2,2,2,1)$ (see [2, Ch. 2, Sect. 8]). The horn condition is an important element of the proofs of the imbedding theorems for anisotropic Sobolev spaces used in this paper. Moreover we assume Ω is not axially symmetric.

Now we formulate the main result of this paper. Let $g = f_{,x_3}$, $h = v_{,x_3}$, $q = p_{,x_3}$, $\vartheta = \theta_{,x_3}$, $\chi = (\operatorname{rot} v)_3$, $F = (\operatorname{rot} f)_3$. Assume that $\|\theta_0\|_{L_\infty(\Omega)} < \infty$.

Define

$$a : [0,\infty) \to [0,\infty), \qquad a(x) = \sup\{|\alpha(y)| + |\alpha'(y)| + |\alpha''(y)| : |y| \le x\}$$

and $c_1 = a(\|\theta_0\|_{L_\infty})$. Moreover assume that $5/3 < \sigma < \infty$, $5/3 < \varrho < \infty$, $5/\varrho - 5/\sigma < 1$ and for $t \le T$:

1. $c_1\|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1c_0\|f\|_{L_\infty(0,t;L_3(\Omega))} + c_1\|F\|_{L_2(0,t;L_{6/5}(\Omega))}$ $+ c_1\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h_0\|_{L_2(\Omega)} + \|\vartheta_0\|_{L_2(\Omega)} + \|\chi_0\|_{L_2(\Omega)}$ $+ c_0^2(c_1\|f\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v_0\|_{L_2(\Omega)}) + \psi(c_0) \le k_1 < \infty$,
2. $\|f\|_{L_2(0,t;L_3(\Omega))} \le k_2 < \infty$,
3. $\|f\|_{L_2(\Omega^t)} + \|v_0\|_{H^1(\Omega)} \le k_3 < \infty$,
4. $c_1\|f\|_{L_\infty(\Omega^t)}e^{cc_1^2k_2^2}k_1 + c_1\|g\|_{L_\sigma(\Omega^t)} + \|\vartheta_0\|_{W_\sigma^{2-2/\sigma}(\Omega)} + \|h_0\|_{W_\sigma^{2-2/\sigma}(\Omega)} \le$ $k_4 < \infty$,

5. $c_1\|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h_0\|_{L_2(\Omega)} + \|\vartheta_0\|_{L_2(\Omega)} \le d < \infty$,
6. $c_1\|f\|_{L_\sigma(\Omega^t)} + \|v_0\|_{W_\varrho^{2-2/\varrho}(\Omega)} + \|\theta_0\|_{W_\varrho^{2-2/\varrho}(\Omega)} \le k_5 < \infty$,

where c_0 is the constant from Lemma 2.4, $\psi(c_0)$ is an increasing function (see Lemma 3.3 in [8]) and $k_1, \dots, k_5$ are given constants. Assume

$$f \in L_\sigma(\Omega^T), \qquad g \in L_\sigma(\Omega^T), \qquad \vartheta_0 \in W_\sigma^{2-2/\sigma}(\Omega).$$

THEOREM 1.1. *Let the above assumptions hold. Assume that d is sufficiently small (see [5, Main Theorem]). Then there exists a strong solution (v, p, θ) to (1.1) such that $v, \theta \in W_\varrho^{2,1}(\Omega^T)$, $\nabla p \in L_\varphi(\Omega^T)$, $h, \vartheta \in W_\sigma^{2,1}(\Omega^T)$, $\nabla q \in L_\sigma(\Omega^T)$.*

The result follows by applying the methods developed in [6] to the more complicated system (1.1). However, the proof of existence is now much clearer than in [6], because the mapping ϕ is constructed in a simpler way. This, however, needs more regularity. Therefore in this paper we prove the existence of much more regular solutions than in [6].

Problem (1.1) in the case of inflow-outflow was generalized in [3, 4]. Papers [3, 4] base on [13], where the inflow-outflow problem was considered for the Navier–Stokes equations in a cylindrical pipe.

2. Preliminaries. The notation used in this paper is the same as in [8, Sect. 2.1]. The definition of weak solution is introduced in [8, Sect. 2.2]. Now we recall some important results employed in this paper.

LEMMA 2.1 (see [12], Korn inequality). *Assume that*

$$E_\Omega(v) = \|\mathbb{D}(v)\|^2_{L_2(\Omega)} < \infty, \qquad v \cdot \bar{n}|_S = 0, \qquad \operatorname{div} v = 0. \tag{2.1}$$

If Ω is not axially symmetric there exists a constant c_1 independent of v such that

$$\|v\|^2_{H^1(\Omega)} \le cE_\Omega(v). \tag{2.2}$$

If Ω is axially symmetric, $\eta = (-x_2, x_1, 0)$, $\alpha = \int_\Omega v \cdot \eta \, dx$, then there exists a constant c independent of v such that

$$\|v\|^2_{H^1(\Omega)} \le c(E_\Omega(v) + |\alpha|^2). \tag{2.3}$$

Let us consider the problem

$$\begin{aligned}
&h_{,t} - \operatorname{div} \mathbb{T}(h, q) = f && \text{in } \Omega^T,\\
&\operatorname{div} h = 0 && \text{in } \Omega^T,\\
&\bar{n} \cdot h = 0, \quad \bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_1^T,\\
&h_i = 0, \quad i = 1, 2, \quad h_{3,x_3} = 0 && \text{on } S_2^T,\\
&h|_{t=0} = h_0 && \text{in } \Omega.
\end{aligned} \tag{2.4}$$

THEOREM 2.2. *Let $f \in L_p(\Omega^T)$, $h(0) \in W_p^{2-2/p}(\Omega)$, $S_i \in C^2$, $i = 1,2$, $1 < p < \infty$. Then there exists a solution to problem* (2.4) *such that $h \in W_p^{2,1}(\Omega^T)$, $\nabla q \in L_p(\Omega^T)$ and there exists a constant c depending on S and p such that*

$$\|h\|_{W_p^{2,1}(\Omega^T)} + \|\nabla q\|_{L_p(\Omega^T)} \le c(\|f\|_{L_p(\Omega^T)} + \|h_0\|_{W_p^{2-2/p}(\Omega)}). \tag{2.5}$$

The proof follows from considerations in [5, Ch. 4].

Let us consider the problem

$$\begin{aligned} &v_{,t} - \operatorname{div} \mathbb{T}(v,q) = f && \text{in } \Omega^T,\\ &\operatorname{div} v = 0 && \text{in } \Omega^T,\\ &\bar n \cdot v = 0, \quad \bar n \cdot \mathbb{D}(v) \cdot \bar\tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\ &v|_{t=0} = v_0 && \text{in } \Omega. \end{aligned} \tag{2.6}$$

THEOREM 2.3. *Let $f \in L_p(\Omega^T)$, $v_0 \in W_p^{2-2/p}(\Omega)$, $S_i \in C^2$, $i = 1,2$, $1 < p < \infty$. Then there exists a solution to problem* (2.6) *such that $v \in W_p^{2,1}(\Omega^T)$, $\nabla p \in L_p(\Omega^T)$ and there exists a constant c depending on S and p such that*

$$\|v\|_{W_p^{2,1}(\Omega^T)} + \|\nabla q\|_{L_p(\Omega^T)} \le c(\|f\|_{L_p(\Omega^T)} + \|v_0\|_{W_p^{2-2/p}(\Omega)}). \tag{2.7}$$

The proof is similar to the proof from [1].

LEMMA 2.4 (see [8, Lemma 2.3]). *Assume that $v_0 \in L_2(\Omega)$, $\theta_0 \in L_\infty(\Omega)$, $f \in L_2(0,T;L_{6/5}(\Omega))$, $T < \infty$. Assume that Ω is not axially symmetric. Assume that there exist constants θ_*, θ^* such that $\theta_* < \theta^*$ and*

$$\theta_* \le \theta_0(x) \le \theta^*, \qquad x \in \Omega.$$

Then there exists a weak solution to problem (1.1) *such that $(v,\theta) \in V_2^0(\Omega^T) \times V_2^0(\Omega^T)$, $\theta \in L_\infty(\Omega^T)$ and*

$$\theta_* \le \theta(x,t) \le \theta^*, \qquad (x,t) \in \Omega^T, \tag{2.8}$$

and there exist positive constants c, c_0 independent of v and θ such that

$$\|v\|_{V_2^0(\Omega^T)} \le c(a(\|\theta_0\|_{L_\infty(\Omega)})\|f\|_{L_2(0,T;L_{6/5}(\Omega))} + \|v_0\|_{L_2(\Omega)}) \le c_0, \tag{2.9}$$

$$\|\theta\|_{V_2^0(\Omega^T)} \le c\|\theta_0\|_{L_2(\Omega)} \le c_0. \tag{2.10}$$

REMARK 2.5. If $\theta(0) \ge 0$, then $\theta(t) \ge 0$ for $t \ge 0$.

3. Existence. For $\xi, \eta, \sigma, \varrho \ge 1$ define

$$\begin{aligned} \|(v,\theta)\|_{\mathcal{M}(\Omega^T)} &= \|v\|_{L_\infty(0,T;W_\eta^1(\Omega))} + \|\theta\|_{L_\infty(0,T;W_\eta^1(\Omega))}\\ &\quad + \|v_{,x_3}\|_{L_\infty(0,T;W_\xi^1(\Omega))} + \|\theta_{,x_3}\|_{L_\infty(0,T;W_\xi^1(\Omega))}, \end{aligned}$$

$$
\begin{aligned}
\mathcal{M}(\Omega^T) &= \{(v,\theta) : \|(v,\theta)\|_{\mathcal{M}(\Omega^T)} < \infty\},\\
\|(v,\theta)\|_{\mathcal{N}(\Omega^T)} &= \|v\|_{W^{2,1}_\varrho(\Omega^T)} + \|\theta\|_{W^{2,1}_\varrho(\Omega^T)}\\
&\quad + \|v_{,x_3}\|_{W^{2,1}_\sigma(\Omega^T)} + \|\theta_{,x_3}\|_{W^{2,1}_\sigma(\Omega^T)},\\
\mathcal{N}(\Omega^T) &= \{(v,\theta) : \|(v,\theta)\|_{\mathcal{N}(\Omega^T)} < \infty\}.
\end{aligned}
$$

LEMMA 3.1.

1. $(\mathcal{M}(\Omega^T), \| \ \|_{\mathcal{M}(\Omega^T)})$ *is a Banach space.*
2. $(\mathcal{N}(\Omega^T), \| \ \|_{\mathcal{N}(\Omega^T)})$ *is a Banach space.*
3. $\|u\|_{\mathcal{M}(\Omega^T)} \le c\|u\|_{\mathcal{N}(\Omega^T)}$ *for* $u \in \mathcal{N}(\Omega^T)$ *and the imbedding* $\mathcal{N}(\Omega^T) \subset \mathcal{M}(\Omega^T)$ *is compact for* $\varrho < \eta$, $5/\varrho - 3/\eta < 1$, $\sigma < \xi$, $5/\sigma - 3/\xi < 1$.

Let us consider the problems

$$
\begin{aligned}
&v_t - \operatorname{div} \mathbb{T}(v,p) = -\lambda[\tilde v \cdot \nabla \tilde v + \alpha(\tilde\theta) f],\\
&\operatorname{div} v = 0,\\
&v \cdot \bar n|_S = 0, \quad \bar n \cdot \mathbb{D}(v) \cdot \bar\tau_\alpha|_S = 0, \quad \alpha = 1,2,\\
&v|_{t=0} = v_0
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
&\theta_t - \varkappa\Delta\theta = -\lambda \tilde v \cdot \nabla\tilde\theta,\\
&\bar n \cdot \nabla\theta|_S = 0,\\
&\theta|_{t=0} = \theta_0,
\end{aligned}
\tag{3.2}
$$

where $\lambda \in [0,1]$ is a parameter and $\tilde v, \tilde\theta$ are treated as given functions. We will assume that $\alpha \in C^2(\mathbb{R})$.

LEMMA 3.2. *Assume that*

$$
\begin{aligned}
&(\tilde v, \tilde\theta) \in \mathcal{M}(\Omega^T), \quad 3 < \eta < \infty,\\
&f \in L_\varrho(\Omega^T), \quad 1 < \varrho < \infty,\\
&v_0 \in W^{2-2/\varrho}_\varrho(\Omega),\\
&S_i \in C^2, \ \ i = 1,2, \quad 5/\varrho - 3/\eta < 1, \quad \varrho < \eta.
\end{aligned}
$$

Then there exists a unique solution to problem (3.1) *such that*

$$
v \in W^{2,1}_\varrho(\Omega^T) \subset L_\infty(0,T; W^1_\eta(\Omega))
$$

and

$$
\begin{aligned}
&\|v\|_{L_\infty(0,T;W^1_\eta(\Omega))} \le c\|v\|_{W^{2,1}_\varrho(\Omega^T)}\\
&\quad \le c(\lambda\|(\tilde v,\tilde\theta)\|^2_{\mathcal{M}(\Omega^T)} + \lambda a(c\|(\tilde v,\tilde\theta)\|_{\mathcal{M}(\Omega^T)})\|f\|_{L_\varrho(\Omega^T)} + \|v_0\|_{W^{2-2/\varrho}_\varrho(\Omega)}).
\end{aligned}
$$

Proof. We have

$$\begin{aligned}\|\tilde v\cdot\nabla\tilde v\|_{L_\varrho(\Omega^T)} &\le c\|\tilde v\|_{L_\infty(\Omega^T)}\|\nabla\tilde v\|_{L_\eta(\Omega^T)}\le c\|\tilde v\|^2_{L_\infty(0,T;W^1_\eta(\Omega))}\\ &\le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^T)}\end{aligned}$$

and

$$\begin{aligned}\|\alpha(\tilde\theta)f\|_{L_\varrho(\Omega^T)} &\le a(c\|\tilde\theta\|_{L_\infty(0,T;W^1_\eta(\Omega))})\|f\|_{L_\varrho(\Omega^T)}\\ &\le a(c\|(\tilde v,\tilde\theta)\|_{\mathcal M(\Omega^T)})\|f\|_{L_\varrho(\Omega^T)}.\end{aligned}$$

By Theorem 2.2 the proof is complete. ∎

Lemma 3.3. *Assume that*

$$3<\eta<\infty,\quad 1<\varrho<\infty,\quad \varrho<\eta,\quad 5/\varrho-3/\eta<1,$$
$$(\tilde v,\tilde\theta)\in\mathcal M(\Omega^T),\quad \theta_0\in W^{2-2/\varrho}_\varrho(\Omega).$$

Then there exists a unique solution to problem (3.2) *such that*

$$\theta\in W^{2,1}_\varrho(\Omega^T)\subset L_\infty(0,T;W^1_\eta(\Omega))$$

and

$$\|\theta\|_{L_\infty(0,T;W^1_\eta(\Omega))}\le c\|\theta\|_{W^{2,1}_\varrho(\Omega^T)}\le c(\lambda\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^T)}+\|\theta_0\|_{W^{2-2/\varrho}_\varrho(\Omega)}).$$

Proof. We have

$$\begin{aligned}\|\tilde v\cdot\nabla\tilde\theta\|_{L_\varrho(\Omega^T)} &\le \|\tilde v\|_{L_\infty(\Omega^T)}\|\nabla\tilde\theta\|_{L_\eta(\Omega^T)}\\ &\le c\|\tilde v\|_{L_\infty(0,T;W^1_\eta(\Omega))}\|\tilde\theta\|_{L_\infty(0,T;W^1_\eta(\Omega))}\le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^T)}.\end{aligned}$$

Then we argue as for Theorem 9.1 in [5, Ch. 4, Sect. 9] (see also [9, Theorem 17]). ∎

Lemma 3.4. *Let*

$$(\tilde v,\tilde\theta)\in\mathcal M(\Omega^T),\quad 3<\xi<\infty,\quad 3<\eta<\infty,$$
$$f\in L_\sigma(\Omega^T),\quad g\in L_\sigma(\Omega^T),\quad 1<\sigma<\infty\quad (\textit{where } g=f_{,x_3}),$$
$$\sigma<\eta,\quad S_i\in C^2,\quad i=1,2,\quad \sigma<\xi,\quad 5/\sigma-3/\xi<1.$$

Let v,p be a unique solution to problem (3.1). *Let $h=v_{,x_3}$, $q=p_{,x_3}$. Assume $h_0\in W^{2-2/\sigma}_\sigma(\Omega)$. Then*

$$h\in W^{2,1}_\sigma(\Omega^T)\subset L_\infty(0,T;W^1_\xi(\Omega))$$

and

$$\begin{aligned}\|h\|_{L_\infty(0,T;W^1_\xi(\Omega))} &\le c\|h\|_{W^{2,1}_\sigma(\Omega^T)}\\ &\le c(\lambda\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^T)}+\lambda a(c\|(\tilde v,\tilde\theta)\|_{\mathcal M(\Omega^T)})\|(\tilde v,\tilde\theta)\|_{\mathcal M(\Omega^T)}\|f\|_{L_\sigma(\Omega^T)}\\ &\quad+\lambda a(c\|(\tilde v,\tilde\theta)\|_{\mathcal M(\Omega^T)})\|g\|_{L_\sigma(\Omega^T)}+\|h_0\|_{W^{2-2/\sigma}_\sigma(\Omega)}).\end{aligned}$$

Proof. The function h is a solution of the following problem:

$$
\begin{aligned}
&h_{,t} - \operatorname{div}\mathbb{T}(h,q) = \lambda[-\tilde v\cdot\nabla\tilde h - \tilde h\cdot\nabla\tilde v + \alpha_\theta(\tilde\theta)\tilde v f + \alpha(\tilde\theta)g] && \text{in } \Omega^T,\\
&\operatorname{div} h = 0 && \text{in } \Omega^T,\\
&\bar n\cdot h = 0,\quad \bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_1^T,\\
&h_i = 0,\quad i=1,2,\quad h_{3,x_3}=0 && \text{on } S_2^T,\\
&h|_{t=0} = h_0 && \text{in } \Omega,
\end{aligned}
$$

where $\tilde h = \tilde v_{,x_3}$, $\tilde\vartheta = \tilde\theta_{,x_3}$. We have

$$
\begin{aligned}
\|\tilde v\cdot\nabla\tilde h\|_{L_\sigma(\Omega^T)} &\le c\|\tilde v\|_{L_\infty(\Omega^T)}\|\nabla\tilde h\|_{L_\xi(\Omega^T)}\\
&\le c\|\tilde v\|_{L_\infty(0,T;W^1_\eta(\Omega))}\|\tilde h\|_{L_\infty(0,T;W^1_\xi(\Omega))} \le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal{M}(\Omega^T)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tilde h\cdot\nabla\tilde v\|_{L_\sigma(\Omega^T)} &\le c\|\tilde h\|_{L_\infty(\Omega^T)}\|\nabla\tilde v\|_{L_\eta(\Omega^T)}\\
&\le c\|\tilde h\|_{L_\infty(0,T;W^1_\xi(\Omega))}\|\tilde v\|_{L_\infty(0,T;W^1_\eta(\Omega))} \le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal{M}(\Omega^T)}.
\end{aligned}
$$

Next

$$
\begin{aligned}
\|\alpha_\theta(\tilde\theta)\tilde\vartheta f\|_{L_\sigma(\Omega^T)} &\le ca(c\|\tilde\theta\|_{L_\infty(0,T;W^1_\eta(\Omega))})\|\tilde\vartheta\|_{L_\infty(0,T;W^1_\xi(\Omega))}\|f\|_{L_\sigma(\Omega^T)}\\
&\le ca(c\|(\tilde v,\tilde\theta)\|_{\mathcal{M}(\Omega^T)})\|(\tilde v,\tilde\theta)\|_{\mathcal{M}(\Omega)}\|f\|_{L_\sigma(\Omega^T)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\alpha(\tilde\theta)g\|_{L_\sigma(\Omega^T)} &\le a(c\|\tilde\theta\|_{L_\infty(0,T;W^1_\eta(\Omega))})\|g\|_{L_\sigma(\Omega^T)}\\
&\le a(c\|(\tilde v,\tilde\theta)\|_{\mathcal{M}(\Omega^T)})\|g\|_{L_\sigma(\Omega^T)}.
\end{aligned}
$$

By Theorem 2.1 the proof is complete. ∎

LEMMA 3.5. *Assume that* $3<\eta<\infty$, $1<\sigma<\infty$, $\sigma<\eta$, $5/\sigma-3/\xi<1$, $3<\xi<\infty$, $\sigma<\xi$, $(\tilde v,\tilde\theta)\in\mathcal{M}(\Omega^T)$. *Let* θ *be a unique solution to problem* (3.2). *Let* $\vartheta=\theta_{,x_3}$. *Assume that* $\vartheta(0)\in W^{2-2/\sigma}_\sigma(\Omega)$. *Then*

$$
\vartheta\in W^{2,1}_\sigma(\Omega^T)\subset L_\infty(0,T;W^1_\xi(\Omega))
$$

and

$$
\|\vartheta\|_{L_\infty(0,T;W^1_\xi(\Omega))}\le c\|\vartheta\|_{W^{2,1}_\sigma(\Omega^T)}\le c(\lambda\|(\tilde v,\tilde\theta)\|^2_{\mathcal{M}(\Omega^T)}+\|\vartheta(0)\|_{W^{2-2/\sigma}_\sigma(\Omega)}).
$$

Proof. The function ϑ is a solution of the problem

$$
\begin{aligned}
&\vartheta_{,t}-\varkappa\Delta\vartheta = -\lambda[\tilde h\cdot\nabla\tilde\theta+\tilde v\cdot\nabla\tilde\vartheta] && \text{in } \Omega^T,\\
&\bar n\cdot\nabla\vartheta = 0 && \text{on } S_1^T,\\
&\vartheta = 0 && \text{on } S_2^T,\\
&\vartheta|_{t=0}=\vartheta_0 && \text{in } \Omega,
\end{aligned}
$$

where $\tilde{\vartheta} = \tilde{\theta}_{,x_3}$. We have

$$\begin{aligned}\|\tilde{h}\cdot\nabla\tilde{\theta}\|_{L_\sigma(\Omega^T)} &\le \|\tilde{h}\|_{L_\infty(\Omega^T)}\|\nabla\tilde{\theta}\|_{L_\eta(\Omega^T)} \le c\|\tilde{h}\|_{L_\infty(0,T;W^1_\xi(\Omega))}\|\tilde{\theta}\|_{L_\infty(0,T;W^1_\eta(\Omega))}\\ &\le c\|(\tilde{v},\tilde{\theta})\|^2_{\mathcal{M}(\Omega^T)}\end{aligned}$$

and

$$\begin{aligned}\|\tilde{v}\cdot\nabla\tilde{\vartheta}\|_{L_\sigma(\Omega^T)} &\le \|\tilde{v}\|_{L_\infty(\Omega^T)}\|\nabla\tilde{\vartheta}\|_{L_\xi(\Omega^T)} \le c\|\tilde{v}\|_{L_\infty(0,T;W^1_\eta(\Omega))}\|\tilde{\vartheta}\|_{L_\infty(0,T;W^1_\xi(\Omega))}\\ &\le c\|(\tilde{v},\tilde{\theta})\|^2_{\mathcal{M}(\Omega^T)}.\end{aligned}$$

Then we argue as for Theorem 9.1 in [5, Ch. 4, Sect. 9] (see also [9, Theorem 17]). ■

From Lemmas 3.1–3.5 it follows that if $(\tilde{v},\tilde{\theta}) \in \mathcal{M}(\Omega^T)$, then there exists a unique solution (v,θ) to problems (3.1)–(3.2) such that $(v,\theta) \in \mathcal{M}(\Omega^T)$.

To prove the existence of solutions to problem (1.1) we apply the Leray–Schauder fixed point theorem (see [7, 10, 11]). Therefore we introduce the mapping $\phi : [0,1]\times\mathcal{M}(\Omega^T) \to \mathcal{M}(\Omega^T)$, $(\lambda,\tilde{v},\tilde{\theta}) \mapsto \phi(\lambda,\tilde{v},\tilde{\theta}) = (v,\theta)$, where (v,θ) is a solution to problems (3.1)–(3.2).

For $\lambda = 0$ we have the existence of a unique solution. For $\lambda = 1$ every fixed point is a solution to problem (1.1).

LEMMA 3.6. *Let the assumptions of Lemmas* 3.2–3.5 *be satisfied. Then the mappings* $\phi(\lambda,\cdot) : \mathcal{M}(\Omega^T) \to \mathcal{M}(\Omega^T)$, $\lambda \in [0,1]$, *are completely continuous.*

Proof. By Lemmas 3.1–3.5 the mappings $\phi(\lambda,\cdot)$, $\lambda \in [0,1]$, are compact. It follows that bounded sets in $\mathcal{M}(\Omega^T)$ are transformed into bounded sets in $\mathcal{M}(\Omega^T)$. Let $(\tilde{v}_i,\tilde{\theta}_i) \in \mathcal{M}(\Omega^T)$, $i = 1,2$, be two given elements. Then (v_i,θ_i), $i = 1,2$, are solutions to the problems

$$\begin{aligned}&v_{it} - \operatorname{div}\mathbb{T}(v_i,p_i) = -\lambda(\tilde{v}_i\cdot\nabla\tilde{v}_i + \alpha(\tilde{\theta}_i)f),\\ &\operatorname{div} v_i = 0,\\ &\bar{n}\cdot\mathbb{D}(v_i)\cdot\bar{\tau}|_S = 0, \quad \bar{n}\cdot v_i|_S = 0,\\ &v_i|_{t=0} = v_0, \quad i = 1,2,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\theta_{it} - \varkappa\Delta\theta_i = -\lambda\tilde{v}_i\cdot\nabla\tilde{\theta}_i,\\ &\bar{n}\cdot\nabla\theta_i|_S = 0,\\ &\theta_i|_{t=0} = \theta_0, \quad i = 1,2.\end{aligned} \tag{3.4}$$

To show continuity we introduce the differences

$$V = v_1 - v_2, \qquad P = p_1 - p_2, \qquad \mathcal{T} = \theta_1 - \theta_2, \tag{3.5}$$

which are solutions to the problems

$$
\begin{aligned}
& V_t - \operatorname{div} \mathbb{T}(V,P) = -\lambda[\tilde{V} \cdot \nabla \tilde{v}_1 + \tilde{v}_2 \cdot \nabla \tilde{V} + (\alpha(\tilde{\theta}_1) - \alpha(\tilde{\theta}_2))f] \\
& \operatorname{div} V = 0, \\
& V \cdot \bar{n}|_S = 0, \quad \bar{n} \cdot \mathbb{D}(V) \cdot \bar{\tau}|_S = 0, \\
& V|_{t=0} = 0,
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
& \mathcal{T}_t - \varkappa \Delta \mathcal{T} = -\lambda[\tilde{V} \cdot \nabla \tilde{\theta}_1 + \tilde{v}_2 \cdot \nabla \tilde{\mathcal{T}}], \\
& \bar{n} \cdot \nabla \mathcal{T}|_S = 0, \\
& \mathcal{T}|_{t=0} = 0,
\end{aligned}
\tag{3.7}
$$

where $\tilde{V} = \tilde{v}_1 - \tilde{v}_2$, $\tilde{\mathcal{T}} = \tilde{\theta}_1 - \tilde{\theta}_2$. In view of [6] and [10, 11] we have

$$
\begin{aligned}
& \|V\|_{W^{2,1}_\varrho(\Omega^T)} + \|\mathcal{T}\|_{W^{2,1}_\varrho(\Omega^T)} \\
& \quad \le c[\|\tilde{V}\|_{L_\infty(\Omega^T)} \|\nabla \tilde{v}_1\|_{L_\varrho(\Omega^T)} + \|\tilde{v}_2\|_{L_\infty(\Omega^T)} \|\nabla \tilde{V}\|_{L_\varrho(\Omega^T)} \\
& \qquad + ca(\max\{\|\tilde{\theta}_1\|_{L_\infty(\Omega^T)}, \|\tilde{\theta}_2\|_{L_\infty(\Omega^T)}\}) \|\tilde{\mathcal{T}}\|_{L_\infty(\Omega^T)} \|f\|_{L_\varrho(\Omega^T)} \\
& \qquad + \|\tilde{v}_2\|_{L_\infty(\Omega^T)} \|\nabla \tilde{\mathcal{T}}\|_{L_\varrho(\Omega^T)} + \|\tilde{V}\|_{L_\infty(\Omega^T)} \|\nabla \tilde{\theta}_1\|_{L_\varrho(\Omega^T)}] \\
& \quad \le c(\|\tilde{V}\|_{\mathcal{M}(\Omega^T)} + \|\tilde{\mathcal{T}}\|_{\mathcal{M}(\Omega^T)}).
\end{aligned}
\tag{3.8}
$$

Let $h_i = v_{i,x_3}$, $q_i = p_{i,x_3}$, $\vartheta_i = \theta_{i,x_3}$, $\tilde{h}_i = \tilde{v}_{i,x_3}$, $\tilde{\vartheta}_i = \tilde{\theta}_{i,x_3}$. The functions h_i, ϑ_i, $i = 1, 2$, are solutions to the following problems:

$$
\begin{aligned}
& h_{i,t} - \operatorname{div} \mathbb{T}(h_i, q_i) = -\lambda[\tilde{h}_i \cdot \nabla \tilde{v}_i + \tilde{v}_i \cdot \nabla \tilde{h}_i + \alpha_\theta(\tilde{\theta}_i)\tilde{\vartheta}_i f + \alpha(\tilde{\theta}_i) g] && \text{in } \Omega^T, \\
& \operatorname{div} h_i = 0 && \text{in } \Omega^T, \\
& \bar{n} \cdot h_i = 0, \quad \bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha, \quad \alpha = 1, 2, \quad i = 1, 2, && \text{on } S_1^T, \\
& h_{ij} = 0, \quad i = 1, 2, \quad j = 1, 2, && \text{on } S_2^T, \\
& h_{i3,x_3} = 0, \quad i = 1, 2, && \text{on } S_2^T, \\
& h_i|_{t=0} = h_0 && \text{in } \Omega
\end{aligned}
$$

and

$$
\begin{aligned}
& \vartheta_{i,t} - \varkappa \Delta \vartheta_i = -\lambda[\tilde{h}_i \cdot \nabla \tilde{\theta}_i + \tilde{v}_i \cdot \nabla \tilde{\vartheta}_i] && \text{in } \Omega^T, \\
& \bar{n} \cdot \nabla \vartheta_i = 0 && \text{on } S_1^T, \\
& \vartheta_i = 0 && \text{on } S_2^T, \\
& \vartheta_i|_{t=0} = \vartheta_0 && \text{in } \Omega.
\end{aligned}
$$

We introduce the differences

$$
H = h_1 - h_2, \qquad Q = q_1 - q_2, \qquad R = \vartheta_1 - \vartheta_2
$$

which are solutions to the problems

$$
\begin{aligned}
&H_{,t} - \operatorname{div} \mathbb{T}(H,Q) = -\lambda[\tilde{H}\cdot\nabla\tilde{v}_1 + \tilde{h}_2\cdot\nabla\tilde{V} + \tilde{V}\cdot\nabla\tilde{h}_1 + \tilde{v}_2\cdot\nabla\tilde{H} \\
&\qquad\qquad\qquad\qquad + (\alpha_\theta(\tilde{\theta}_1) - \alpha_\theta(\tilde{\theta}_2))\tilde{\vartheta}_1 f + \alpha_\theta(\tilde{\theta}_2)\tilde{R}f \\
&\qquad\qquad\qquad\qquad + (\alpha(\tilde{\theta}_1) - \alpha(\tilde{\theta}_2))g && \text{in } \Omega^T, \\
&\operatorname{div} H = 0 && \text{in } \Omega^T, \\
&\bar{n}\cdot H = 0, \quad \bar{n}\cdot\mathbb{D}(H)\cdot\bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T, \\
&H_j = 0, \quad j = 1,2, \quad H_{3,x_3} = 0 && \text{on } S_2^T, \\
&H|_{t=0} = 0 && \text{in } \Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
&R_{,t} - \varkappa\Delta R = -\lambda[\tilde{H}\cdot\nabla\tilde{\theta}_1 + \tilde{h}_2\cdot\nabla\tilde{\mathcal{T}} + \tilde{V}\cdot\nabla\tilde{\vartheta}_1 + \tilde{v}_2\cdot\nabla\tilde{R}] && \text{in } \Omega^T, \\
&\bar{n}\cdot\nabla R = 0 && \text{on } S_1^T, \\
&R = 0 && \text{on } S_2^T, \\
&R|_{t=0} = 0 && \text{in } \Omega,
\end{aligned}
$$

where $\tilde{H} = \tilde{h}_1 - \tilde{h}_2$, $\tilde{R} = \tilde{\vartheta}_1 - \tilde{\vartheta}_2$. In view of [6] and [10, 11] we have

$$
\begin{aligned}
&\|H\|_{W_\sigma^{2,1}(\Omega^T)} + \|R\|_{W_\sigma^{2,1}(\Omega^T)} \\
&\quad \le c[\|\tilde{H}\|_{L_\infty(\Omega^T)}\|\nabla\tilde{v}_1\|_{L_\eta(\Omega^T)} + \|\tilde{h}_2\|_{L_\infty(\Omega^T)}\|\nabla\tilde{V}\|_{L_\eta(\Omega^T)} \\
&\qquad + \|\tilde{V}\|_{L_\infty(\Omega^T)}\|\nabla\tilde{h}_1\|_{L_\xi(\Omega^T)} + \|\tilde{v}_2\|_{L_\infty(\Omega^T)}\|\nabla\tilde{H}\|_{L_\xi(\Omega^T)}] \\
&\qquad + c(\max\{\|\tilde{\theta}_1\|_{L_\infty(\Omega^T)}, \|\tilde{\theta}_2\|_{L_\infty(\Omega^T)}\})\|\mathcal{T}\|_{L_\infty(\Omega^T)}\|\tilde{\vartheta}_1\|_{L_\infty(\Omega^T)}\|f\|_{L_\sigma(\Omega^T)} \\
&\qquad + c(\|\tilde{\theta}_2\|_{L_\infty(\Omega^T)}\|\tilde{R}\|_{L_\infty(\Omega^T)}\|f\|_{L_\sigma(\Omega^T)} \\
&\qquad + a(\max\{\|\tilde{\theta}_1\|_{L_\infty(\Omega^T)}, \|\tilde{\theta}_2\|_{L_\infty(\Omega^T)}\})\|\tilde{\mathcal{T}}\|_{L_\infty(\Omega^T)}\|g\|_{L_\sigma(\Omega^T)} \\
&\qquad + \|\tilde{H}\|_{L_\infty(\Omega^T)}\|\nabla\tilde{\theta}_1\|_{L_\eta(\Omega^T)} + \|\tilde{h}_2\|_{L_\infty(\Omega^T)}\|\nabla\tilde{\mathcal{T}}\|_{L_\eta(\Omega^T)} \\
&\qquad + \|\tilde{V}\|_{L_\infty(\Omega^T)}\|\nabla\tilde{\vartheta}_1\|_{L_\xi(\Omega^T)} + \|\tilde{v}_2\|_{L_\infty(\Omega^T)}\|\nabla\tilde{R}\|_{L_\xi(\Omega^T)}) \\
&\quad \le c(\|\tilde{V}, \tilde{\mathcal{T}}\|_{\mathcal{M}(\Omega^T)})
\end{aligned}
$$

and from (3.8) and Lemma 3.1 we obtain $\|(V,\mathcal{T})\|_{\mathcal{M}(\Omega^T)} \le c\|(\tilde{V},\tilde{\mathcal{T}})\|_{\mathcal{M}(\Omega^T)}$. Hence continuity of ϕ follows. This concludes the proof. ∎

Lemma 3.7. *Let the assumptions of Lemmas* 3.2–3.5 *be satisfied. Then for every bounded subset $\mathcal{M}_0$ of $\mathcal{M}(\Omega^T)$, the family of maps*

$$\phi(\cdot,\tilde{v},\tilde{\theta}) : [0,1] \to \mathcal{M}(\Omega^T), \qquad (\tilde{v},\tilde{\theta}) \in \mathcal{M}_0,$$

is uniformly equicontinuous.

Proof. Let $(\tilde{v},\tilde{\theta}) \in \mathcal{M}_0$, $\lambda_i \in [0,1]$, $i = 1,2$, $\lambda_1 \ge \lambda_2$ and let v_i, θ_i be solutions to the problems

$$
\begin{aligned}
&v_{it} - \operatorname{div} \mathbb{T}(v_i, p_i) = -\lambda_i(\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta}) f),\\
&\operatorname{div} v_i = 0,\\
&\bar{n} \cdot \mathbb{D}(v_i) \cdot \bar{\tau}|_S = 0, \quad \bar{n} \cdot v_i|_S = 0,\\
&v_i|_{t=0} = v_0, \quad i = 1, 2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\theta_{it} - \varkappa \Delta \theta_i = -\lambda_i \tilde{v} \cdot \nabla \tilde{\theta},\\
&\bar{n} \cdot \nabla \theta_i|_S = 0,\\
&\theta_i|_{t=0} = \theta_0, \quad i = 1, 2.
\end{aligned}
$$

To show uniform equicontinuity we introduce the differences

$$
V = v_1 - v_2, \qquad P = p_1 - p_2, \qquad \mathcal{T} = \theta_1 - \theta_2
$$

which are solutions to the problems

$$
\begin{aligned}
&V_t - \operatorname{div} \mathbb{T}(V, P) = -(\lambda_1 - \lambda_2)(\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta}) f),\\
&\operatorname{div} V = 0,\\
&\bar{n} \cdot \mathbb{D}(V) \cdot \bar{\tau}|_S = 0, \quad \bar{n} \cdot V|_S = 0,\\
&V|_{t=0} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{T}_{,t} - \varkappa \Delta \mathcal{T} = -(\lambda_1 - \lambda_2) \tilde{v} \cdot \nabla \tilde{\theta},\\
&\bar{n} \cdot \nabla \mathcal{T}|_S = 0,\\
&\mathcal{T}|_{t=0} = 0.
\end{aligned}
$$

In view of Lemmas 3.2–3.3,

$$
\begin{aligned}
(3.9) \qquad \|V\|_{L_\infty(0,T;W^1_\eta(\Omega))} + \|\mathcal{T}\|_{L_\infty(0,T;W^1_\eta(\Omega))} &\le c((\lambda_1 - \lambda_2)\|(\tilde{v}, \tilde{\theta})\|^2_{\mathcal{M}(\Omega)}\\
&\quad + (\lambda_1 - \lambda_2) a(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}) \|f\|_{L_\sigma(\Omega^T)}).
\end{aligned}
$$

Let $h_i = v_{i,x_3}$, $\vartheta_i = \theta_{i,x_3}$. We introduce the differences

$$
H = h_1 - h_2, \qquad R = \vartheta_1 - \vartheta_2,
$$

which satisfy $H = V_{,x_3}$, $R = \mathcal{T}_{,x_3}$. In view of Lemmas 3.4 and 3.5,

$$
\begin{aligned}
(3.10) \qquad \|H\|_{L_\infty(0,T;W^1_\xi(\Omega))} + \|R\|_{L_\infty(0,T;W^1_\xi(\Omega))} &\le c((\lambda_1 - \lambda_2)\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}\\
&\quad + (\lambda_1 - \lambda_2) a(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}) \|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)} \|f\|_{L_\sigma(\Omega^T)}\\
&\quad + (\lambda_1 - \lambda_2) a(c\|(\tilde{v}, \theta)\|_{\mathcal{M}(\Omega^T)}) \|g\|_{L_\sigma(\Omega^T)}).
\end{aligned}
$$

From (3.9) and (3.10) the uniform equicontinuity of $\phi(\cdot, \tilde{v}, \tilde{\theta})$ follows.

Proof of Theorem 1.1. In view of the above considerations and [8, Main Theorem] the assumptions of the Leray–Schauder fixed point theorem are satisfied. Hence Theorem 1.1 is proved. ∎

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