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ITERATIVE METHODS FOR PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. This paper is concerned with iterative methods for parabolic functional differential equations with initial boundary conditions. Monotone iterative methods are discussed. We prove a theorem on the existence of solutions for a parabolic problem whose right-hand side admits a Jordan type decomposition with respect to the function variable. It is shown that there exist Newton sequences which converge to the solution of the initial problem. Differential equations with deviated variables and differential integral equations can be obtained from our general model by specializing given operators.

1. Introduction. Let $\Omega \subset \mathbb{R}^2$ be an open domain and suppose that $f : \Omega \to \mathbb{R}$ is a continuous function and $(t_0, x_0) \in \Omega$. Let us denote by $\tilde{\omega} : [t_0, a] \to \mathbb{R}$ the solution of the Cauchy problem

$$\omega'(t) = f(t, \omega(t)), \quad \omega(t_0) = x_0. \tag{1.1}$$

Under natural assumptions on f there exist sequences $\{\alpha_k\}$, $\{\beta_k\}$ of functions $[t_0, a] \to \mathbb{R}$ for $k \geq 0$ such that:

(i) For each $k \geq 1$ the functions α_k and β_k are solutions of linear problems generated by (1.1). More precisely, α_k is a solution of the problem

$$\omega'(t) = f(t, \alpha_{k-1}(t)) + \frac{\partial f}{\partial x}(t, \alpha_{k-1}(t))(\omega(t) - \alpha_{k-1}(t)), \quad \omega(t_0) = x_0,$$

and β_k is a solution of the problem

$$\omega'(t) = f(t, \beta_{k-1}(t)) + \frac{\partial f}{\partial x}(t, \alpha_{k-1}(t))(\omega(t) - \beta_{k-1}(t)), \quad \omega(t_0) = x_0.$$

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(ii) For $k \geq 0$ we have

$$\alpha_k(t) \leq \alpha_{k+1}(t) \leq \tilde{\omega}(t) \leq \beta_{k+1}(t) \leq \beta_k(t), \qquad t \in [t_0, a],$$

and

$$\lim_{k\to\infty} \alpha_k(t) = \lim_{k\to\infty} \beta_k(t) = \tilde{\omega}(t) \quad \text{uniformly on } [t_0, a].$$

(iii) The convergence that we get is of the Newton type, which means that there is $A \geq 0$ such that

$$\tilde{\omega}(t) - \alpha_k(t) \leq \frac{A}{2^{2^k}} \quad \text{and} \quad \beta_k(t) - \tilde{\omega}(t) \leq \frac{A}{2^{2^k}}, \qquad t \in [t_0, a],\ k \geq 0.$$

The construction of the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ was given by S. A. Chaplygin [5]. There exist many generalizations and extensions of the above classical result. It is not our aim to give a full review of results concerning the above problem.

Iterative methods and monotone iterative methods for differential systems and for boundary value problems generated by second order differential equations have been considered in [6], [15]. The monographs [12], [14], [15] contain an exposition of classical developments on monotone iterative methods for partial differential equations.

The papers [4], [11] introduced the monotone iterative method for evolution functional differential equations. Initial boundary value problems of the Dirichlet type for parabolic functional differential equations were investigated in [4]. Monotone iterative methods for the Cauchy problem for an infinite system of parabolic type equations were studied in [20]. The papers [2], [9], [11] concern initial or initial boundary value problems for Hamilton–Jacobi functional differential equations. A theorem on convergence of the Newton method for the Darboux problem related to a hyperbolic functional differential equation can be found in [8]. Theorems on Chaplygin sequences and on the Newton method for hyperbolic functional differential problems are given in [10]. The Chaplygin method is proposed in [19] as a tool of proving existence results for an infinite system of first order partial functional differential equations. It is easy to see that the results given in [19] are not applicable to differential integral systems of Volterra type and to systems with deviated variables.

The aim of the paper is to construct two monotone iterative methods for parabolic functional differential equations with initial boundary conditions. Now we formulate our functional differential problems.

Let $S \subset \mathbb{R}^n$ be a bounded domain with boundary ∂S of class C^1. Write

$$Q_0 = [-b_0, 0] \times \bar{S}, \quad Q = (0, a) \times \bar{S},$$

where $a > 0, b_0 \in \mathbb{R}_+$ and $\bar{S}$ is the closure of S. For each $(t, x) \in [0, a] \times \bar{S}$

we define

$$D[t,x] = \{(\tau,y) \in \mathbb{R}^{n+1} : \tau \le 0,\ (t+\tau, x+y) \in Q_0 \cup Q\}.$$

There is $[c,d] \subset \mathbb{R}^n$ such that

$$D[t,x] \subset [-b_0 - a, 0] \times [c,d] \quad \text{for } (t,x) \in [0,a) \times \bar{S}.$$

Write $I = [-b_0 - a, 0]$ and $B = [-b_0 - a, 0] \times [c,d]$. For a function $z : Q_0 \cup Q \to \mathbb{R}$ and a point $(t,x) \in [0,a) \times \bar{S}$ we define $z_{(t,x)} : D[t,x] \to \mathbb{R}$ by

$$z_{(t,x)}(\tau,y) = z(t+\tau, x+y), \quad (\tau,y) \in D[t,x].$$

In words, we restrict z to $(Q_0 \cup Q) \cap ([-b_0, t] \times \mathbb{R}^n)$ and then shift the restriction to $D[t,x]$. Let $\phi_0 : [0,a) \to \mathbb{R}$ and $\phi = (\phi_1, \dots, \phi_n) : Q \to \mathbb{R}^n$ be given functions. Write $\varphi(t,x) = (\phi_0(t), \phi(t,x))$ for $(t,x) \in Q$. We assume that $0 \le \phi_0(t) \le t$ and $\phi(t,x) \in S$ for $(t,x) \in Q$.

Let $F : \bar{Q} \times C(B,\mathbb{R}) \to \mathbb{R}$, $\psi : Q_0 \to \mathbb{R}$, and β, γ, $\Psi : [0,a) \times \partial S \to \mathbb{R}$ be given. Write

$$\Lambda[z](t,x) = \beta(t,x) z(t,x) + \gamma(t,x) \frac{\partial z(t,x)}{\partial n(x)},$$

where $n(x)$ is the unit outward normal to ∂S at $x \in \partial S$. We write ∂n instead of $\partial n(x)$.

Let $a_{ij}, b_i : \bar{Q} \to \mathbb{R}$, $i,j = 1, \dots, n$, be given functions. Write

$$\mathbf{L}[z](t,x) = \partial_t z(t,x) - \sum_{i,j=1}^{n} a_{ij}(t,x) \partial_{x_i x_j} z(t,x) + \sum_{i=1}^{n} b_i(t,x) \partial_{x_i} z(t,x).$$

We consider the functional differential equation

$$\mathbf{L}[z](t,x) = F(t,x,z_{\varphi(t,x)}) \tag{1.2}$$

with initial boundary conditions

$$\Lambda[z](t,x) = \Psi(t,x) \quad \text{on } [0,a) \times \partial S, \qquad z(t,x) = \psi(t,x) \quad \text{on } Q_0. \tag{1.3}$$

We will say that the function F satisfies *condition* (V) if for each $(t,x) \in Q$ and any $\omega, \tilde{\omega} \in C(B,\mathbb{R})$ such that $\omega(\tau,y) = \tilde{\omega}(\tau,y)$ for $(\tau,y) \in D[\varphi(t,x)]$, we have $F(t,x,\omega) = F(t,x,\tilde{\omega})$. Condition (V) means that the value of F at $(t,x,w) \in \bar{Q} \times C(B,\mathbb{R})$ depends on (t,x) and on the restriction of w to $D[\varphi(t,x)]$ only.

Remark 1.1. Consider the functional differential equation

$$L[z](t,x) = F(t,x,z_{(t,x)}) \tag{1.4}$$

with initial boundary conditions (1.3). It is clear that (1.4) is a particular case of (1.2) with $\varphi(t,x) = (t,x)$.

Differential equations with deviated variables are obtained from (1.4) in the following way. Write

$$F(t,x,w) = \tilde{F}(t,x,w(\varphi(t,x) - (t,x))), \tag{1.5}$$

where $\tilde{F} : \bar{Q} \times \mathbb{R} \to \mathbb{R}$ is a given function. Then (1.4) is equivalent to

$$L[z](t,x) = \tilde{F}(t,x,z(\varphi(t,x))).$$

Later we assume that $F(\cdot,w) \in C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$ for every $w \in C(B,\mathbb{R})$. On the other hand, the function F defined by (1.5) does not satisfy this condition (in general). That is why we consider problem (1.2), (1.3) instead of (1.4), (1.3).

We give examples of equations which can be derived from (1.2) by specializing the operator F.

Example 1.1. Let $f : \bar{Q} \times \mathbb{R} \to \mathbb{R}$ be given. Write $\varphi(t,x) = (t,x)$ and

$$F(t,x,w) = f(t,x, \int_{D[t,x]} w(\tau,y)dy\,d\tau) \quad \text{on } \bar{Q} \times C(B,\mathbb{R}).$$

Then (1.2) reduces to the differential integral equation

$$\mathbf{L}[z](t,x) = f\Big(t,x, \int_{D[t,x]} z(t+\tau,x+y)\,dy\,d\tau\Big).$$

Example 1.2. For the above f we put

$$F(t,x,w) = f(t,x,w(0,O_{[n]})) \quad \text{on } \bar{Q} \times C(B,\mathbb{R}),$$

where $O_{[n]} = (0,\ldots,0) \in \mathbb{R}^n$. Then (1.2) reduces to the equation with deviated variables

$$\mathbf{L}[z](t,x) = f(t,x,z(\psi(t,x))).$$

The paper is organized as follows: In Section 2 we give sufficient conditions for the existence of two monotone sequences which converge to extremal solutions of problem (2.2), (1.3); uniqueness of solutions is also considered. In Section 3 we investigate the Newton method. We prove that there exists a Newton type sequence which converges to the unique solution of (1.2), (1.3).

2. Monotone iterative methods. Suppose that

$$F(t,x,w) = H(t,x,w) + G(t,x,w) \quad \text{on } \bar{Q} \times C(B,\mathbb{R}), \tag{2.1}$$

where $H(t,x,\cdot)$ is non-decreasing and $G(t,x,\cdot)$ is non-increasing. We prove a theorem on the existence of solutions for a parabolic problem whose right-hand side admits a Jordan type decomposition with respect to the function variable (see [17]).

Suppose that $H,G : \bar{Q}\times C(B,\mathbb{R}) \to \mathbb{R}$ are given. We consider the problem consisting of the functional differential equation

$$\mathbf{L}[z](t,x) = H(t,x,z_{\varphi(t,x)}) + G(t,x,z_{\varphi(t,x)}) \tag{2.2}$$

and the initial boundary conditions (1.3).

We prove that there are $u^*, v^* : Q_0 \cup Q \to \mathbb{R}$ such that

$$\mathbf{L}[u^*](t,x) = H(t,x,u^*_{\varphi(t,x)}) + G(t,x,v^*_{\varphi(t,x)}) \quad \text{in } Q, \tag{2.3}$$

$$\mathbf{L}[v^*](t,x) = H(t,x,v^*_{\varphi(t,x)}) + G(t,x,u^*_{\varphi(t,x)}) \quad \text{in } Q, \tag{2.4}$$

and u^*, v^* satisfy (1.3).

Now we define some function spaces. Let $A \subset \mathbb{R}^{1+n}$ be a bounded domain and $0 < \alpha < 1$. We denote by $C^{\alpha/2,\alpha}(A,\mathbb{R})$ the space of all continuous functions $f : A \to \mathbb{R}$ with the finite norm

$$\|f\|_{C^{\alpha/2,\alpha}(A,\mathbb{R})} = \|f\|_{C(A,\mathbb{R})} + H^{\alpha/2,\alpha}[f]$$

where

$$\begin{aligned} \|f\|_{C(A,\mathbb{R})} &= \sup\{|f(t,x)| : (t,x) \in A\}, \\ H^{\alpha/2,\alpha}[f] &= \sup\left\{\frac{|f(t,x) - f(\tilde{t},x)|}{|t-\tilde{t}|^{\alpha/2}} : (t,x), (\tilde{t},x) \in A,\ t \neq \tilde{t}\right\} \\ &\quad + \sup\left\{\frac{|f(t,x) - f(t,\tilde{x})|}{\|x-\tilde{x}\|^{\alpha}} : (t,x), (t,\tilde{x}) \in A,\ x \neq \tilde{x}\right\} \end{aligned}$$

and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. Let $C^{1+\alpha/2,2+\alpha}(A,\mathbb{R})$ denote the space of all continuous functions $f : A \to \mathbb{R}$ satisfying:

(i) $\partial_t f, \partial_x f = (\partial_{x_1} f, \ldots, \partial_{x_n} f)$, $\partial_{xx} f = [\partial_{x_i x_j} f]_{i,j=1}^n$ exist on A, and $\partial_t f$, $\partial_x f$, $\partial_{xx} f$ are continuous,

(ii) the following norm is finite:

$$\begin{aligned} &\|f\|_{C^{1+\alpha/2,2+\alpha}(A,\mathbb{R})} \\ &\quad = \|f\|_{C(A,\mathbb{R})} + \|\partial_t f\|_{C(A,\mathbb{R})} + \sum_{i=1}^n \|\partial_{x_i} f\|_{C(A,\mathbb{R})} \\ &\qquad + \sum_{i,j=1}^n \|\partial_{x_i x_j} f\|_{C(A,\mathbb{R})} + H^{\alpha/2,\alpha}[\partial_t f] + \sum_{i,j=1}^n H^{\alpha/2,\alpha}[\partial_{x_i x_j} f]. \end{aligned}$$

In a similar way we define the space $C^{(1+\alpha)/2,1+\alpha}(A,\mathbb{R})$, $0 < \alpha < 1$. Let $C^{1,2}(A,\mathbb{R})$ be the space of all continuous functions $f : A \to \mathbb{R}$ satisfying:

(i) $\partial_t f$, $\partial_x f$, $\partial_{xx} f$ exist and are continuous on A,

(ii) the following norm is finite:

$$\begin{aligned} &\|f\|_{C^{1,2}(A,\mathbb{R})} \\ &= \|f\|_{C(A,\mathbb{R})} + \|\partial_t f\|_{C(A,\mathbb{R})} + \sum_{i=1}^n \|\partial_{x_i} f\|_{C(A,\mathbb{R})} + \sum_{i,j=1}^n \|\partial_{x_i x_j} f\|_{C(A,\mathbb{R})}. \end{aligned}$$

Let $L^q(A,\mathbb{R})$, $q \geq 1$, be the Banach space of all equivalence classes of Lebesgue measurable functions f defined on A into $\mathbb{R}$ with a finite norm

$$\|f\|_{L^q(A,\mathbb{R})} = \Big(\int_A |f(\tau,y)|^q \, dy \, d\tau\Big)^{1/q}.$$

We denote by $W_q^{1,2}(A,\mathbb{R})$ the Banach space consisting of all $f \in L^q(A,\mathbb{R})$ having generalized derivatives $\partial_t f, \partial_x f, \partial_{xx} f = [\partial_{x_i x_j} f]_{i,j=1}^n$ and such that the following norm is finite:

$$\|f\|_{W_q^{1,2}(A,\mathbb{R})} = \|f\|_{L^q(A,\mathbb{R})} + \|\partial_t f\|_{L^q(A,\mathbb{R})} + \sum_{i=1}^n \|\partial_{x_i} f\|_{L^q(A,\mathbb{R})} + \sum_{i,j=1}^n \|\partial_{x_i x_j} f\|_{L^q(A,\mathbb{R})}.$$

For non-integral α, the Banach space $W_q^{\alpha/2,\alpha}(A,\mathbb{R})$ is defined analogously (see [14]).

Let $S \subset \mathbb{R}^n$ be a bounded domain with boundary ∂S. We will say that ∂S is of *class* $C^{2+\alpha}$, $0 < \alpha < 1$, if for every $x \in \partial S$ there exists a neighborhood U_x of x and $i \in \{1,\dots,n\}$ such that $\partial S \cap U_x$ can be represented in the form

$$x_i = h(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n), \qquad (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \in \tilde{U}_x,$$

where $\tilde{U}_x \subset \mathbb{R}^{n-1}$ is an open set and $h \in C^{2+\alpha}(\tilde{U}_x,\mathbb{R})$.

For $w \in C(B,\mathbb{R})$, $\tilde{w} \in C(Q_0 \cup Q,\mathbb{R})$ we put

$$\|w\|_B = \max\{|w(\tau,y)| : (\tau,y) \in B\},$$
$$\|\tilde{w}_{(t,x)}\|_{D[t,x]} = \max\{|\tilde{w}_{(t,x)}(\tau,y)| : (\tau,y) \in D[t,x],\ (t,x) \in Q\}.$$

Assumption $\mathbf{H}_*$.

(1) $a_{ij} \in C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$, $b_i \in C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$ for $i,j = 1,\dots,n$ and there exist $K_1, K_2 > 0$ such that for $(t,x) \in \bar{Q}$ we have

$$K_1\|\lambda\|^2 \le \sum_{i,j=1}^n a_{ij}(t,x)\lambda_i\lambda_j \le K_2\|\lambda\|^2, \quad \lambda = (\lambda_1,\dots,\lambda_n),$$

(2) $S \subset \mathbb{R}^n$ is a bounded domain and ∂S is of class $C^{2+\alpha}$,

(3) $\beta,\gamma \in C^{(1+\alpha)/2,1+\alpha}([0,a]\times\partial S,\mathbb{R}_+)$, $\beta(t,x) > 0$ for $(t,x) \in [0,a]\times\partial S$,

(4) $\Psi \in C^{(1+\alpha)/2,1+\alpha}([0,a]\times\partial S,\mathbb{R})$, $\psi \in C^{2+\alpha}(Q_0,\mathbb{R})$ and the following compatibility conditions hold for all $x \in \partial S$:

$$\beta(0,x)\psi(0,x) + \gamma(0,x)\frac{\partial\psi(0,x)}{\partial n(x)} = \Psi(0,x), \tag{2.5}$$

$$\mathbf{L}[\psi](0,x) = H(0,x,\psi_{\varphi(0,x)}) + G(0,x,\psi_{\varphi(0,x)}). \tag{2.6}$$

Given $\mathbf{f} : \bar{Q} \to \mathbb{R}$, consider the linear parabolic equation

$$\mathbf{L}[z](t,x) = \mathbf{f}(t,x) \tag{2.7}$$

with the initial boundary conditions

$$\Lambda[z](t,x) = \Psi(t,x) \text{ on } [0,a]\times\partial S, \quad z(0,x) = \psi(0,x) \text{ for } x \in \partial S. \tag{2.8}$$

THEOREM 2.1. *Suppose that Assumption* $\mathbf{H}_*$ *is satisfied and* $\mathbf{f} \in C^{\alpha/2,\alpha}(\bar{Q}, \mathbb{R})$. *Then there exists exactly one solution* $\tilde{u} : \bar{Q} \to \mathbb{R}$ *of* (2.7), (2.8) *and* $\tilde{u} \in C^{1+\alpha/2,2+\alpha}(\bar{Q}, \mathbb{R})$.

The proof can be found in [13, Chapter IV, Theorem 5.3]; see also [14, Appendix A, Section 3] and [16, Chapter V, Theorem 5.18].

ASSUMPTION $\mathbf{H}[\varphi]$. The function $\varphi : \bar{Q} \to \mathbb{R}^{1+n}$, $\varphi(t,x) = (\phi_0(t), \phi(t,x))$ for $(t,x) \in Q$, satisfies the conditions

(1) $\phi_0 \in C([0,a], \mathbb{R}_+)$, $\phi \in C(\bar{Q}, \mathbb{R}^n)$, $0 \le \phi_0(t) \le t$ for $t \in [0,a]$, $\phi(t,x) \in \bar{S}$ for $(t,x) \in \bar{Q}$,
(2) there is $C_0 \ge 0$ such that

$$|\phi_0(t) - \phi_0(\tilde{t})| \le C_0 |t - \tilde{t}|, \qquad t, \tilde{t} \in [0,a],$$
$$\|\phi(t,x) - \phi(\tilde{t}, \tilde{x})\| \le C_0(|t - \tilde{t}| + \|x - \tilde{x}\|), \qquad (t,x),\ (\tilde{t}, \tilde{x}) \in \bar{Q}.$$

ASSUMPTION $\mathbf{H}[H,G]$. The functions $H, G : \bar{Q} \times C(B, \mathbb{R}) \to \mathbb{R}$ satisfy condition (V) and

(1) $H,\ G \in C(\bar{Q} \times C(B, \mathbb{R}), \mathbb{R})$ and for every $w \in C(B, \mathbb{R})$ we have $H(\cdot, w),\ G(\cdot, w) \in C^{\alpha/2,\alpha}(\bar{Q}, \mathbb{R})$,
(2) for each $(t,x) \in \bar{Q}$ the function $G(t,x,\cdot)$ is non-increasing and $H(t,x,\cdot)$ is non-decreasing,
(3) there is $L \ge 0$ such that for all $(t,x,w) \in \bar{Q} \times C(B, \mathbb{R})$,

$$|H(t,x,w) - H(t,x,\tilde{w})| \le L \|w - \tilde{w}\|_B^\alpha,$$
$$|G(t,x,w) - G(t,x,\tilde{w})| \le L \|w - \tilde{w}\|_B^\alpha.$$

EXAMPLE 2.1. Suppose that $d, k \in C^{\alpha/2,\alpha}(\bar{Q}, \mathbb{R}_+)$ are given functions. Write $H(t,x,w) = d(t,x)w$, and $G(t,x,w) = -k(t,x)w$. Then H and G satisfy Assumption $\mathbf{H}[H,G]$.

EXAMPLE 2.2. Suppose that $G(t,x,w) = 0$ on $\bar{Q} \times C(B, \mathbb{R})$ and $H : \bar{Q} \times C(B, \mathbb{R}) \to \mathbb{R}$ satisfy condition (V) and condition (1) of Assumption $\mathbf{H}[H,G]$ and there are $L_1, L_2, L_3 \in \mathbb{R}_+$ such that

$$\|\partial_w H(t,x,w) - \partial_w H(\bar{t}, \bar{x}, \bar{w})\|_{CL} \le L_1(|t - \bar{t}|^{\alpha/2} + \|x - \bar{x}\|^\alpha) + L_2 \|w - \bar{w}\|_B$$

for all $(t,x,w), (\bar{t}, \bar{x}, \bar{w}) \in Q \times C(B, \mathbb{R})$, and

$$\|\partial_w H(t,x,\theta)\|_{CL} \le L_3,$$

for all $(t,x,\theta) \in Q \times C^{1,2}(B, \mathbb{R})$ with $\theta(t,x) = 0$ for $(t,x) \in B$. Moreover suppose that for $(t,x,w) \in Q \times C(B, \mathbb{R})$ and $\bar{w} \in C(B, \mathbb{R})$ we have

$$\partial_w H(t,x,w)h \ge 0 \qquad \text{if } h \in C(B, \mathbb{R}_+),$$
$$\partial_w H(t,x,w)h \ge \partial_w H(t,x,\bar{w})h \qquad \text{if } h \in C(B, \mathbb{R}_+),\ w \ge \bar{w} \text{ on } B.$$

Then H and G satisfy Assumption $\mathbf{H}[H,G]$.

Assumption $\mathbf{H}[u,v]$. There are $u, v \in C^{1,2}(Q_0 \cup \bar{Q}, \mathbb{R})$ such that $u(t,x) \le v(t,x)$ for $(t,x) \in Q_0 \cup \bar{Q}$ and

$$\mathbf{L}[u](t,x) \le H(t,x,u_{\varphi(t,x)}) + G(t,x,v_{\varphi(t,x)}) \quad \text{on } Q,$$
$$\Lambda[u](t,x) \le \Psi(t,x) \quad \text{on } [0,a] \times \partial S, \qquad u(t,x) \le \psi(t,x) \quad \text{on } Q_0,$$

and

$$\mathbf{L}[v](t,x) \ge H(t,x,v_{\varphi(t,x)}) + G(t,x,u_{\varphi(t,x)}) \quad \text{on } Q,$$
$$\Lambda[v](t,x) \ge \Psi(t,x) \quad \text{on } [0,a] \times \partial S, \qquad v(t,x) \ge \psi(t,x) \quad \text{on } Q_0.$$

Theorem 2.2. *Suppose that Assumptions* $\mathbf{H}_*$, $\mathbf{H}[u,v]$, $\mathbf{H}[\varphi]$ *and* $\mathbf{H}[H,G]$ *are satisfied. Then there exist functions* $U[u,v], V[u,v] \in C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R})$ *which are solutions of*

$$\mathbf{L}[U[u,v]](t,x) = H(t,x,u_{\varphi(t,x)}) + G(t,x,v_{\varphi(t,x)}),$$
$$\mathbf{L}[V[u,v]](t,x) = H(t,x,v_{\varphi(t,x)}) + G(t,x,u_{\varphi(t,x)}),$$

with the initial boundary conditions (1.3). *Moreover*

$$u(t,x) \le U[u,v](t,x) \le V[u,v](t,x) \le v(t,x) \quad \textit{for } (t,x) \in \bar{Q}. \tag{2.9}$$

and for $\tilde{u} = U[u,v]$, $\tilde{v} = V[u,v]$ *we have*

$$\mathbf{L}[\tilde{u}](t,x) \le H(t,x,\tilde{u}_{\varphi(t,x)}) + G(t,x,\tilde{v}_{\varphi(t,x)}) \quad \textit{on } Q, \tag{2.10}$$

$$\mathbf{L}[\tilde{v}](t,x) \ge H(t,x,\tilde{v}_{\varphi(t,x)}) + G(t,x,\tilde{u}_{\varphi(t,x)}) \quad \textit{on } Q. \tag{2.11}$$

Proof. Consider problem (2.7), (2.8) with

$$\mathbf{f}(t,x) = H(t,x,u_{\varphi(t,x)}) + G(t,x,v_{\varphi(t,x)}), \qquad (t,x) \in Q. \tag{2.12}$$

We prove that $\mathbf{f} \in C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$. Note that the functions $u_{\varphi(t,x)}$ and $u_{\varphi(\bar{t},\bar{x})}$, where $(t,x), (\bar{t},\bar{x}) \in Q$, have different domains. Therefore, we need the following construction. Write $Y = [-b_0,a] \times [\tilde{c},\tilde{d}]$ where $\tilde{c} = (\tilde{c}_1,\ldots,\tilde{c}_n)$, $\tilde{d} = (\tilde{d}_1,\ldots,\tilde{d}_n)$, $\tilde{c}_i = c_i - |d_i - c_i|$, $\tilde{d}_i = d_i + |d_i - c_i|$ for $i = 1,\ldots,n$. There is $\tilde{u} : Y \to \mathbb{R}$ such that $\tilde{u} \in C^{1,2}(Y,\mathbb{R})$ and $\tilde{u}(t,x) = u(t,x)$ for $(t,x) \in Q_0 \cup \bar{Q}$. Then the function $\tilde{u}_{\varphi(t,x)}$ is defined on B for $(t,x) \in \bar{Q}$. There is $\tilde{C}_0 > 0$ such that for $(t,x), (\bar{t},\bar{x}) \in Q$ we have

$$\|\tilde{u}(t,x) - \tilde{u}(\bar{t},\bar{x})\|_B \le \tilde{C}_0(|t-\bar{t}| + \|x-\bar{x}\|),$$
$$\|\tilde{v}(t,x) - \tilde{v}(\bar{t},\bar{x})\|_B \le \tilde{C}_0(|t-\bar{t}| + \|x-\bar{x}\|).$$

It follows from Assumptions $\mathbf{H}[H,G]$ and $\mathbf{H}[\varphi]$ that there is $\tilde{C} > 0$ such that for $(t,x), (\bar{t},\bar{x}) \in Q$ we have

$$\begin{aligned}
&|\mathbf{f}(t,x) - \mathbf{f}(\bar{t},\bar{x})| \\
&= |H(t,x,\tilde{u}_{\varphi(t,x)}) + G(t,x,\tilde{v}_{\varphi(t,x)}) - H(\bar{t},\bar{x},\tilde{u}_{\varphi(\bar{t},\bar{x})}) - G(\bar{t},\bar{x},\tilde{v}_{\varphi(\bar{t},\bar{x})})| \\
&\le L\|\tilde{u}_{\varphi(t,x)} - \tilde{u}_{\varphi(\bar{t},\bar{x})}\|_B^\alpha + L\|\tilde{v}_{\varphi(t,x)} - \tilde{v}_{\varphi(\bar{t},\bar{x})}\|_B^\alpha \\
&\quad + 2\tilde{C}[|t-\bar{t}|^{\alpha/2} + \|x-\bar{x}\|^\alpha] \le C_*[|t-\bar{t}|^{\alpha/2} + \|x-\bar{x}\|^\alpha],
\end{aligned}$$

where $C_* = 2\tilde{C}+2L\tilde{C}_0^\alpha(1+2C_0)^\alpha[\max\{a^{\alpha/2},1\}]$. This gives $\mathbf{f} \in C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$. It follows from Theorem 2.1 that there is exactly one solution of problem (2.7), (2.8) with $\mathbf{f}$ given by (2.12). Let us denote this solution by $U[u,v]$. Analogously we prove that there is exactly one solution $V[u,v]$ of problem (2.7), (2.8) with $\mathbf{f}$ given by $\mathbf{f}(t,x) = H(t,x,v_{\varphi(t,x)})+G(t,x,u_{\varphi(t,x)})$. It follows from Theorem 2.1 that $U[u,v], V[u,v] \in C^{1+\alpha/2,2+\alpha}(Q_0 \cup \bar{Q},\mathbb{R})$. Let $z = u - U[u,v]$. We conclude from Assumptions $\mathbf{H}[u,v]$, $\mathbf{H}[\varphi]$, $\mathbf{H}[H,G]$ that

$$\mathbf{L}[z](t,x) \le 0 \quad \text{on } Q,$$

and $\Lambda[z](t,x) = 0$ on $[0,a) \times \partial S$ and $z(t,x) = 0$ on Q_0. The theorem on differential inequalities for mixed problems (see [14, Theorem 2.2.1]) now shows that $z(t,x) \le 0$ on Q. Hence $u \le U[u,v]$. In the same manner we can see that $V[u,v] \le v$ on Q. Let $\tilde{z} = V[u,v] - U[u,v]$. We have $\Lambda[\tilde{z}](t,x) = 0$ on $[0,a) \times \partial S$ and $\tilde{z}(t,x) = 0$ on Q_0. Moreover

$$\begin{aligned}\mathbf{L}[\tilde{z}](t,x) = H(t,x,v_{\varphi(t,x)}) &+ G(t,x,u_{\varphi(t,x)}) \\ &- H(t,x,u_{\varphi(t,x)}) - G(t,x,v_{\varphi(t,x)}) \ge 0.\end{aligned}$$

An application of the theorem on differential inequalities for mixed problems yields $\tilde{z}(t,x) \ge 0$ on Q. Consequently, we have

$$u \le U[u,v] \le V[u,v] \le v \quad \text{on } Q,$$

and (2.9) is proved. From (2.9) it is obvious that inequalities (2.10), (2.11) are satisfied. This completes the proof.

THEOREM 2.3. *Suppose that Assumptions* $\mathbf{H}_*$, $\mathbf{H}[u,v]$, $\mathbf{H}[\varphi]$ *and* $\mathbf{H}[H,G]$ *are satisfied. Then there exist monotone sequences* $\{u^{(k)}\}$, $\{v^{(k)}\}$ *which converge to* u^*, v^* *respectively, and* u^*, v^* *satisfy* (2.3), (2.4), *and the initial boundary conditions* (1.3).

Proof. First we formulate a monotone iterative method for problem (2.2), (1.3). Let us consider the sequences $\{u^{(k)}\}$, $\{v^{(k)}\}$, where $u^{(k)}, v^{(k)} : Q_0 \cup \bar{Q} \to \mathbb{R}$, defined in the following way:

- $u^{(0)} = u$, $v^{(0)} = v$, where u, v are given in $\mathbf{H}[u,v]$,
- if $u^{(k)}$ and $v^{(k)}$ are known functions, then we put $u^{(k+1)} = U[u^{(k)},v^{(k)}]$, $v^{(k+1)} = V[u^{(k)},v^{(k)}]$.

It follows from Theorem 2.2 that

$$u^{(0)} \le \cdots \le u^{(k)} \le u^{(k+1)} \le v^{(k+1)} \le v^{(k)} \le \cdots \le v^{(0)} \quad \text{on } Q. \tag{2.13}$$

We will show that the sequences $\{u^{(k)}\}$ and $\{v^{(k)}\}$ are convergent. Let

$$h_k(t,x) = H(t,x,u^{(k)}_{\varphi(t,x)}) + G(t,x,v^{(k)}_{\varphi(t,x)}), \qquad (t,x) \in Q,\ k \ge 1.$$

We will show that there exists $\tilde{K} > 0$ such that

$$\|u^{(k)}\|_{C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R})} \le \tilde{K}.$$

It follows from [13, Theorem 5.3] (see also [14, Theorem 2.3.1]) that

$$\|u^{(k)}\|_{C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R})} \le C[\|h_k\|_{C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})} + \|\Psi\|_{C^{(1+\alpha)/2,1+\alpha}([0,a]\times\partial S,\mathbb{R})} + \|\psi\|_{C^{2+\alpha}(\bar{Q}_0,\mathbb{R})}].$$

We need to show that there exists $\tilde{K}_1 > 0$ such that

$$\|h_k\|_{C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})} = \|h_k\|_{C(\bar{Q},\mathbb{R})} + H^{\alpha/2,\alpha}[h_k] \le \tilde{K}_1.$$

It follows from Assumptions $\mathbf{H}[H,G]$ and $\mathbf{H}[\varphi]$ that

$$\begin{aligned}
&|h_k(t,x) - h_k(\bar{t},\bar{x})| \\
&\quad \le |H(t,x,u^{(k)}_{\varphi(t,x)}) - H(\bar{t},\bar{x},u^{(k)}_{\varphi(\bar{t},\bar{x})})| + |G(t,x,v^{(k)}_{\varphi(t,x)}) - G(\bar{t},\bar{x},v^{(k)}_{\varphi(\bar{t},\bar{x})})| \\
&\quad \le 2C[|t-\bar{t}|^{\alpha/2} + \|x-\bar{x}\|^{\alpha}] + L(\|\tilde{u}^{(k)}_{\varphi(t,x)} - \tilde{u}^{(k)}_{\varphi(\bar{t},\bar{x})}\|^{\alpha}_B + \|\tilde{v}^{(k)}_{\varphi(t,x)} - \tilde{v}^{(k)}_{\varphi(\bar{t},\bar{x})}\|^{\alpha}_B).
\end{aligned}$$

Notice that $C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R}) \subset W^{1,2}_q(\bar{Q},\mathbb{R})$ where $q \ge (n+2)/(1-\alpha)$. It follows from Theorem A.3.4 in [14] that $W^{1,2}_q(\bar{Q},\mathbb{R})$ is embedded in $C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R})$ and

$$\|u^{(k)}\|_{C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R})} \le C\|u^{(k)}\|_{W^{1,2}_q(\bar{Q},\mathbb{R})}, \quad k \ge 1.$$

By Theorem A.3.3 in [14] we have

$$\begin{aligned}
(2.14)\quad &\|u^{(k)}\|_{W^{1,2}_q(\bar{Q},\mathbb{R})} \\
&\le C[\|h_k\|_{L^q(\bar{Q},\mathbb{R})} + \|\Psi\|_{W^{1/2-1/2q,1-1/q}([0,a]\times\partial S,\mathbb{R})} + \|\psi\|_{W^{2-2/q}_q(Q_0,\mathbb{R})}].
\end{aligned}$$

As $C(\bar{Q},\mathbb{R})$ is dense in $L^q(\bar{Q},\mathbb{R})$, from the conditions (2), (3) of $\mathbf{H}[H,G]$, it follows that $\{h_k\}$ is a bounded sequence in $L^q(\bar{Q},\mathbb{R})$. Hence there exists $\tilde{K}_2 > 0$ such that

$$\|u^{(k)}\|_{C^{(1+\alpha)/2,1+\alpha}(\bar{Q},\mathbb{R})} \le \tilde{K}_2 \quad \text{for } k \ge 0.$$

Using this estimate we have

$$\begin{aligned}
|h_k(t,x) - h_k(\bar{t},\bar{x})| &\\
&\le 2C[|t-\bar{t}|^{\alpha/2} + \|x-\bar{x}\|^{\alpha}] + 2L\tilde{K}_2[|\phi_0(t) - \phi_0(\bar{t})|^{(1+\alpha)/2} \\
&\quad + \|\phi(t,x) - \phi(\bar{t},\bar{x})\|]^{\alpha} \\
&\le 2C[|t-\bar{t}|^{\alpha/2} + \|x-\bar{x}\|^{\alpha}] + M[|t-\bar{t}|^{\alpha/2} + \|x-\bar{x}\|^{\alpha}] \\
&= N[|t-\bar{t}|^{\alpha/2} + \|x-\bar{x}\|^{\alpha}],
\end{aligned}$$

where $M = 2L\tilde{K}_2 C_0^{\alpha} \max\{a^{\alpha^2/2} + a^{\alpha/2}, 1\}$ and $N = 2C + M$. Hence

$$\|u^{(k)}\|_{C^{1+\alpha/2,2+\alpha}(\bar{Q},\mathbb{R})} \le \tilde{K}.$$

It follows from the Arzelà–Ascoli Theorem that there exists a subsequence of $\{u^{(k)}\}$ which converges uniformly to u^* in $C^{1,2}(\bar{Q},\mathbb{R})$. Since $\{u^{(k)}\}$ is monotone, it converges uniformly in $C^{1,2}(\bar{Q},\mathbb{R})$. Hence

$$\lim_{n\to\infty}\|u^{(k)}-u^*\|_{C^{1,2}(\bar{Q},\mathbb{R})}=0,$$

and $u^{(0)}(t,x)\le u^*(t,x)\le v^{(0)}(t,x)$ for $(t,x)\in\bar{Q}$. Similarly, we can prove that $v^{(k)}$ converges to v^* in $C^{1,2}(\bar{Q},\mathbb{R})$ and $u^{(0)}(t,x)\le v^*(t,x)\le v^{(0)}(t,x)$ for $(t,x)\in\bar{Q}$. It follows from (2.13) that

$$u^{(0)}(t,x)\le u^*(t,x)\le v^*(t,x)\le v^{(0)}(t,x),\qquad (t,x)\in\bar{Q}. \tag{2.15}$$

It is easy to see that u^*,v^* satisfy

$$\mathbf{L}[u^*](t,x)=H(t,x,u^*_{\varphi(t,x)})+G(t,x,v^*_{\varphi(t,x)})\quad\text{on } Q,$$
$$\mathbf{L}[v^*](t,x)=H(t,x,v^*_{\varphi(t,x)})+G(t,x,u^*_{\varphi(t,x)})\quad\text{on } Q,$$

and the initial boundary conditions (1.3). This completes the proof.

THEOREM 2.4. *Suppose that Assumptions* $\mathbf{H}_*$, $\mathbf{H}[u,v]$, $\mathbf{H}[\varphi]$ *and* $\mathbf{H}[H,G]$ *are satisfied and for any* $w,\tilde{w}\in C(B,\mathbb{R})$ *with* $w(t,x)\ge\tilde{w}(t,x)$ *on* B,

$$H(t,x,w)-H(t,x,\tilde{w})\le L_1(w-\tilde{w}),\qquad G(t,x,w)-G(t,x,\tilde{w})\ge -L_2(w-\tilde{w}),$$

for all $(t,x)\in Q$ *with some* $L_1,L_2>0$. *Then* $u^*=v^*=\tilde{z}$, *where* $\tilde{z}$ *is solution of* (2.2), (1.3) *such that* $u^{(0)}\le\tilde{z}\le v^{(0)}$ *on* $\bar{Q}$.

Proof. It follows from Theorem 2.3 that $u^*\le v^*$ on $\bar{Q}$. Suppose that $\tilde{z}$ is any solution of (2.2), (1.3) such that $u^{(0)}(t,x)\le\tilde{z}(t,x)\le v^{(0)}(t,x)$ on $\bar{Q}$. Assume that for some $k>1$ we have $u^{(k)}(t,x)\le\tilde{z}(t,x)\le v^{(k)}(t,x)$ on $\bar{Q}$. Let $w=u^{(k+1)}-\tilde{z}$. Then $\Lambda[w](t,x)=0$ on $(0,a]\times\partial S$ and $w(t,x)=0$ for all $(t,x)\in Q_0$. It follows from Assumption $\mathbf{H}[H,G]$ that

$$\begin{aligned}\mathbf{L}[w](t,x)&=H(t,x,u^{(k)}_{\varphi(t,x)})+G(t,x,v^{(k)}_{\varphi(t,x)})\\&\quad-H(t,x,\tilde{z}_{\varphi(t,x)})-G(t,x,\tilde{z}_{\varphi(t,x)})\le 0,\qquad (t,x)\in\bar{Q}.\end{aligned}$$

Then $u^{(k+1)}(t,x)\le\tilde{z}(t,x)$ on $\bar{Q}$. Similarly we show that $\tilde{z}(t,x)\le v^{(k+1)}(t,x)$ on $\bar{Q}$. It follows by induction that

$$u^{(k)}(t,x)\le\tilde{z}(t,x)\le v^{(k)}(t,x)\quad\text{ on } \bar{Q},\ k\ge 0.$$

Hence $u^*(t,x)\le\tilde{z}(t,x)\le v^*(t,x)$ on $\bar{Q}$. Let $\tilde{w}=v^*-u^*$. We have $\Lambda[\tilde{w}](t,x)=0$ on $[0,a)\times\partial S$ and $\tilde{w}(t,x)=0$ on Q_0. Moreover

$$\begin{aligned}\mathbf{L}[\tilde{w}](t,x)&=H(t,x,v^*_{\varphi(t,x)})+G(t,x,u^*_{\varphi(t,x)})-H(t,x,u^*_{\varphi(t,x)})-G(t,x,v^*_{\varphi(t,x)})\\&\le(L_1+L_2)\tilde{w}_{\varphi(t,x)}\quad\text{on } Q.\end{aligned}$$

It follows from the theorem on differential inequalities for mixed problems (see for instance [18, Th. 2.2]) that $v^*\le u^*$ on $\bar{Q}$. This completes the proof.

3. Newton method for functional differential problems. In the proof of existence and convergence of monotone iterative methods we assume the monotonicity of the right hand side of the equation. In this section we will omit this assumption and define a sequence which converges to the solution of the initial problem.

We denote by $CL(B,\mathbb{R})$ the class of all linear and continuous real functions defined on $C(B,\mathbb{R})$. Let $\|\cdot\|_{CL}$ denote the norm in the space $CL(B,\mathbb{R})$ generated by the maximum norm in $C(B,\mathbb{R})$.

Assumption $H_C[F]$. The function $F:\bar{Q}\times C(B,\mathbb{R})\to\mathbb{R}$ of the variables (t,x,w) satisfies condition (V) and

(1) F is continuous and $F(\cdot,w)\in C^{\alpha/2,\alpha}(\bar{Q},\mathbb{R})$ for every $w\in C(B,\mathbb{R})$,
(2) for every $(t,x,w)\in Q\times C(B,\mathbb{R})$ the Fréchet derivative $\partial_w F(t,x,w)$ exists, and $\partial_w F(t,x,w)\in CL(B,\mathbb{R})$,
(3) there are $L_1,L_2,L_3\in\mathbb{R}_+$ such that

$$\|\partial_w F(t,x,w)-\partial_w F(\bar{t},\bar{x},\bar{w})\|_{CL} \le L_1(|t-\bar{t}|^{\alpha/2}+\|x-\bar{x}\|^{\alpha})+L_2\|w-\bar{w}\|_B$$

for all $(t,x,w),(\bar{t},\bar{x},\bar{w})\in Q\times C(B,\mathbb{R})$, and

$$\|\partial_w F(t,x,\tilde{w})\|_{CL}\le L_3 \quad \text{for all } (t,x,\tilde{w})\in Q\times C^{1,2}(B,\mathbb{R}).$$

For given $u\in C(Q_0\cup Q,\mathbb{R})$, we consider the function

$$\mathbb{F}(t,x,w;u)=F(t,x,u_{\varphi(t,x)})+\partial_w F(t,x,u_{\varphi(t,x)})(w-u_{\varphi(t,x)}).$$

Suppose that the sequence $\{u^{(k)}\}$, where $u^{(k)}:Q_0\cup Q\to\mathbb{R}$ for $k\in\mathbb{N}$, satisfies the conditions

(a) $u^{(0)}\in C^{1,2}(Q_0\cup\bar{Q},\mathbb{R})$,
(b) for every $k\ge 1$ the function $u^{(k)}$ is a solution of the equation

$$L[z](t,x)=\mathbb{F}(t,x,z_{\varphi(t,x)};u^{(k-1)})$$

with the initial boundary conditions

$$\Lambda[z](t,x)=\Psi^{(k)}(t,x) \quad \text{for } (t,x)\in[0,a]\times\partial S,$$
$$z(t,x)=\psi^{(k)}(t,x) \quad \text{for } (t,x)\in Q_0,$$

where $\Psi^{(k)}\in C^{((1+\alpha)/2,1+\alpha)}([0,a]\times\partial S,\mathbb{R})$ and $\psi^{(k)}\in C^{(2+\alpha)}(Q_0,\mathbb{R})$.

It follows from Assumptions $\mathbf{H}_*$, $\mathbf{H}_C[F]$, $\mathbf{H}[\varphi]$ and from Theorem 2.1 that such a sequence exists.

Now we will prove that the sequence $\{u^{(k)}\}$ converges to a solution of (1.2), (1.3). In the proof we will use some ideas from the proof of the Newton–Kantorovich theorem.

Theorem 3.1. *Suppose that Assumptions* $\mathbf{H}_*$, $\mathbf{H}_C[F]$, $\mathbf{H}[\varphi]$ *are satisfied and*

(1) $\tilde{z}:Q_0\cup\bar{Q}\to\mathbb{R}$ *is a solution of* (1.2), (1.3),

(2) $u^{(0)} \in C^{1,2}(Q_0 \cup \bar{Q}, \mathbb{R})$ *and*

$$|\tilde{z}(t,x) - u^{(0)}(t,x)| \le \varepsilon_0 \quad \text{on } Q_0 \cup \bar{Q},$$

where $\varepsilon_0 = \{4aL_2K\}^{-1}$ *and* $K = \exp(L_3 a)$,

(3) *the inequalities*

$$|\psi(t,x) - \psi^{(k)}(t,x)| \le \varepsilon_0[2^{2^k}K]^{-1}, \qquad (t,x) \in Q_0, \tag{3.1}$$

$$|\Psi(t,x) - \Psi^{(k)}(t,x)| \le \Lambda[\omega_k(\cdot))](t,x), \quad (t,x) \in [0,a) \times \partial S, \tag{3.2}$$

hold for $k \ge 1$, *where* $\omega_k(\cdot)$ *is the solution of the Cauchy problem*

$$\eta'(t) = L_2\left(\frac{2\varepsilon_0}{2^{2^{(k-1)}}}\right)^2 + L_3\eta(t), \qquad \eta(0) = \varepsilon_0[2^{2^k}K]^{-1}. \tag{3.3}$$

Then for $k \ge 0$ *we have*

$$|\tilde{z}(t,x) - u^{(k)}(t,x)| \le \frac{2\varepsilon_0}{2^{2^k}} \quad \text{on } \bar{Q}. \tag{3.4}$$

Proof. We prove (3.4) by induction. It is easy to see that (3.4) is satisfied for $k = 0$. Suppose that it holds for some $k \ge 1$. Then for $(t,x) \in Q$ we have

$$\begin{aligned}\mathbf{L}[\tilde{z} - u^{(k+1)}](t,x) &= F(t,x,\tilde{z}_{\varphi(t,x)}) - F(t,x,u^{(k)}_{\varphi(t,x)}) \\ &\quad - \partial_w F(t,x,u^{(k)}_{\varphi(t,x)})[u^{(k+1)}_{\varphi(t,x)} - u^{(k)}_{\varphi(t,x)}].\end{aligned}$$

It follows from the Hadamard mean value theorem that

$$\begin{aligned}&\mathbf{L}[\tilde{z} - u^{(k+1)}](t,x) \\ &\quad = \int_0^1 [\partial_w F(t,x,u^{(k)}_{\varphi(t,x)} + \tau(\tilde{z}_{\varphi(t,x)} - u^{(k)}_{\varphi(t,x)})) - \partial_w F(t,x,u^{(k)}_{\varphi(t,x)})] \\ &\qquad\qquad \times [\tilde{z}_{\varphi(t,x)} - u^{(k)}_{\varphi(t,x)}]\, d\tau \\ &\qquad + \partial_w F(t,x,u^{(k)}_{\varphi(t,x)})[u^{(k+1)}_{\varphi(t,x)} - \tilde{z}_{\varphi(t,x)}].\end{aligned}$$

Set $Z^{(k)} = \tilde{z} - u^{(k)}$. We conclude from the above relations and from Assumption $\mathbf{H}_C[F]$ that

$$|L[Z^{(k+1)}](t,x)| \le L_2\|Z^{(k)}_{\varphi(t,x)}\|_B^2 + L_3\|Z^{(k+1)}_{\psi(t,x)}\|_B, \qquad (t,x) \in \bar{Q}.$$

It follows from a comparison theorem for parabolic functional differential inequalities (see [18]) and from (3.1), (3.2) that

$$|Z^{(k+1)}(t,x)| \le \omega_{k+1}(t), \qquad (t,x) \in \bar{Q},$$

where ω_{k+1} is the solution of (3.3). It follows that

$$\omega_{k+1}(t) = \varepsilon_0 \{2^{2^{k+1}} K\}^{-1} \exp(L_3 t) + L_2 \left(\frac{2\varepsilon_0}{2^{2^k}}\right)^2 \int_0^t \exp(L_3(t-s))\, ds$$
$$\leq \frac{\varepsilon_0}{2^{2^{k+1}}} + \frac{4\varepsilon_0^2 L_2 a K}{2^{2^{k+1}}} = \frac{2\varepsilon_0}{2^{2^{k+1}}}.$$

Hence

$$|Z^{(k+1)}(t,x)| \leq \frac{2\varepsilon_0}{2^{2^{k+1}}}.$$

This completes the proof.

REMARK 3.1. It is clear that conditions (3.1), (3.2) are satisfied if we put

$$\psi^{(k)}(t,x) = \psi(t,x) \quad \text{on } Q_0, \qquad \Psi^{(k)}(t,x) = \Psi(t,x) \quad \text{on } [0,a) \times \partial S.$$

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