APPLICATIONES MATHEMATICAE
33,3-4 (2006), pp. 293–304

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SHARP UPPER BOUNDS FOR THE BEST PREDICTOR OF FUTURE MEAN OF SOME ORDER STATISTICS

Abstract. We provide sharp upper bounds for the mean of the future order statistics based on observed r order statistics. These bounds are expressed in terms of various scale units. We also determine the probability distributions for which the bounds are attained.

1. Introduction. Suppose that n components are put on test and that their lifetimes $X_1, \ldots, X_n$ are independent identically distributed (iid) random variables (r.v.'s) with a common distribution function (cdf) F, probability density function (pdf) f, quantile function F^{-1} defined by

$$F^{-1}(x) = \sup\{y : F(y) \le x\}, \quad 0 < x < 1,$$

and finite mean $\mu = \int_0^1 F^{-1}(x)\,dx$.

Of the n items put on test, suppose r failure times are observed and the remaining $n-r$ failure times are not observed. Let $\mathbf{X} = (X_{1:n} \le X_{2:n} \le \cdots \cdots \le X_{r:n})$, $1 \le r \le n-1$, be the first observed failure times from F. In a sample-prediction problem, prediction of order statistics or a function of order statistics is of interest. One might be interested in predicting the average strength of survivors having observed the first r failure times. The behavior of the future mean of the remaining failure times will help in setting up warranty for the items sent out to the market.

Generally, it is of interest to predict the mean of some future failure times which is defined by

$$T_{j,k,n} = \frac{1}{k-j+1} \sum_{s=j}^{k} X_{s:n}, \quad 1 \le r < j \le k \le n.$$

2000 *Mathematics Subject Classification*: 62G30, 62G32, 62E10, 60E15.

Key words and phrases: order statistics, quantile, sharp bound, Hölder inequality, pth central absolute moment, convex minorant approximation, predictor.

In the context of reliability theory, $X_{s:n}$ represents the life length of an $(n-s+1)$-out-of-n system made up of n identical components with independent life lengths. When $s = n$, it is better known as the parallel system. For more discussion on this subject, see Barlow and Proschan (1981).

Since F is continuous, the conditional distribution of $X_{s:n}$ given $\mathbf{X}$ is just the distribution of the $X_{s:n}$ given $X_{r:n}$, $r < s \le n$. This is the well known Markov property of the order statistics (see, for example, Arnold *et al.*, 1992, p. 24). The best unbiased predictor (BUP) of $T_{j,k,n}$, $E(T_{j,k,n} \mid \mathbf{X})$, is nothing but $E(T_{j,k,n} \mid X_{r:n})$.

The expectation of the ith order statistic $X_{i:n}$ $(1 \le i \le n)$ is given by

$$E(X_{i:n}) = \int_0^1 F^{-1}(x) f_{i:n}(x)\, dx,$$

where

$$f_{i:n}(x) = n\binom{n-1}{i-1} x^{i-1}(1-x)^{n-i}, \qquad 0 \le x \le 1,$$

is the pdf of the ith order statistic from the standard uniform iid sample of size n (cf., e.g., Arnold *et al.*, 1992). The respective cdf $F_{i:n}$ can be written as

$$F_{i:n}(x) = \sum_{k=i}^{n} \binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \le x \le 1. \tag{1.1}$$

In fact, the distribution of $X_{s:n}$ given $X_{r:n} = w$ is like the unconditional distribution of $X_{s-r:n-r}$ from the truncated distribution of Y given $Y > w$, that is,

$$G_{|w}(y) = \frac{F(y) - F(w)}{1 - F(w)}, \qquad -\infty < w < y < \infty.$$

Now we have

$$E_F(X_{s:n} \mid X_{r:n} = w) = E_{G_{|w}}(X_{s-r:n-r}) = \int_0^1 G_{|w}^{-1}(u) f_{s-r:n-r}(u)\, du.$$

Our purpose in this paper is to provide sharp upper bounds of the mean of $T_{j,k:n}$ given $X_{r:n} = F^{-1}(\xi)$ for $0 < \xi < 1$ for distributions having finite pth moments, $1 \le p \le \infty$. The pth moments of absolute deviations from the quantile are defined as follows:

$$\begin{aligned} \sigma_p^p(\xi) &= E_F|X_1 - F^{-1}(\xi)|^p, \qquad 1 \le p < \infty, \\ \sigma_\infty(\xi) &= \operatorname{ess\,sup} |X_1 - F^{-1}(\xi)| = \sup_{x \in (0,1)} |F^{-1}(x) - F^{-1}(\xi)| \\ &= \max\{F^{-1}(1-) - F^{-1}(\xi), F^{-1}(\xi) - F^{-1}(0)\}. \end{aligned} \tag{1.2}$$

The conditional expectation of $X_{s:n}$ given $X_{r:n} = F^{-1}(\xi)$ can be written as follows:

$$E_F(X_{s:n} \mid X_{r:n} = F^{-1}(\xi)) = \int_0^1 F^{-1}(\xi + x(1-\xi)) f_{s-r:n-r}(x)\, dx$$
$$= \frac{1}{1-\xi} \int_\xi^1 F^{-1}(v) f_{s-r:n-r}\left(\frac{v-\xi}{1-\xi}\right) dv.$$

Therefore

$$(1.3) \qquad E_F(T_{j,k:n} - F^{-1}(\xi) \mid X_{r:n} = F^{-1}(\xi)) = \int_0^1 [F^{-1}(x) - F^{-1}(\xi)] \varphi_{j,k:n}(x)\, dx,$$

where

$$(1.4) \qquad \varphi_{j,k:n}(x) = \frac{1}{k-j+1} \sum_{s=j}^{k} f_{s-r:n-r}\left(\frac{x-\xi}{1-\xi}\right) \frac{I_{[\xi,1)}(x)}{1-\xi},$$

with $I_A(x) = 1$ if $x \in A$, and 0 otherwise. Let us consider the function

$$f_{j,k:n}(x) = \frac{1}{k-j+1} \sum_{s=j}^{k} f_{s-r:n-r}(x), \qquad 0 \le x \le 1.$$

The anti-derivative of $f_{j,k:n}(x)$ is denoted by $F_{j,k:n}(x)$, which can be written as

$$F_{j,k:n}(x) = \frac{1}{k-j+1} \sum_{s=j}^{k} F_{s-r:n-r}(x), \qquad 0 \le x \le 1.$$

The anti-derivative of $\varphi_{j,k:n}(x)$ can be written as

$$(1.5) \qquad \Phi_{j,k:n}(x) = \frac{1}{k-j+1} \sum_{s=j}^{k} F_{s-r:n-r}\left(\frac{x-\xi}{1-\xi}\right) I_{[\xi,1]}(x).$$

Distribution-free bounds on order and record statistics can be found in Raqab (1997), Raqab and Rychlik (2002), Rychlik (2001), Danielak and Rychlik (2003). Recently Raqab (2005) established bounds for the mean of the total time on test using type II censored samples. Klimczak and Rychlik (2005) provided optimal bounds for the increments of order and record statistics under the condition that the values of future order statistics and records are known.

The aim of this paper is to present sharp moment bounds for the expectations of the future order statistics on the basis of observing r order statistics. These sharp bounds are obtained by combining the principle of Moriguti monotone approximations (Moriguti, 1953) with Hölder's inequality.

2. Auxiliary results. Let us first present some auxiliary results that are helpful in establishing sharp upper bounds on $E(T_{j,k:n} \mid X_{r:n} = F^{-1}(\xi))$.

LEMMA 2.1. *Let $\overline{g}$ be the right derivative of the greatest convex function $\overline{G}(x) = \int_a^x \overline{g}(u)\,du$, not greater than the integral $G(x) = \int_a^x g(u)\,du$ of g. For every nondecreasing function w on $[a,b]$ for which both integrals in* (2.1) *are finite, we have*

$$\int_a^b w(u)g(u)\,du \le \int_a^b w(u)\overline{g}(u)\,du. \tag{2.1}$$

Equality holds in (2.1) *iff w is constant on every open interval where $G > \overline{G}$.*

Lemma 2.1 follows from Moriguti (1953, Theorem 1). The function $\overline{g}(x)$ is called the projection of $g(x)$ onto the convex cone of nondecreasing functions in $L^2([a,b], dx)$ (cf. Rychlik, 2001, pp. 12–16).

The lemma below known as the variability diminishing property (VDP) of densities of order statistics was presented in Gajek and Rychlik (1998).

LEMMA 2.2. *The number of zeros of a linear combination of Bernstein polynomials $\sum_{k=0}^m a_k f_{k:m}$ in $(0,1)$ does not exceed the number of sign changes of the sequence $a_0, a_1, \ldots, a_m$. The first and the last sign of $\sum_{k=0}^m a_k f_{k:m}$ are identical with the signs of the first and the last nonzero elements of $a_0, a_1, \ldots, a_m$, respectively.*

LEMMA 2.3. *If $r+1 < j < k = n$ then $\varphi_{j,k:n}(\xi) = 0$,*

$$\lim_{x \nearrow 1-} \varphi_{j,k:n}(x) = \frac{n-r}{(n-j+1)(1-\xi)} > 0$$

and $\varphi_{j,k:n}$ is increasing. If $r+1 = j < k < n$, then $\varphi_{j,k:n}$ is decreasing with

$$\varphi_{j,k:n}(\xi) = \frac{n-r}{(k-r)(1-\xi)}$$

and $\lim_{x \searrow 1-} \varphi_{j,k:n}(x) = 0$. If $r+1 < j < k < n$ then $\varphi_{j,k:n}$ is first increasing, and then decreasing with $\varphi_{j,k:n}(\xi) = 0$, and $\lim_{x \searrow 1-} \varphi_{j,k:n}(x) = 0$, and it has a unique maximum at $x_0 = \xi + \theta(1-\xi)$, where

$$\theta = \frac{1}{1 + \tau^{1/(k-j+1)}} \quad \textit{with} \quad \tau = \frac{\binom{n-r-2}{k-r-1}}{\binom{n-r-2}{j-r-2}}.$$

Proof. The derivative of $f_{i:n}$ can be written as

$$f'_{i:n}(x) = n(n-1)[B_{i-2:n-2}(x) - B_{i-1:n-2}(x)],$$

where $B_{j:m}(x) = \binom{m}{j} x^j (1-x)^{m-j}$, $j = 0, 1, \ldots, m$, $m = 0, 1, \ldots$, are the Bernstein polynomials. We adopt the convention that $B_{l,m} = 0$ if $l > m$ or $l < 0$.

Straightforward algebra leads to the following representation:

$$\varphi'_{j,k:n}(x) = \frac{(n-r)(n-r+1)}{(k-j+1)(1-\xi)^2}\Big\{B_{j-r-2:n-r-2}\Big(\frac{x-\xi}{1-\xi}\Big) - B_{k-r-1:n-r-2}\Big(\frac{x-\xi}{1-\xi}\Big)\Big\}I_{[\xi,1)}(x).$$

If $k=n$, $\varphi_{j,k:n}(x)$ is positive and its derivative is also positive. That is, $\varphi_{j,k:n}$ is increasing from 0 at $x=\xi$ to $\varphi_{j,n:n}(1) = (n-r)/((n-j+1)(1-\xi)) > 0$. If $j=r+1$, then $\varphi'_{r+1,k:n}(x)$ is negative. So $\varphi_{r+1,k:n}(x)$ is decreasing from $\varphi_{r+1,k:n}(\xi) = (n-r)/((k-r)(1-\xi)) > 0$ to $\varphi_{r+1,k:n}(1) = 0$. If $r+1<j<k<n$, then by VDP, $\varphi'_{j,k:n}(x)$ is first positive and then negative ($+\,-$, for short). Therefore each $\varphi_{j,k:n}(x)$ is increasing-decreasing and it has a maximum at x_0.

It is clear that for $j=k$, $T_{j,k:n} = X_{k:n}$ and the problem is of finding optimal evaluation for the mean of a future order statistic (cf. Moriguti, 1953 and Rychlik, 2001). For $j=r+1$ and $k=n$, $T_{j,k:n} = \sum_{s=r+1}^{n} X_{s:n}/(n-r)$, which is the mean of all unobserved failure times. In this case,

$$E_F(T_{r+1,n:n}\,|\,X_{r:n} = F^{-1}(\xi)) = \frac{1}{n-r}\sum_{s=r+1}^{n} E_{G|F^{-1}(\xi)}(X_{s-r:n-r}),$$

where $G_{|w}$ is the truncated distribution of Y given $Y > F^{-1}(\xi)$. This implies that

$$E_F(T_{r+1,n:n}\,|\,X_{r:n} = F^{-1}(\xi)) = E_{G|F^{-1}(\xi)}(X_1) = m(F^{-1}(\xi)),$$

where $m(\varrho) = E_F(X\,|\,X>\varrho)$ is the expectation of X under the condition that it exceeds the level ϱ.

Lemma 2.4. *For given $r+1 \le j < k \le n$, the derivative of the greatest convex minorant of $\Phi_{j,k:n}(x)$ is*

$$\overline{\varphi}_{j,k:n}(x) = \begin{cases} (1-\xi)^{-1}I_{[\xi,1]}(x) & \text{if } r+1=j<k<n,\\ \varphi_{j,k:n}(x) & \text{if } r+1<j<k=n,\\ \varphi_{k,j:n}(\min\{\xi(1-\alpha^*)+\alpha^*, x\}) & \text{if } r+1<j<k<n, \end{cases} \tag{2.2}$$

where α^* is the solution to

$$1-F_{j,k:n}(\alpha^*) = (1-\alpha^*)f_{j,k:n}(\alpha^*). \tag{2.3}$$

Proof. The function $F^{-1}(x) - F^{-1}(\xi)$, $0<x<1$, is an element of the convex cone of nondecreasing functions and changes sign at $x=\xi$. We need to determine the projection $\overline{\varphi}_{j,k:n}$ of $\varphi_{j,k:n}(x)$ onto the family of nondecreasing functions in $L^2([0,1],dx)$. It is enough to show that $\overline{\varphi}_{j,k:n}(x)$ is the derivative of the greatest convex minorant $\overline{\Phi}_{j,k:n}(x)$ of the anti-derivative $\Phi_{j,k:n}$.

For $r+1 < j < k < n$, $\Phi_{j,k:n}(\xi) = 0$, $\Phi_{j,k:n}(1) = 1$, and $\Phi_{j,k:n}(x)$ is increasing convex on $[\xi, \xi + \theta(1-\xi)]$, increasing concave on $[\xi + \theta(1-\xi), 1]$ and constant on $[0, \xi]$. Thus its greatest convex minorant is linear in $[\alpha, 1]$ for some $\alpha \in [\xi, \xi + (1-\xi)\theta]$. That is,

$$\bar{\Phi}_{j,k:n}(x) = \begin{cases} 0 & \text{if } 0 \le x < \xi, \\ \Phi_{j,k:n}(x) & \text{if } \xi \le x < \alpha, \\ \varphi_{j,k:n}(\alpha)(x-1) + 1 & \text{if } \alpha \le x \le 1, \end{cases} \tag{2.4}$$

where α is the solution to

$$1 - \Phi_{j,k:n}(\alpha) = \varphi_{j,k:n}(\alpha)(1-\alpha). \tag{2.5}$$

Setting $\alpha^* = (\alpha - \xi)/(1-\xi)$ turns (2.5) into (2.3).

For $r+1 = j < k < n$, the anti-derivative $\Phi_{j,k:n}(x)$ is concave increasing with $\Phi_{j,k:n}(\xi) = 0$, $\Phi_{j,k:n}(1) = 1$. Therefore the greatest convex minorant is linear in $[\xi, 1]$ with slope $(1-\xi)^{-1}$. If $r+1 < j < k = n$, $\Phi_{j,n:n}(x)$ is convex increasing with $\Phi_{j,n:n}(\xi) = 0$, $\Phi_{j,n:n}(1) = 1$ and then $\bar{\Phi}_{j,n:n}(x) = \Phi_{j,n:n}(x)$. Summing up, the derivative of $\bar{\Phi}_{j,k:n}(x)$, $r+1 \le j < k \le n$, is described in (2.2).

3. The main results. Here we use the preceding auxiliary results to evaluate optimal sharp upper bounds for $E_F(T_{j,k:n} \mid X_{r:n} = F^{-1}(\xi))$, $r+1 \le j < k \le n$, in terms of different scale units generated by various central absolute moments about the quantile function. Applying Moriguti's inequality and Hölder's inequality to (1.3), we obtain

$$\begin{aligned} &E_F(T_{j,k:n} - F^{-1}(\xi) \mid X_{r:n} = F^{-1}(\xi)) \\ &\qquad \le \int_0^1 [F^{-1}(x) - F^{-1}(\xi)]\overline{\varphi}_{j,k:n}(x)\,dx \qquad (3.1) \\ &\qquad \le \|\overline{\varphi}_{j,k:n}\|_q \sigma_p(\xi), \qquad (3.2) \end{aligned}$$

where

$$\|g\|_p = \Big(\int_0^1 |g(x)|^p\,dx\Big)^{1/p},$$

defines the pth norm of $g \in L^p([0,1], dx)$ and $\|g\|_q$ is defined analogously for the conjugate exponent $q = p/(p-1)$. First we consider the case $1 < p < \infty$.

THEOREM 3.1. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be order statistics from n iid random variables with continuous cdf F, $\sigma_p^p(\xi) < \infty$, and $1 < p < \infty$. Let α^* be defined by* (2.3). *Then*

$$\frac{E_F(T_{j,k:n} - F^{-1}(\xi) \mid X_{r:n} = F^{-1}(\xi))}{\sigma_p(\xi)} \le B_q(j,k,n),$$

where

$$(3.3)\qquad B_q(j,k,n)=\begin{cases}(1-\xi)^{-1/p} & \text{if } r+1=j<k<n,\\ (1-\xi)^{-1/p}\|f_{j,n:n}\|_q & \text{if } r+1<j<k=n,\\ (1-\xi)^{-1/(p-1)}[D(\alpha^*)]^{1/q} & \text{if } r+1<j<k<n,\end{cases}$$

with

$$D(z)=\Big(\int_0^z f^q_{j,k:n}(x)\,dx+(1-z)f^q_{j,k:n}(z)\Big).$$

For $r+1=j<k<n$, the bound in (3.3) is attained in the limit by the two-point distribution

$$(3.4)\qquad \begin{aligned}&P(X=F^{-1}(\xi))=\xi,\\ &P(X=F^{-1}(\xi)+\sigma_p(\xi)(1-\xi)^{-1/p})=1-\xi,\end{aligned}$$

and for $r+1<j<k<n$, the bound is attained in the limit by continuous cdf's converging to F of the following form:

$$(3.5)\qquad F(x)=\begin{cases}0 & \text{if } (x-F^{-1}(\xi))/\sigma_p(\xi)<0,\\ \xi+(1-\xi)f^{-1}_{j,k:n}\Big((1-\xi)B_q(j,k,n)\Big[\dfrac{x-F^{-1}(\xi)}{\sigma_p(\xi)}\Big]^{p/q}\Big) & \\ & \text{if } 0\le (x-F^{-1}(\xi))/\sigma_p(\xi)<\nu,\\ 1 & \text{if } (x-F^{-1}(\xi))/\sigma_p(\xi)\ge\nu,\end{cases}$$

where

$$\nu=\Big[\frac{f_{j,k:n}(\alpha^*)}{(1-\xi)B_q(j,k,n)}\Big]^{q/p},$$

and $\alpha^*=1$ for $k=n$.

Proof. For $r+1<j<k<n$, the norm $\|\overline{\varphi}_{j,k:n}\|_q^q$ can be written as

$$\|\overline{\varphi}_{j,k:n}\|_q^q=\int_\xi^\alpha[\varphi_{j,k:n}(x)]^q\,dx+(1-\alpha)\varphi^q_{j,k:n}(\alpha).$$

Now

$$(3.6)\qquad \int_\xi^\alpha[\overline{\varphi}_{j,k:n}(x)]^q\,dx$$
$$=\int_\xi^{\xi(1-\alpha^*)+\alpha^*}\varphi^q_{j,k:n}(x)\,dx+(1-\alpha^*)(1-\xi)[\varphi_{j,k:n}(\alpha^*+\xi(1-\alpha^*))]^q$$

$$= (1-\xi)^{-1/(p-1)} \int_0^{\alpha^*} [f_{j,k:n}(x)]^q \, dx + (1-\alpha^*)(1-\xi)^{-1/(p-1)} f^q_{j,k:n}(\alpha^*)$$

$$= (1-\xi)^{-1/(p-1)} \Big[\int_0^{\alpha^*} [f_{j,k:n}(x)]^q \, dx + (1-\alpha^*) f^q_{j,k:n}(\alpha^*) \Big].$$

If $r+1 = j < k < n$, the bound becomes

$$B_q(r+1,k,n) = \|\overline{\varphi}_{r+1,k:n}\|_q = \Big(\int_\xi^1 \frac{1}{(1-\xi)^q} \, dx \Big)^{1/q} = (1-\xi)^{-1/p}. \tag{3.7}$$

If $r+1 < j < k = n$, the bound can be written as

$$\begin{aligned} B_q(j,k,n) = \|\overline{\varphi}_{j,n:n}\|_q &= (1-\xi)^{-1/p} \Big(\int_0^1 [f_{j,n:n}(x)]^q \, dx \Big)^{1/q} \\ &= (1-\xi)^{-1/p} \|f_{j,n:n}\|_q. \end{aligned} \tag{3.8}$$

From (3.6), (3.7) and (3.8), we immediately obtain the bound in (3.3).

The bound is attained if $F^{-1}(x) = \text{const}$ on $(\xi, 1]$ for $j = r+1$. It follows that the bound is attained by the two-point distribution as defined in (3.4). For $r+1 < j < k < n$, the equality in (3.1) holds if $F^{-1}(x) = \text{const}$ on $[\xi(1-\alpha^*)+\alpha^*, 1]$ (see Lemma 2.1). Equality (3.2) holds if

$$F^{-1}(x) - F^{-1}(\xi) = \lambda(\overline{\varphi}_{j,k:n}(x))^{q/p}, \quad 0 < x < 1,$$

for some $\lambda \geq 0$. Condition $E_F|X_1 - F^{-1}(\xi)|^p = \sigma_p^p(\xi)$ forces

$$\lambda = \frac{\sigma_p(\xi)}{B_q^{q/p}(j,k,n)},$$

and then

$$\frac{F^{-1}(x) - F^{-1}(\xi)}{\sigma_p(\xi)} = \Big[\frac{\overline{\varphi}_{j,k:n}(x)}{B_q(j,k,n)} \Big]^{1/(p-1)}. \tag{3.9}$$

The quantile function in (3.9) is a nondecreasing function, constant on the interval $\{x : \overline{\varphi}_{j,k:n}(x) \neq \varphi_{j,k:n}(x)\}$, which is necessary and sufficient for Moriguti's equality. Substituting (2.3) into (3.9) and simplifying the resulting expression, we establish the distribution (3.5) for which the bound is attained. For $j = n$, $\overline{\varphi}_{j,n:n} = \varphi_{j,n:n}$, the distribution function for which the bound is attained is the same as the one in (3.5) except on the extended support interval $[\xi, 1]$. That is, the cdf $F(x)$ is obtained by setting $\alpha^* = 1$ in (3.5).

The distribution function in (3.5) has two atoms of measures ξ and $\xi + (1-\xi)\alpha^*$ at the left and right ends of the support intervals, respectively.

Using the fact that

$$f_{i:m}(x)f_{j:n}(x) = \frac{n\binom{i+j-2}{i-1}\binom{m+n-i-j}{m-i}}{\binom{m+n-1}{m}}\, f_{i+j-1:m+n-1}(x),$$

we can show the following corollary.

COROLLARY 3.1. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be order statistics from n iid random variables with continuous cdf F, $EX_1 = \mu$ and $\sigma_2^2 = \mathrm{Var}(X_1) < \infty$. Then for $r+1 < j < k < n$,*

$$\frac{E_F(T_{j,k:n} - F^{-1}(\xi)\,|\,X_{r:n} = F^{-1}(\xi))}{\sigma_2(\xi)} \le B_2(j,k,n), \tag{3.10}$$

where

$$B_2^2(j,k,n) = \frac{n-r}{(k-j+1)^2\binom{2n-2r-1}{n-r}}\,[(1-\alpha^*)f_{j,k:n}^2(\alpha^*) + S_{j,k,n}(\alpha^*)], \tag{3.11}$$

with

$$\begin{aligned} S_{j,k,n}(x) = &\sum_{i=j}^{k}\binom{2s-2r-2}{s-r-1}\binom{2n-2s}{n-s}F_{2s-2r-1:2n-2r-1}(x)\\ &+2\sum_{i=j+1}^{k}\sum_{l=j}^{i-1}\binom{i+l-2r-2}{i-r-1}\binom{2n-i-l}{n-i}F_{i+l-2r-1:2n-2r-1}(x). \end{aligned}$$

The distribution function $F(x)$ attaining the bound is of the form

$$F(x) = \begin{cases} 0 & \text{if } (x - F^{-1}(\xi))/\sigma_2(\xi) < 0,\\ \xi + (1-\xi)f_{j,k:n}^{-1}\Big((1-\xi)B_2(j,k,n)\Big(\dfrac{x-F^{-1}(\xi)}{\sigma_2(\xi)}\Big)\Big) & \\ & \text{if } 0 \le (x - F^{-1}(\xi))/\sigma_2(\xi) < \nu,\\ 1 & \text{if } x \ge \nu. \end{cases} \tag{3.12}$$

Now we study the extreme cases $p = 1$ and $p = \infty$.

THEOREM 3.2. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be order statistics from n iid r.v.'s, with continuous cdf F, and $\sigma_1(\xi) < \infty$. With α^* defined in (2.4), we have*

$$\frac{E_F(T_{j,k:n} - F^{-1}(\xi)\,|\,X_{r:n} = F^{-1}(\xi))}{\sigma_1(\xi)} \le B_\infty(j,k,n),$$

where

$$B_\infty(j,k,n) = \begin{cases} (1-\xi)^{-1} & \textit{if } r+1 = j < k < n,\\ \dfrac{n-r}{(n-j+1)(1-\xi)} & \textit{if } r+1 < j < k = n,\\ (1-\xi)^{-1}f_{j,k:n}(\alpha^*) & \textit{if } r+1 < j < k < n. \end{cases} \tag{3.13}$$

For $r+1=j<k<n$, the bound is attained in the limit by continuous distribution functions converging to the two-point distribution

$$P(X=F^{-1}(\xi))=\xi \\ =1-P\Big(X=F^{-1}(\xi)+\frac{\sigma_1(\xi)}{1-\xi}\Big). \tag{3.14}$$

If $r+1<j<k=n$, the bound is attained in the limit by the sequence of two-point distributions

$$P(X=F^{-1}(\xi))=\epsilon_r=1-P\Big(X=F^{-1}(\xi)+\frac{\sigma_1(\xi)}{1-\epsilon_r}\Big), \tag{3.15}$$

where the sequence $\epsilon_r\in(\xi,1)$, $r=1,2,\ldots$, converges to 1. If $r+1<j<k<n$ the bound is attained by

$$\begin{aligned} &P(X=F^{-1}(\xi))=\xi+\alpha^*(1-\xi),\\ &P\Big(X=F^{-1}(\xi)+\frac{\sigma_1(\xi)}{(1-\xi)(1-\alpha^*)}\Big)=(1-\xi)(1-\alpha^*). \end{aligned} \tag{3.16}$$

Proof. Here we have

$$\begin{aligned} &E_F(T_{j,k:n}-F^{-1}(\xi)\,|\,X_{r:n}=F^{-1}(\xi))\\ &\qquad\le\int_0^1[F^{-1}(x)-F^{-1}(\xi)]\overline{\varphi}_{j,k:n}(x)\,dx\\ &\qquad\le\sup_{0\le x\le 1}\overline{\varphi}_{j,k:n}(x)\sigma_1(\xi)=\overline{\varphi}_{j,k:n}(1)\sigma_1(\xi). \end{aligned} \tag{3.17}$$

From (2.2) and (3.17), we immediately obtain (3.13). The second equality holds if

$$F^{-1}(x)=F^{-1}(\xi)$$

a.e. except for a subset of $\{x:\overline{\varphi}_{j,k:n}(x)=\overline{\varphi}_{j,k:n}(1)\}$. Therefore $F^{-1}(x)=F^{-1}(\xi)$ a.e. except on the sets $(\xi,1]$, $\{0\}$ and $(\xi+\alpha^*(1-\xi),1]$, for $j=r+1<k<n, r+1<j<k=n$ and $r+1<j<k<n$, respectively.

In the first and third cases, the conditions for equality in the first inequality of (3.17) impose that

$$F^{-1}(x)=\text{const}\ge F^{-1}(\xi)$$

on

$$x\in(\xi,1] \quad\text{and}\quad (\xi+\alpha^*(1-\xi),1],$$

respectively. This implies that the bounds are attained by two-point distributions with respective probabilities $1-\xi$ and $(1-\xi)(1-\alpha^*)$. The probability distributions are as in (3.15) and (3.16).

If $r+1<j<k=n$, the inequality becomes an equality if $F^{-1}(x)=F^{-1}(\xi)$, $0<x<1$. That is, X is a degenerate r.v. with $\sigma_1(\xi)=0$. However, for $k=n$, the bound is attained in the limit by the sequence of two-point distributions in (3.15).

THEOREM 3.3. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be order statistics from n iid r.v.'s, with continuous cdf F. If X_1 is bounded almost surely, then for $r+1\le j<k\le n$, we have*

$$\frac{E_F(T_{j,k:n}-F^{-1}(\xi)\,|\,X_{r:n}=F^{-1}(\xi))}{\sigma_\infty(\xi)} \le B_1(j,k,n)=1.$$

The bound is attained in the limit by continuous distribution functions converging weakly to the following two-point distribution:

$$\begin{aligned} &P(F^{-1}(\xi)-\sigma_\infty \le X \le F^{-1}(\xi))=\xi,\\ &P(X=F^{-1}(\xi)+\sigma_\infty(\xi))=1-\xi. \end{aligned} \tag{3.18}$$

Proof. Arguments similar to those in Theorem 3.2 yield

$$\begin{aligned} E_F(T_{j,k:n}-F^{-1}(\xi)\,|\,X_{r:n}=F^{-1}(\xi)) &\le \int_0^1 [F^{-1}(x)-F^{-1}(\xi)]\,\overline{\varphi}_{j,k:n}(x)\,dx\\ &\le \sup_{0\le x\le 1}|F^{-1}(x)-F^{-1}(\xi)|\int_0^1 \overline{\varphi}_{j,k:n}(x)\,dx=\bar{\Phi}_{j,k:n}(1)\sigma_\infty(\xi)=\sigma_\infty(\xi). \end{aligned}$$

The above equalities hold if

$$F^{-1}(x)-F^{-1}(\xi)=\sigma_\infty(\xi) \quad \text{on } \{x:\overline{\varphi}_{j,k:n}(x)\ne 0\},$$

and

$$|F^{-1}(x)-F^{-1}(\xi)|\le\sigma_\infty(\xi) \quad \text{on } \{x:\overline{\varphi}_{j,k:n}(x)=0\}.$$

almost surely. Here

$$\{x:\overline{\varphi}_{j,k:n}(x)\ne 0\}=[\xi,1), \qquad \{x:\overline{\varphi}_{j,k:n}(x)=0\}=[0,\xi).$$

This implies that

$$\begin{aligned} F^{-1}(\xi)-\sigma_\infty \le F^{-1}(x) \le F^{-1}(\xi), &\quad 0\le x<\xi,\\ F^{-1}(x)=F^{-1}(\xi)+\sigma_\infty, &\quad \xi\le x<1. \end{aligned}$$

As a consequence, the bound is attained in the limit by the distribution described in (3.18).

Acknowledgements. The author would like to thank the University of Jordan for supporting this research work. The author is also grateful to the referee for his comments and useful suggestions.

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Received on 15.11.2005;
revised version on 16.11.2006 (1796)