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ON A FAMILY OF BAYESIAN ESTIMATORS AND PREDICTORS FOR A GUMBEL MODEL BASED ON THE kTH LOWER RECORDS

Abstract. Bayesian estimation for the two parameters of a Gumbel distribution are obtained based on kth lower record values. Prediction, either point or interval, for future kth lower record values is also presented from a Bayesian view point. Some of the results of [4] can be obtained as special cases of our results ($k = 1$).

1. Introduction. Let $\{X_n,\, n \geq 1\}$ be a sequence of independent identically distributed (iid) random variables with a cumulative distribution function (cdf) $F(x)$ and a probability density function (pdf) $f(x)$. The jth order statistic of a sample $(X_1, \ldots, X_n)$ is denoted by $X_{j:n}$. For a fixed $k \geq 1$ we define the sequence $L_k(n),\ n \geq 1$, of kth *lower record times* of $\{X_n,\, n \geq 1\}$ as follows:

$$L_k(1) = 1,$$
$$L_k(n+1) = \min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}, \qquad n \geq 1.$$

The sequence $\{Z_n^{(k)}, n \geq 1\}$ with

$$Z_n^{(k)} = X_{k:L_k(n)+k-1}, \qquad n \geq 1,$$

is called the sequence of kth *lower record values* of $\{X_n,\, n \geq 1\}$. Note that $Z_1^{(k)} = \max\{X_1, \ldots, X_k\}$ and $Z_n^{(1)} = X_{L(n)},\ n \geq 1$, are lower record values.

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It is known that

$$
(1)\quad
\begin{aligned}
&f_{Z_n^{(k)}}(z) = \frac{k^n}{(n-1)!}[-\ln F(z)]^{n-1}(F(z))^{k-1}f(z), \qquad z \in \mathbb{R},\\
&f_{Z_1^{(k)},\ldots,Z_n^{(k)}}(z_1,\ldots,z_n) = k^n(F(z_n))^{k-1}f(z_n)\prod_{i=1}^{n-1}\frac{f(z_i)}{F(z_i)},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z_1 > \ldots > z_n
\end{aligned}
$$

(cf. [6]). A random variable X is said to have a *Gumbel distribution*, which we shall denote by $G(\mu,\sigma)$, if its cdf is

$$
(2)\quad F(x;\mu,\sigma) = \exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right), \qquad -\infty < x < \infty \qquad (-\infty < \mu < \infty,\ \sigma > 0).
$$

The Gumbel pdf may be written in the form

$$
(3)\quad f(x;\mu,\sigma) = \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right)F(x;\mu,\sigma), \qquad -\infty < x < \infty.
$$

In [1], [2] the maximum likelihood (ML), best linear invariant (BLI) and minimum variance unbiased (MVU) estimators of the Gumbel parameters μ, σ were obtained. In those papers there are also given two types of predictors of the sth record values based on the first m ($m < s$) record values. The Bayesian estimators of the Gumbel parameters μ and σ based on record values were furnished in [4]. Bayesian prediction of the sth lower record, both point or interval, was also presented.

In this note, the Bayesian estimators of the Gumbel parameters μ and σ are obtained via the kth lower record values. Point and interval Bayesian prediction of the sth one of the kth record values is also obtained. In fact, families of the Bayesian estimators and predictors are given.

2. Bayesian estimation of the parameters. Suppose we observe m kth lower record values $Z_1^{(k)} = x_1^{(k)}$, $Z_2^{(k)} = x_2^{(k)}, \ldots, Z_m^{(k)} = x_m^{(k)}$ from the Gumbel distribution $G(\mu,\sigma)$, with cdf and pdf given by (2) and (3), respectively. By (1) the likelihood function is as follows:

$$
(4)\quad
\begin{aligned}
L(\mu,\sigma\,|\,\underline{x}^{(k)}) &= k^m\left(\prod_{i=1}^{m-1}\frac{f(x_i^{(k)})}{F(x_i^{(k)})}\right)[F(x_m^{(k)})]^{k-1}f(x_m^{(k)})\\
&= \frac{k^m}{\sigma^m}\exp\left[-m\left(\frac{\overline{x}^{(k)}-\mu}{\sigma}\right) - k\exp\left(-\frac{x_m^{(k)}-\mu}{\sigma}\right)\right],\\
&\qquad\qquad\qquad\qquad x_1^{(k)} > x_2^{(k)} > \ldots > x_m^{(k)},
\end{aligned}
$$

where

$$\underline{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_m^{(k)}), \qquad \overline{x}^{(k)} = \sum_{i=1}^{m} x_i^{(k)}/m.$$

Assume that a bivariate prior distribution of the parameters μ and σ has the form

$$g(\mu, \sigma) = g_1(\mu \,|\, \sigma) g_2(\sigma), \tag{5}$$

where

$$g_1(\mu \,|\, \sigma) \propto 1/\sigma, \qquad -\infty < \mu < \infty, \tag{6}$$

which is the Jeffreys non-informative prior distribution (cf. [5]) of μ for a fixed value of σ, i.e. the distribution with pdf proportional to the square root of the Fisher information function ($I(\sigma) = 1/\sigma^2$), and

$$g_2(\sigma) = \frac{\beta^\alpha}{\Gamma(\alpha)\sigma^{\alpha+1}}\, e^{-\beta/\sigma}, \qquad \sigma > 0;\ \alpha > 0,\ \beta > 0, \tag{7}$$

which is the conjugate prior distribution of σ for a fixed value of μ. Substituting (6) and (7) in (5), we get

$$g(\mu, \sigma) \propto \frac{\beta^\alpha}{\Gamma(\alpha)\sigma^{\alpha+2}}\, e^{-\beta/\sigma}, \qquad -\infty < \mu < \infty;\ \sigma > 0. \tag{8}$$

By the Bayes theorem, the posterior distribution of μ and σ is

$$h(\mu, \sigma \,|\, \underline{x}^{(k)}) = AL(\mu, \sigma \,|\, \underline{x}^{(k)}) g(\mu, \sigma), \qquad -\infty < \mu < \infty,\ \sigma > 0, \tag{9}$$

where $L(\mu, \sigma \,|\, \underline{x}^{(k)})$ is the likelihood function given by (4), $g(\mu, \sigma)$ is the joint prior density given by (8) and A is the normalizing constant. If we apply (4) and (8) in (9), then the joint posterior density is

$$\begin{aligned} &h(\mu, \sigma \,|\, \underline{x}^{(k)}) \\ &\quad = \frac{Ak^m}{\sigma^{m+\alpha+2}} \exp\left[-\frac{m}{\sigma}\left((\overline{x}^{(k)} - \mu) + \frac{\beta}{m}\right) - k\exp\left(-\frac{x_m^{(k)} - \mu}{\sigma}\right)\right], \\ &\qquad\qquad -\infty < \mu < \infty,\ \sigma > 0, \end{aligned} \tag{10}$$

where

$$A = \frac{(\eta(\underline{x}^{(k)}))^{m+\alpha}}{\Gamma(m)\Gamma(m+\alpha)}$$

with

$$\eta(\underline{x}^{(k)}) = m(\overline{x}^{(k)} - x_m^{(k)}) + \beta. \tag{11}$$

Assuming a squared error loss function, the Bayes estimate of a parameter is its posterior mean. Therefore, the Bayes estimate of the parameter σ is given by

$$\widehat{\sigma}_B^{(k)} = \int\limits_0^\infty \sigma h_1(\sigma \,|\, \underline{x}^{(k)})\, d\sigma, \tag{12}$$

where $h_1(\sigma \,|\, \underline{x}^{(k)})$ is the marginal posterior density of σ obtained from (10) by integrating out the parameter μ. Thus

$$\begin{aligned}\widehat{\sigma}_B^{(k)} &= \int\limits_0^\infty \int\limits_{-\infty}^\infty \sigma h(\mu, \sigma \,|\, \underline{x}^{(k)})\, d\mu\, d\sigma \\ &= Ak^m \int\limits_0^\infty \int\limits_{-\infty}^\infty \frac{1}{\sigma^{\alpha+m+1}} \exp\biggl(-\biggl(\frac{m}{\sigma}(\overline{x}^{(k)} - \mu) + \frac{\beta}{\sigma}\biggr)\biggr) \\ &\qquad \times \exp\biggl(-k \exp\biggl(-\frac{x_m^{(k)} - \mu}{\sigma}\biggr)\biggr)\, d\mu\, d\sigma \\ &= \frac{\eta(\underline{x}^{(k)})}{m + \alpha - 1},\end{aligned} \tag{13}$$

where $\eta(\underline{x}^{(k)})$ is given by (11). Similarly, the Bayes estimate of μ is given by

$$\begin{aligned}\widehat{\mu}_B^{(k)} &= \int\limits_0^\infty \int\limits_{-\infty}^\infty \mu h(\mu, \sigma \,|\, \underline{x}^{(k)})\, d\mu\, d\sigma \\ &= Ak^m \int\limits_0^\infty \int\limits_{-\infty}^\infty \mu \frac{1}{\sigma^{\alpha+m+2}} \exp\biggl(-\biggl(\frac{m}{\sigma}(\overline{x}^{(k)} - \mu) + \frac{\beta}{\sigma}\biggr)\biggr) \\ &\qquad \times \exp\biggl(-k \exp\biggl(-\frac{x_m^{(k)} - \mu}{\sigma}\biggr)\biggr)\, d\mu\, d\sigma.\end{aligned}$$

Hence after standard evaluations we get

$$\widehat{\mu}_B^{(k)} = x_m^{(k)} - (\nu(m) + \ln k)\widehat{\sigma}_B^{(k)}, \tag{14}$$

where

$$\nu(m) = \nu(m-1) - \frac{1}{m-1}, \qquad m \ge 2,$$

and $\nu(1) = \gamma$, which is Euler's constant ($\gamma = 0.57722$). Note that $\widehat{\sigma}_B^{(1)}$ and $\widehat{\mu}_B^{(1)}$ are the estimators given in [4], i.e.

$$\widehat{\sigma}_B^{(1)} = \frac{\eta(\underline{x}^{(1)})}{m + \alpha - 1}, \qquad \widehat{\mu}_B^{(1)} = x_m^{(1)} - \nu(m)\widehat{\sigma}_B^{(1)}.$$

When $m = 1$,

$$\widehat{\sigma}_B^{(1)} = \frac{\eta(\underline{x}^{(1)})}{\alpha} = \frac{\beta}{\alpha}, \qquad \widehat{\mu}_B^{(1)} = X_1 - \gamma\,\frac{\beta}{\alpha}$$

are the estimators based on a sample of size 1. Our approach allows us to give the Bayesian estimators of μ and σ using a sample of size k. Namely, for $m = 1$ we have

$$\widehat{\sigma}_B^{(k)} = \frac{\beta}{\alpha},$$

$$\widehat{\mu}_B^{(k)} = x_1^{(k)} - (\gamma + \ln k)\widehat{\sigma}_B^{(k)} = \max\{X_1, \ldots, X_k\} - (\gamma + \ln k)\,\frac{\beta}{\alpha}.$$

Note that as α and β tend to zero, the estimators (13), (14) tend to the estimators

$$\widehat{\sigma}_B^{(k)} = \frac{m(\overline{x}^{(k)} - x_m^{(k)})}{m-1} = (m-1)^{-1}\sum_{i=1}^{m-1} x_i^{(k)} - x_m^{(k)},$$

$$\widehat{\mu}_B^{(k)} = x_m^{(k)} - (\nu(m) + \ln k)\,\frac{m(\overline{x}^{(k)} - x_m^{(k)})}{m-1},$$

respectively, which are for $k = 1$ the minimum variance unbiased estimators (MVUE) of the two parameters σ and μ, given in [2], [3], i.e.

$$\widehat{\sigma}_B^{(1)} = \frac{m(\overline{x}^{(1)} - x_m^{(1)})}{m-1},$$

$$\widehat{\mu}_B^{(1)} = x_m^{(1)} - \nu(m)\,\frac{m(\overline{x}^{(1)} - x_m^{(1)})}{m-1}.$$

3. Bayesian prediction of future records. Assume that we have m kth lower records $Z_1^{(k)} = x_1^{(k)}$, $Z_2^{(k)} = x_2^{(k)}, \ldots, Z_m^{(k)} = x_m^{(k)}$ from the Gumbel distribution $G(\mu, \sigma)$. Based on such a sample, prediction, either point or interval, is needed for the sth one of the kth lower record values, $1 < m < s$. Now let $Y^{(k)} = Z_s^{(k)}$ be the sth lower record value, $1 < m < s$. The conditional pdf of $Y^{(k)}$ given the parameters μ and σ and the observed value $x_m^{(k)}$ of $Z_m^{(k)}$ is

$$\begin{aligned}
(15)\quad & f^*(y_s^{(k)} \mid \mu, \sigma) \\
&= \frac{k^{s-m}}{\Gamma(s-m)}\,[H(y_s^{(k)}) - H(x_m^{(k)})]^{s-m-1}\left(\frac{F(y_s^{(k)})}{F(x_m^{(k)})}\right)^{k-1}\frac{f(y_s^{(k)})}{F(x_m^{(k)})} \\
&= \frac{k^{s-m}}{\sigma\Gamma(s-m)}\left[\exp\left(-\frac{y_s^{(k)} - \mu}{\sigma}\right) - \exp\left(-\frac{x_m^{(k)} - \mu}{\sigma}\right)\right]^{s-m-1} \\
&\quad \times \exp\left(-\frac{y_s^{(k)} - \mu}{\sigma}\right)\exp\left[-k\exp\left(-\frac{y_s^{(k)} - \mu}{\sigma}\right)\right] \\
&\quad \times \exp\left[k\exp\left(-\frac{x_m^{(k)} - \mu}{\sigma}\right)\right], \qquad -\infty < y_s^{(k)} < x_m^{(k)} < \infty,
\end{aligned}$$

where $f(\cdot), F(\cdot)$ are the pdf and cdf, respectively, and $H(\cdot) = -\ln F(\cdot)$. Combining the posterior density, given by (10), and the conditional density, given by (15), and integrating out the parameters μ and σ, one may get the Bayesian predictive density function of $Y^{(k)} = Z_s^{(k)}$ given the past m kth lower record values, in the form

$$\begin{aligned} q(y_s^{(k)} \mid \underline{x}^{(k)}) &= \int_0^\infty \int_{-\infty}^\infty f^*(y_s^{(k)} \mid \mu,\sigma) h(\mu,\sigma \mid \underline{x}^{(k)})\, d\mu\, d\sigma \\ &= \frac{Ak^s}{\Gamma(s-m)} \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sigma^{m+\alpha+3}} \\ &\quad \times \left[\exp\left(-\frac{y_s^{(k)}-\mu}{\sigma}\right) - \exp\left(-\frac{x_m^{(k)}-\mu}{\sigma}\right)\right]^{s-m-1} \\ &\quad \times \exp\left(-\frac{y_s^{(k)}-\mu}{\sigma}\right) \exp\left[-k\exp\left(-\frac{y_s^{(k)}-\mu}{\sigma}\right)\right] \\ &\quad \times \exp\left[k\exp\left(-\frac{x_m^{(k)}-\mu}{\sigma}\right)\right] \\ &\quad \times \exp\left[-\frac{m}{\sigma}\left((\overline{x}^{(k)}-\mu)+\frac{\beta}{m}\right) - k\exp\left(-\frac{x_m^{(k)}-\mu}{\sigma}\right)\right] d\mu\, d\sigma. \end{aligned}$$

We obtain

$$\begin{aligned} (16)\qquad q(y_s^{(k)} \mid \underline{x}^{(k)}) &= \frac{\alpha+m}{B(m,s-m)} \sum_{i=0}^{s-m-1} \binom{s-m-1}{i} (-1)^i \\ &\quad \times \frac{(\eta(\underline{x}^{(k)}))^{\alpha+m}}{[m(\overline{x}^{(k)}-y_s^{(k)}) + i(x_m^{(k)}-y_s^{(k)}) + \beta]^{\alpha+m+1}}, \end{aligned}$$

where $\eta(\underline{x}^{(k)})$ is given by (11), and $B(a,b)$ is the beta function, i.e.

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt, \qquad a>0,\ b>0.$$

The Bayes point predictor of the sth one of the kth lower record values is given by

$$\begin{aligned} E(y_s^{(k)} \mid \underline{x}^{(k)}) &= \int_{-\infty}^{x_m^{(k)}} y_s^{(k)} q(y_s^{(k)} \mid \underline{x}^{(k)})\, dy_s^{(k)} \\ &= \frac{\alpha+m}{B(m,s-m)} \int_{-\infty}^{x_m^{(k)}} y_s^{(k)} \Bigg\{ \sum_{i=0}^{s-m-1} \binom{s-m-1}{i} (-1)^i \\ &\quad \times \frac{(\eta(\underline{x}^{(k)}))^{\alpha+m}}{[m(\overline{x}^{(k)}-y_s^{(k)}) + i(x_m^{(k)}-y_s^{(k)}) + \beta]^{\alpha+m+1}} \Bigg\}\, dy_s^{(k)}. \end{aligned}$$

Thus

$$E(y_s^{(k)}\,|\,\underline{x}^{(k)}) = \frac{1}{B(m,s-m)}\sum_{i=0}^{s-m-1}\binom{s-m-1}{i}(-1)^i \times\left[\frac{x_m^{(k)}}{i+m}-\frac{\eta(\underline{x}^{(k)})}{(\alpha+m-1)(m+i)^2}\right]. \tag{17}$$

The Bayesian prediction bounds for $Y^{(k)} = Z_s^{(k)}$ are obtained by evaluating $\Pr(Y^{(k)} \ge \Theta\,|\,\underline{x}^{(k)})$ for some given value of Θ. It follows from (16) that

$$\begin{aligned}\Pr(Y^{(k)} \ge \Theta\,|\,\underline{x}^{(k)}) &= \int_{\Theta}^{x_m^{(k)}} q(y_s^{(k)}\,|\,\underline{x}^{(k)})dy_s^{(k)}\\ &= \frac{\alpha+m}{B(m,s-m)}\int_{\Theta}^{x_m^{(k)}}\sum_{i=0}^{s-m-1}\binom{s-m-1}{i}(-1)^i\\ &\quad\times\frac{(\eta(\underline{x}^{(k)}))^{\alpha+m}}{[m(\overline{x}^{(k)}-y_s^{(k)})+i(x_m^{(k)}-y_s^{(k)})+\beta]^{\alpha+m+1}}\,dy_s^{(k)}.\end{aligned}$$

Thus

$$\begin{aligned}\Pr(Y^{(k)} \ge \Theta\,|\,\underline{x}^{(k)}) &= \frac{1}{B(m,s-m)}\sum_{i=0}^{s-m-1}\binom{s-m-1}{i}(-1)^i\\ &\quad\times\frac{1}{m+i}\left[1-\left(\frac{\eta(\underline{x}^{(k)})}{\eta(\underline{x}^{(k)})+(m+i)(x_m^{(k)}-\Theta)}\right)^{\alpha+m}\right],\\ &\qquad -\infty<\theta<x_m^{(k)}.\end{aligned} \tag{18}$$

The $(1-\tau)100\%$ predictive interval for $Y^{(k)} = Z_s^{(k)}$ is obtained by evaluating both the lower, $L(\underline{x}^{(k)})$, and upper, $U(\underline{x}^{(k)})$, limits which satisfy

$$\Pr(Y^{(k)} > L(\underline{x}^{(k)})\,|\,\underline{x}^{(k)}) = 1-\frac{\tau}{2},\qquad \Pr(Y^{(k)} > U(\underline{x}^{(k)})\,|\,\underline{x}^{(k)}) = \frac{\tau}{2}. \tag{19}$$

Thus, one may obtain $L(\underline{x}^{(k)})$ and $U(\underline{x}^{(k)})$ by equating (18) to $1-\tau/2$ and $\tau/2$, respectively, and solving, numerically, the resulting equations. For the special case when $s = m+1$, which is of special practical interest, (16) simplifies to

$$q(y_{m+1}^{(k)}\,|\,\underline{x}^{(k)}) = m(m+\alpha)\frac{(\eta(\underline{x}^{(k)}))^{\alpha+m}}{[m(\overline{x}^{(k)}-y_{m+1}^{(k)})+\beta]^{\alpha+m+1}},\qquad -\infty<y_{m+1}^{(k)}<x_m^{(k)}<\infty. \tag{20}$$

This gives the Bayes point predictor of the next kth lower record value

$Y_{m+1}^{(k)} = Z_{m+1}^{(k)}$ in the form

$$E(y_{m+1}^{(k)} \,|\, \underline{x}^{(k)}) = x_m^{(k)} - \frac{\eta(\underline{x}^{(k)})}{m(\alpha+m-1)}, \tag{21}$$

and the $(1-\tau)100\%$ Bayesian predictive bounds $L(\underline{x}^{(k)})$ and $U(\underline{x}^{(k)})$ for $Y_{m+1}^{(k)}$ are given by

$$L(\underline{x}^{(k)}) = x_m^{(k)} + \frac{\eta(\underline{x}^{(k)})}{m}\left[1 - \left(\frac{\tau}{2}\right)^{-1/(m+\alpha)}\right],$$
$$U(\underline{x}^{(k)}) = x_m^{(k)} + \frac{\eta(\underline{x}^{(k)})}{m}\left[1 - \left(1-\frac{\tau}{2}\right)^{-1/(m+\alpha)}\right].$$

Note that for $m = 1$ and $k = 1$, $E(y_2^{(1)} \,|\, \underline{x}^{(1)}) = X_1 - \beta/\alpha$ is the Bayesian point predictor based on one observation. From (21) the point predictor based on a sample of size k is

$$E(y_2^{(k)} \,|\, \underline{x}^{(k)}) = \max\{X_1, \ldots, X_k\} - \beta/\alpha.$$

Note that as α and β tend to zero, the predictor (21) tends to the Bayesian point predictor

$$E(y_{m+1}^{(k)} \,|\, \underline{x}^{(k)}) = x_m^{(k)} - \frac{\overline{x}^{(k)} - x_m^{(k)}}{m-1},$$

which is for $k = 1$ the best linear unbiased predictor (BLUP), given in [2].

4. Characterization result. In this section we use a recurrence relation for conditional moments of nonadjacent kth record values to characterize the Gumbel distribution, $G(\mu,\sigma)$. By (1) we see that the conditional pdf of $Z_m^{(k)} = x^{(k)}$ for given $Z_s^{(k)} = t^{(k)}$, $1 < m < s$, is

$$f(x^{(k)} \,|\, t^{(k)}) = D_{m,s}(t^{(k)}) H^{m-1}(x^{(k)})[H(t^{(k)}) - H(x^{(k)})]^{s-m-1} r(x^{(k)}),$$
$$-\infty < t^{(k)} < x^{(k)} < \infty,$$

where

$$r(x^{(k)}) = -\frac{dH(x^{(k)})}{dx^{(k)}}, \qquad D_{m,s}(t^{(k)}) = \frac{\Gamma(s)}{\Gamma(m)\Gamma(s-m)H^{s-1}(t^{(k)})}.$$

Following the argument of Section 4 in [4] we have immediately a more general characterization of $G(\mu,\sigma)$.

THEOREM. *The random variable X has the $G(\mu,\sigma)$ distribution if and only if, for $t^{(k)} < x^{(k)}$ and $j = 1, 2, 3, \ldots$, the recurrence relation*

$$(m-1)E[\exp(-(j/\sigma)(Z_m^{(k)} - \mu)) \,|\, Z_s^{(k)} = t^{(k)}]$$
$$= (j+m-1)E[\exp(-(j/\sigma)(Z_{m-1}^{(k)} - \mu)) \,|\, Z_s^{(k)} = t^{(k)}]$$

is satisfied for some $k \geq 1$.

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