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EXISTENCE RESULTS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS WITH TWO LOWER ORDER TERMS

Abstract. We investigate the existence of renormalized solutions for some nonlinear parabolic problems associated to equations of the form

$$\begin{cases} \frac{\partial(e^{\beta u}-1)}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \operatorname{div}(c(x,t)|u|^{s-1}u) + b(x,t)|\nabla u|^r = f \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } Q = \Omega \times (0,T), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ (e^{\beta u}-1)(x,0) = (e^{\beta u_0}-1)(x) \quad \text{in } \Omega. \end{cases}$$

with $s = \frac{N+2}{N+p}(p-1)$, $c(x,t) \in (L^\tau(Q_T))^N$, $\tau = \frac{N+p}{p-1}$, $r = \frac{N(p-1)+p}{N+2}$, $b(x,t) \in L^{N+2,1}(Q_T)$ and $f \in L^1(Q)$.

1. Introduction. Let Ω be a bounded subset of $\mathbb{R}^N$, $N \geq 1$, and let $T > 0$ be a real constant. Let us define the cylinder $Q = \Omega \times (0,T)$ and its lateral surface $\Gamma = \partial\Omega \times (0,T)$. Our main purpose in this paper is to study the following problem:

$$\begin{cases} \frac{\partial b(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + \operatorname{div}(\phi(x,t,u)) + H(x,t,\nabla u) = f \quad \text{in } Q_T, \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ b(u(x,0)) = b(u_0(x)) \quad \text{in } \Omega. \end{cases} \tag{1.1}$$

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Here b is a strictly increasing C^1-function, the data f and $b(u_0)$ are in $L^1(Q)$ and $L^1(\Omega)$ respectively, $-\mathrm{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator defined on $W_0^{1,p}(\Omega)$ (see assumptions (2.2)–(2.4) of Section 2), and $\phi(x,t,u)$ and $H(x,t,\nabla u)$ are Carathéodory functions assumed to be continuous on u (see assumptions (2.5)–(2.9)).

Under our assumptions, problem (1.1) does not admit, in general, a solution in the sense of distributions since we cannot expect to have the field $\phi(x,t,u)$ in $(L^1_{\mathrm{loc}}(Q_T))^N$ and $H(x,t,\nabla u)$ in $L^1_{\mathrm{loc}}(Q_T)$. For this reason we consider the framework of renormalized solutions (see Definition 3.1).

The notion of renormalized solution was introduced in [9], and has been developed for elliptic problems with L^1 data in [6], [12].

The existence of renormalized solution for (1.1) has been proved by R. Di Nardo [7] for $b(u)=u$ using the symmetrization method, by Y. Akdim et al. [2] in the case where $a(x,t,s,\xi)$ is independent of s and $\phi=0$, by D. Blanchard et al. [4] for $a(x,t,s,\xi)$ only assumed to be non-strictly monotone, and ϕ depending only on s, and by A. Aberqi et al. [1] in the case where $H=0$.

It is our purpose to generalize the result of [2], [7], [1] and prove the existence of a renormalized solution of (1.1).

2. Technical lemma and assumptions on data

2.1. Technical lemma. Throughout, T_k denotes the truncation function at height $k\ge 0$:

$$T_k(r)=\max(-k,\min(k,r)).$$

LEMMA 2.1 (see [7]). *Assume that Ω is an open subset of $\mathbb{R}^N$ of finite measure and $1<p<\infty$. Let u be a measurable function satisfying $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ for every k and such that*

$$\sup_{t\in(0,T)}\int_\Omega |\nabla T_k(u)|^2+\int_{Q_T}|\nabla T_k(u)|^p\le Mk+C,\quad \forall k>0,$$

where M and C are positive constants. Then

$$|u|^{\frac{N(p-1)+p}{N+p}}\in L^{\frac{N+p}{N},\infty}(Q_T)\quad\text{and}\quad |\nabla u|^{\frac{N(p-1)+p}{N+2}}\in L^{\frac{N+2}{N+1},\infty}(Q_T).$$

2.2. Assumptions. Throughout this paper, we assume that the following assumptions hold true:

ASSUMPTIONS (H)

$$b:\mathbb{R}\to\mathbb{R}\text{ is strictly increasing, } C^1,\ b'>\lambda>0,\ b(0)=0, \tag{2.1}$$

$$|a(x,t,s,\xi)|\le \nu[h(x,t)+|\xi|^{p-1}]\quad\text{with }\nu>0\text{ and }h(\cdot,\cdot)\in L^{p'}(Q_T), \tag{2.2}$$

$$a(x,t,s,\xi)\xi\ge\alpha|\xi|^p\quad\text{with }\alpha>0, \tag{2.3}$$

$$(a(x,t,s,\xi)-a(x,t,s,\eta))(\xi-\eta)>0\quad\text{if }\xi\ne\eta, \tag{2.4}$$

$$|\phi(x,t,s)| \le c(x,t)|s|^{\gamma}, \tag{2.5}$$

$$c(\cdot,\cdot) \in (L^{\tau}(Q_T))^N, \quad \tau = \frac{N+p}{p-1}, \tag{2.6}$$

$$\gamma = \frac{N+2}{N+p}(p-1), \tag{2.7}$$

$$|H(x,t,\xi)| \le m(x,t)|\xi|^{\beta}, \tag{2.8}$$

$$m(\cdot,\cdot) \in L^{N+2,1}(Q_T), \quad \beta = \frac{N(p-1)+p}{N+2}, \tag{2.9}$$

for almost every $(x,t) \in Q_T$, for every $s \in \mathbb{R}$ and every $\xi, \eta \in \mathbb{R}^N$. Moreover

$$f \in L^1(Q_T), \tag{2.10}$$

$$u_0 \in L^1(\Omega), \qquad b(u_0) \in L^1(\Omega). \tag{2.11}$$

3. Existence results for noncoercive operators

Definition 3.1. A measurable function u is a *renormalized solution* to problem (1.1) if

$$b(u) \in L^{\infty}(0,T;L^1(\Omega)), \tag{3.1}$$

$$T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)) \quad \text{for any } k > 0, \tag{3.2}$$

$$\lim_{n\to\infty} \int_{\{n \le |u| \le n+1\}} a(x,t,u,\nabla u)\nabla u \, dx \, dt = 0, \tag{3.3}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support,

$$\begin{aligned} \frac{\partial B_S(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)S'(u)) + S''(u)a(x,t,u,\nabla u)\nabla u \\ + \operatorname{div}(\phi(x,t,u)S'(u)) - S''(u)\phi(x,t,u)\nabla u \\ + H(x,t,\nabla u)S'(u) = fS'(u) \quad \text{in } \mathcal{D}'(Q_T), \end{aligned} \tag{3.4}$$

and

$$B_S(u)(t=0) = B_S(u_0) \quad \text{in } \Omega, \tag{3.5}$$

where $B_S(z) = \int_0^z b'(s)S'(s)\,ds$.

Remark 3.2. We notice that equation (3.4) can be formally obtained through pointwise multiplication of (1.1) by $S'(u)$ and all terms have a meaning in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. Moreover $\partial B_S(u)/\partial t$ belongs to $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $B_S(u) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^{\infty}(Q)$. It follows that $B_S(u)$ belongs to $C^0([0,T];L^1(\Omega))$ so the initial condition (3.5) makes sense.

3.1. Existence results

Main Theorem 3.3. *Under Assumptions* (H) *there exists a renormalized solution to problem* (1.1).

Proof. Step 1. *Approximate problem.* For each $\epsilon > 0$, we consider the approximate problem

$$\begin{cases} \frac{\partial b_\epsilon(u_\epsilon)}{\partial t} - \operatorname{div}(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)) \\ \qquad\qquad + \operatorname{div}(\phi_\epsilon(x,t,u_\epsilon)) + H_\epsilon(x,t,\nabla u_\epsilon) = f_\epsilon \quad \text{in } Q_T, \\ u_\epsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ b_\epsilon(u_\epsilon(x,0)) = b_\epsilon(u_{0\epsilon}(x)) \quad \text{in } \Omega. \end{cases} \tag{3.6}$$

where

$$b_\epsilon(r) = T_{1/\epsilon}(b(r)) + \epsilon r \quad \forall r \in \mathbb{R}, \tag{3.7}$$

$$a_\epsilon(x,t,s,\xi) = a(x,t,T_{1/\epsilon}(s),\xi) \quad \text{a.e. in } Q,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{3.8}$$

$$\phi_\epsilon(x,t,r) = \phi(x,t,T_{1/\epsilon}(r)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall r \in \mathbb{R}, \tag{3.9}$$

$$H_\epsilon(x,t,\xi) = T_{1/\epsilon}((x,t,\xi)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall \xi \in \mathbb{R}^N, \tag{3.10}$$

$$f_\epsilon \in L^{p'}(Q_T), \quad f_\epsilon \to f \text{ strongly in } L^1(Q_T), \tag{3.11}$$

$$u_{0\epsilon} \in \mathcal{D}(\Omega), \quad b_\epsilon(u_{0\epsilon}) \to b(u_0) \text{ strongly in } L^1(\Omega). \tag{3.12}$$

Then proving existence of a weak solution $u_\epsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ is an easy task (see [11]).

Step 2. *A priori estimates for solutions and their gradients.* Let $\tau_1 \in (0,T)$ and fix t in $(0,\tau_1)$. Using $T_k(u_\epsilon)\chi_{(0,t)}$ as a test function in (3.6), we integrate between $(0,\tau_1)$, and by the condition (2.5) we have

$$\begin{aligned} \int_\Omega B_k^\epsilon(u_\epsilon(t))\,dx &+ \int_{Q_t} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)\,dx\,ds \\ &\le \int_{Q_t} c(x,t)|u_\epsilon|^\gamma |\nabla T_k(u_\epsilon)|\,dx\,ds + k\int_{Q_t} m(x,t)|\nabla u_\epsilon|^\beta\,dx\,ds \\ &\qquad + \int_{Q_t} f_\epsilon T_k(u_\epsilon)\,dx\,ds + \int_\Omega B_k^\epsilon(u_{0\epsilon})\,dx, \end{aligned} \tag{3.13}$$

where $B_k^\epsilon(r) = \int_0^r T_k(s)b'_\epsilon(s)\,ds$. Due to the definition of B_k^ϵ we have

$$0 \le \int_\Omega B_k^\epsilon(u_{0\epsilon})\,dx \le k\int_\Omega |b_\epsilon(u_{0\epsilon})|\,dx \le k\|b(u_0)\|_{L^1(\Omega)} \quad \forall k > 0. \tag{3.14}$$

Using (3.13) and (2.3) we obtain

$$\begin{aligned}(3.15)\qquad &\int_\Omega B_k^\epsilon(u_\epsilon(t))\,dx + \alpha\int_{Q_t}|\nabla T_k(u_\epsilon)|^p\,dx\,ds\\ &\le \int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,ds\,dx + k\int_{Q_t} m(x,t)|\nabla u_\epsilon|^\beta\,dx\,ds\\ &\quad + k(\|b(u_0)\|_{L^1(\Omega)} + \|f_\epsilon\|_{L^1(Q)}).\end{aligned}$$

We deduce from (3.13)–(3.15) that

$$\begin{aligned}(3.16)\qquad &\frac{\lambda}{2}\int_\Omega |T_k(u_\epsilon)|^2\,dx + \alpha\int_{Q_t}|\nabla T_k(u_\epsilon)|^p\,dx\,ds\\ &\le M_1k + k\int_{Q_t} m(x,t)|\nabla u_\epsilon|^\beta\,dx\,ds + \int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,dx\,ds\end{aligned}$$

for $t\in(0,\tau_1)$, where $M_1 = \sup\|f_\epsilon\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)}$.

► *Estimate of* $\int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,dx\,ds$. By the Gagliardo–Nirenberg and Young inequalities we have

$$\begin{aligned}(3.17)\qquad &\int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,dx\,ds\\ &\le C\frac{\gamma}{N+2}\|c(\cdot,\cdot)\|_{L^\tau(Q_{\tau_1})}\sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx\\ &\quad + C\frac{N+2-\gamma}{N+2}\|c(\cdot,\cdot)\|_{L^\tau(Q_{\tau_1})}\Big(\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,ds\Big)^{(\frac{1}{p}+\frac{N\gamma}{(N+2)p})\frac{N+2}{N+2-\gamma}}.\end{aligned}$$

► *Estimate of* $\int_{Q_t} m(x,t)|\nabla u_\epsilon|^\beta\,dx\,ds$. By the generalized Hölder inequality we have

$$\int_{Q_t} m(x,t)|\nabla u_\epsilon|^\beta\,dx\,ds \le \|m\|_{L^{N+2,1}(Q_t)}\|\nabla u_\epsilon\|^\beta_{L^{\frac{N+2}{N+1}}(Q_t)}.$$

Since $\gamma = \frac{N+2}{N+p}(p-1)$ and $\beta = \frac{N(p-1)+p}{N+2}$, and by using (3.16) and (3.17), we obtain

$$\begin{aligned}&\frac{\lambda}{2}\int_\Omega |T_k(u_\epsilon)|^2\,dx + \alpha\int_{Q_t}|\nabla T_k(u_\epsilon)|^p\,dx\,ds\\ &\le M_1k + C\frac{\gamma}{N+2}\|c(\cdot,\cdot)\|_{L^\tau(Q_{\tau_1})}\sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx\\ &\quad + C\frac{N+2-\gamma}{N+2}\|c(\cdot,\cdot)\|_{L^\tau(Q_{\tau_1})}\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,ds\\ &\quad + \|m\|_{L^{N+2,1}(Q_{\tau_1})}\|\nabla u_\epsilon\|^\beta_{L^{\frac{N+2}{N+1},\infty}(Q_{\tau_1})}.\end{aligned}$$

If τ_1 satisfies

$$\frac{\lambda}{2} - C\frac{\gamma}{N+2}\|c(\cdot,\cdot)\|_{L^\tau(Q_{\tau_1})} > 0, \tag{3.18}$$

$$\alpha - C\frac{N+2-\gamma}{N+2}\|c(\cdot,\cdot)\|_{L^\tau(Q_{\tau_1})} > 0, \tag{3.19}$$

then we have

$$C\Big(\frac{\lambda}{2}\sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx + \int_{Q_t} |\nabla T_k(u_\epsilon)|^p\,dx\,ds\Big) \\ \le M_1 k + \|m\|_{L^{N+2,1}(Q_{\tau_1})}\|\nabla u_\epsilon\|^\beta_{L^{\frac{N+2}{N+1},\infty}(Q_{\tau_1})}.$$

Using [8, Lemma A.1] we have

$$\begin{aligned}\|\nabla u_\epsilon\|^\beta_{L^{\frac{N+2}{N+1},\infty}(Q_{\tau_1})} &= \|\,|\nabla u_\epsilon|^{p-1}\|^{\frac{\beta}{p-1}}_{L^{\frac{N+2}{N+1},\infty}(Q_{\tau_1})} \\ &\le C(M_1 + \|m\|_{L^{N+2,1}(Q_{\tau_1})}\|\nabla u_\epsilon\|^\beta_{L^{\frac{N+2}{N+1},\infty}(Q_{\tau_1})});\end{aligned}$$

then

$$(1 - C\|m\|_{L^{N+2,1}(Q_{\tau_1})})\|\nabla u_\epsilon\|^\beta_{L^{\frac{N+2}{N+1},\infty}(Q_{\tau_1})} < CM_1.$$

If we choose τ_1 such that (3.18) and (3.19) hold and $1-C\|m\|_{L^{N+2,1}(Q_{\tau_1})} > 0$, this leads to

$$\|\nabla u_\epsilon\|^\beta_{L^{\frac{N+2}{N+1},\infty}(Q_{\tau_1})} \le C_1$$

and it follows that

$$\sup_{t\in(0,T)}\int_\Omega |\nabla T_k(u)|^2 + \int_{Q_T} |\nabla T_k(u)|^p \le M_1 k + C_1, \qquad \forall k > 0.$$

Then, by Lemma 2.1, we find that $T_k(u_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and $m(x,t)|\nabla u_\epsilon|^\beta$ is bounded in $L^1(Q_T)$, independently of ϵ and for any $k \ge 0$, so there exists a subsequence still denoted by u_ϵ such that

$$T_k(u_\epsilon) \rightharpoonup \sigma_k \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega)). \tag{3.20}$$

STEP 3. *A.e. convergence of u_ϵ and $b_\epsilon(u_\epsilon)$.* Proceeding as in [3], [4], [1], we prove that for every nondecreasing function $g_k \in C^2(\mathbb{R})$ such that $g_k(s) = s$ for $|s| \le k/2$ and $g_k(s) = k$ for $|s| \ge k$,

$$\frac{\partial g_k(b_\epsilon(u_\epsilon))}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{3.21}$$

Arguing again as in [5], estimates (3.20) and (3.21) imply that, for a subsequence, still indexed by ϵ,

$$u_\epsilon \to u \quad \text{a.e. in } Q_T, \tag{3.22}$$

where u is a measurable function defined on Q_T. Let us prove that $b(u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Taking $T_k(b_\epsilon(u_\epsilon))$ as a test function in (3.6), by (3.9) we have

$$\begin{aligned}(3.23)\qquad &\int_\Omega B_k^\epsilon(u_\epsilon)\,dx + \int_{Q_T} a_\epsilon(x,t,u,\nabla u_\epsilon)\nabla T_k(b_\epsilon(u_\epsilon))\,dx\,dt\\ &\le \int_{Q_T} |m(x,t)|\,|\nabla T_{1/\epsilon}(u_\epsilon)|^\beta \nabla T_k(b_\epsilon(u_\epsilon))\,dx\,dt\\ &\quad+\int_{Q_T} |c(x,t)|\,|T_{1/\epsilon}(u_\epsilon)|^\gamma|\nabla T_k(b_\epsilon(u_\epsilon))|\,dx\,dt\\ &\quad+k(\|f_\epsilon\|_{L^1(Q_T)}+\|b(u_0)\|_{L^1(\Omega)})\end{aligned}$$

with $B_k(r)=\int_0^{b(r)} T_k(s)\,ds$. On the other hand, we have

$$\begin{aligned}(3.24)\qquad &\int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(b_\epsilon(u_\epsilon))\,dx\,ds\\ &=\int_{\{|b_\epsilon(u_\epsilon)|\le k\}} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k'(b_\epsilon(u_\epsilon))b_\epsilon'(u_\epsilon)\nabla u_\epsilon\,dx\,ds\ge 0.\end{aligned}$$

Since $b'(s)\ge\lambda$, for $0<\epsilon<1/k$ and for almost $t\in(0,T)$ we have

$$\begin{aligned}(3.25)\qquad &\int_{Q_T} |c(x,t)|\,|T_{1/\epsilon}(u_\epsilon)|^\gamma|\nabla T_k(b_\epsilon(u_\epsilon))|\,dx\,dt\\ &\le \max_{|s|\le k/\lambda}(b'(s))\|c(\cdot,\cdot)\|_{L^\tau(Q_T)}\\ &\quad\times\Big(\sup_{t\in(0,T)}\Big(\int_\Omega |T_{k/\lambda}(u_\epsilon)|^2\,dx\Big)^{\frac{p-1}{N+p}}\|\nabla T_{k/\lambda}(u_\epsilon)\|_{L^p(Q_T)}^{\frac{p(N+1)}{N+p}}\Big)\le c_k\end{aligned}$$

and

$$\begin{aligned}(3.26)\qquad &\int_{Q_T} |m(x,t)|\,|\nabla T_{1/\epsilon}(u_\epsilon)|^\beta|\nabla T_k(b_\epsilon(u_\epsilon))|\,dx\,dt\\ &\le \max_{|s|\le k/\lambda}(b'(s))\|m(\cdot,\cdot)\|_{L^{N+2,1}(Q_T)}\|\nabla T_{k/\lambda}(b_\epsilon(u_\epsilon))\|_{L^{\frac{N+2}{N+1},\infty}(Q_T)}\le c_k.\end{aligned}$$

Using (3.24), (3.25) and (3.26) in (3.23) we have

$$\int_\Omega B_k^\epsilon(u_\epsilon(t))\,dx\le c_k+k(\|f_\epsilon\|_{L^1(Q_T)}+\|b(u_0)\|_{L^1(\Omega)}).$$

Passing to liminf as $\epsilon \to 0$, we obtain

$$\int_\Omega B_k(u(t))\,dx \le c_k + k(\|f\|_{L^1(Q_T)}) + \|b(u_0)\|_{L^1(\Omega)}) \quad \text{for a.e. } t \in (0,T).$$

Due to the definition of B_k, we have

$$\begin{aligned} k\int_\Omega |b(u(x,t))|\,dx &\le \int_\Omega B_k(u(t))\,dx + \tfrac{3}{2}k^2 \operatorname{meas}(\Omega) \\ &\le k(\|f\|_{L^1(\Omega)} + \|b(u_0)\|_{L^1(\Omega)}) + c_k + \tfrac{3}{2}k^2 \operatorname{meas}(\Omega). \end{aligned}$$

We conclude that $b(u) \in L^\infty(0,T;L^1(\Omega))$.

Lemma 3.4 (see [1]). *A subsequence of u_ϵ defined in Step 1 satisfies*

$$\lim_{n\to\infty} \limsup_{\epsilon\to 0} \int_{\{n\le |u_\epsilon|\le n+1\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon\,dx\,dt = 0.$$

Step 4. In this step we introduce a time regularization of the $T_k(u)$ for $k>0$ in order to apply the monotonicity method (see [10]). Let v_0^μ be a sequence of functions in $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)} \le k$ for all $\mu > 0$ and v_0^μ converges to $T_k(u_0)$ a.e. in Ω and $\frac{1}{\mu}\|v_0^\mu\|_{L^p(\Omega)}$ converges to 0. For $k\ge 0$ and $\mu>0$, let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q)\cap L^p(0,T;W_0^{1,p}(\Omega))$ of the monotone problem

$$\begin{gathered} \frac{\partial (T_k(u))_\mu}{\partial t} + \mu((T_k(u))_\mu - T_k(u)) = 0 \quad \text{in } \mathcal{D}'(\Omega), \\ (T_k(u))_\mu(t=0) = \nu_0^\mu \quad \text{in } \Omega. \end{gathered}$$

Lemma 3.5 (see [5]). *Let $k\ge 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r)=r$ for $|r|\le k$, and $\operatorname{supp} S'$ is compact. Then*

$$\liminf_{\mu\to\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t \left\langle \frac{\partial b_\epsilon(u_\epsilon)}{\partial t}, S'(u_\epsilon)\big(T_k(u_\epsilon) - (T_k(u))_\mu\big)\right\rangle ds\,dt \ge 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega)\cap W^{1,p}(\Omega)$.

Step 5. We prove the following lemma which is the critical point in the development of the monotonicity method.

Lemma 3.6. *A subsequence of u_ϵ satisfies, for any $k\ge 0$,*

$$\limsup_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega a(x,t,u_\epsilon,\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon)\,dx\,ds\,dt \le \int_0^T\int_0^t\int_\Omega \sigma_k \nabla T_k(u)\,dx\,ds\,dt.$$

Proof. Let S_n be a sequence of increasing C^∞-functions such that $S_n(r) = r$ for $|r| \le n$, $\operatorname{supp} S'_n \subset [-(n+1), n+1]$ and $\|S''_n\|_{L^\infty(\mathbb{R})} \le 1$ for any $n \ge 1$. We use the sequence $(T_k(u))_\mu$ of approximations of $T_k(u)$, and plug the test function $S'_n(u_\epsilon)(T_k(u_\epsilon) - (T_k(u))_\mu)$ into (3.4) for $n > 0$ and $\mu > 0$. For fixed $k \ge 0$ let $W^\epsilon_\mu = T_k(u_\epsilon) - (T_k(u))_\mu$. Upon integration over $(0, t)$ and then over $(0, T)$ we obtain

$$\begin{aligned}
(3.27)\qquad & \int_0^T\!\!\int_0^t \left\langle \frac{\partial b_\epsilon(u_\epsilon)}{\partial t}, S'_n(u_\epsilon) W^\epsilon_\mu \right\rangle ds\, dt \\
&\quad + \int_0^T\!\!\int_0^t\!\!\int_\Omega a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) S'_n(u_\epsilon) \nabla W^\epsilon_\mu \, dx\, ds\, dt \\
&\quad + \int_0^T\!\!\int_0^t\!\!\int_\Omega a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) S''_n(u_\epsilon) \nabla u_\epsilon \nabla W^\epsilon_\mu \, dx\, ds\, dt \\
&\quad - \int_0^T\!\!\int_0^t\!\!\int_\Omega \phi_\epsilon(x,t,u_\epsilon) S'_n(u_\epsilon) \nabla W^\epsilon_\mu \, dx\, ds\, dt \\
&\quad - \int_0^T\!\!\int_0^t\!\!\int_\Omega S''_n(u_\epsilon) \phi_\epsilon(x,t,u_\epsilon) \nabla u_\epsilon \nabla W^\epsilon_\mu \, dx\, ds\, dt \\
&\quad + \int_0^T\!\!\int_0^t\!\!\int_\Omega H_\epsilon(x,t,\nabla u_\epsilon) S'_n(u_\epsilon) W^\epsilon_\mu \, dx\, ds\, dt \\
&= \int_0^T\!\!\int_0^t\!\!\int_\Omega f_\epsilon S'_n(u_\epsilon) W^\epsilon_\mu \, dx\, ds\, dt.
\end{aligned}$$

Now we pass to the limit in (3.27) as $\epsilon \to 0$, $\mu \to \infty$ and then $n \to \infty$ for k real fixed. In order to perform this task we prove below the following results for any fixed $k \ge 0$:

$$(3.28)\qquad \liminf_{\mu\to\infty} \lim_{\epsilon\to 0} \int_0^T\!\!\int_0^t \left\langle \frac{\partial b_\epsilon(u_\epsilon)}{\partial t}, W^\epsilon_\mu \right\rangle ds\, dt \ge 0 \quad \text{for any } n \ge k,$$

$$(3.29)\qquad \lim_{\mu\to\infty} \lim_{\epsilon\to 0} \int_0^T\!\!\int_0^t\!\!\int_\Omega \phi_\epsilon(x,t,u_\epsilon) S'_n(u_\epsilon) \nabla W^\epsilon_\mu \, dx\, ds\, dt = 0 \quad \text{for any } n \ge 1,$$

(3.30)

$$\lim_{\mu\to\infty} \lim_{\epsilon\to 0} \int_0^T\!\!\int_0^t\!\!\int_\Omega S''_n(u_\epsilon) \phi_\epsilon(x,t,u_\epsilon) \nabla u_\epsilon \nabla W^\epsilon_\mu \, dx\, ds\, dt = 0 \quad \text{for any } n \ge 1,$$

(3.31)
$$\lim_{n\to\infty}\ \limsup_{\mu\to\infty} \limsup_{\epsilon\to 0} \int_0^T\int_0^t\int_\Omega a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S''_n(u_\epsilon)\nabla u_\epsilon \nabla W^\epsilon_\mu\,dx\,ds\,dt = 0,$$

$$\lim_{\mu\to\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega H_\epsilon(x,t,\nabla u_\epsilon)S'_n(u_\epsilon)W^\epsilon_\mu\,dx\,ds\,dt = 0, \tag{3.32}$$

$$\lim_{\mu\to\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega f_\epsilon S'_n(u_\epsilon)W^\epsilon_\mu\,dx\,ds\,dt = 0. \tag{3.33}$$

We adopt the same proof of [1] for (3.28)–(3.31) and (3.33). It remains to prove (3.32). For any fixed $n \geq 1$ and $0 < \epsilon < 1/(n+1)$,

$$H_\epsilon(x,t,\nabla u_\epsilon)S'_n(u_\epsilon)W^\epsilon_\mu = H_\epsilon(x,t,\nabla T_{n+1}(u_\epsilon))S'_n(u_\epsilon)W^\epsilon_\mu \quad \text{a.e. in } Q_T.$$

It is possible to pass to the limit for $\epsilon \to 0$ since from $\|W^\epsilon_\mu\|_{L^\infty(Q_T)} \leq 2k$ for any $\epsilon,\mu > 0$, and $W^\epsilon_\mu \rightharpoonup T_k(u) - (T_k(u))_\mu$ a.e. in Q_T and weakly-$*$ in $L^\infty(Q_T)$, when $\epsilon \to 0$ we have

$$H_\epsilon(x,t,\nabla T_{n+1}(u_\epsilon))S'_n(u_\epsilon)W^\epsilon_\mu \to H(x,t,\nabla T_{n+1}(u))S'_n(u)W_\mu \quad \text{a.e. in } Q_T.$$

Since

$$|H(x,t,\nabla T_{n+1}(u))S'_n(u)W_\mu| \leq 2k|m(x,t)|(n+1)^\beta \quad \text{a.e. in } Q_T$$

and $(T_k(u))_\mu$ converges to 0 in $L^p(0,T;W^{1,p}_0(\Omega))$, we obtain (3.32).

STEP 6. In this step we prove that the weak limit σ_k of $a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))$ can be identified with $a(x,t,T_k(u),\nabla T_k(u))$. To do so, we recall the following lemmas proved in [1].

LEMMA 3.7. *A subsequence of u_ϵ defined in Step 1 satisfies, for any $k \geq 0$,*

$$\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega \big(a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) - a(x,t,T_k(u_\epsilon),\nabla T_k(u))\big) \times (\nabla T_k(u_\epsilon) - \nabla T_k(u)) = 0. \tag{3.34}$$

LEMMA 3.8. *For fixed $k \geq 0$, we have*

$$\sigma_k = a(x,t,T_k(u),\nabla T_k(u))) \quad \textit{a.e. in } Q_T, \tag{3.35}$$

and as $\epsilon \to 0$,

$$a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)))\nabla T_k(u) \tag{3.36}$$

weakly in $L^1(Q_T)$. ∎

Taking the limit as ϵ tends to 0 and using (3.36) shows that u satisfies (3.3). Our aim is to prove that it satisfies (3.4) and (3.5).

First we prove that u satisfies (3.4). Let $S \in W^{2,\infty}(\mathbb{R})$ with $\operatorname{supp} S' \subset [-k,k]$ where $k > 0$. Pointwise multiplication of the approximate equation (3.6) by $S'(u_\epsilon)$ leads to

$$\begin{aligned}(3.37)\qquad & \frac{\partial B_S^\epsilon(u_\epsilon)}{\partial t} - \operatorname{div}\big(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon)\big) + S''(u_\epsilon)a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \\ &+ \operatorname{div}\big(\phi_\epsilon(x,t,u_\epsilon)S'(u_\epsilon)\big) - S''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon + H_\epsilon(x,t,\nabla u_\epsilon)S'(u_\epsilon) \\ &= f_\epsilon S'(u_\epsilon) \quad \text{in } \mathcal{D}'(\Omega).\end{aligned}$$

where

$$B_S^\epsilon(r) = \int_0^r \frac{\partial b_\epsilon(s)}{\partial s} S'(s)\,ds.$$

In what follows we let $\epsilon \to 0$ in each term of (3.37). Since u_ϵ converges to u a.e. in Q_T, $B_S^\epsilon(u_\epsilon)$ converges to $B_S(u)$ a.e. in Q_T and weakly-$*$ in $L^\infty(Q_T)$. Then $\partial B_S^\epsilon/\partial t$ converges to $\partial B_S/\partial t$ in $\mathcal{D}'(Q_T)$.We observe that the term $a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon)$ can be identified with $a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))S'(u_\epsilon)$ for $\epsilon \le 1/k$, so using the pointwise convergence $u_\epsilon \to u$ in Q_T, and the weak convergence $T_k(u_\epsilon) \rightharpoonup T_k(u)$ in $L^p(0,T;W_0^p(\Omega))$, we get

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))S'(u) \quad \text{in } L^{p'}(Q_T),$$

and

$$\begin{aligned}& S''(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \\ &\qquad \rightharpoonup S''(u)a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \quad \text{in } L^1(Q_T).\end{aligned}$$

Furthermore, since

$$\phi_\epsilon(x,t,u_\epsilon)S'(u_\epsilon) = \phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon) \qquad \text{a.e. in } Q_T,$$

by (3.9) we obtain

$$|\phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon)| \le |c(x,t)|k^\gamma,$$

and it follows that

$$\phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon) \to \phi(x,t,T_k(u))S'(u) \qquad \text{strongly in } L^{p'}(Q_T).$$

Similarly, since $H_\epsilon(x,t,\nabla u_\epsilon)S'(u_\epsilon) = H_\epsilon(x,t,\nabla T_k(u_\epsilon))S'(u_\epsilon)$ a.e. in Q_T, by (3.10) we have $|H_\epsilon(x,t,\nabla T_k(u_\epsilon))S'(u_\epsilon)| \le |m(x,t)|k^\beta$, and it follows that

$$H_\epsilon(x,t,\nabla T_k(u_\epsilon))S'(u_\epsilon) \to H(x,t,\nabla T_k(u))S'(u) \qquad \text{strongly in } L^1(Q_T).$$

In a similar way,

$$S''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon = S''(T_k(u_\epsilon))\phi_\epsilon(x,t,T_k(u_\epsilon))\nabla T_k(u_\epsilon) \qquad \text{a.e. in } Q_T.$$

Using the weak convergence of $T_k(u_\epsilon)$ in $L^p(0,T;W_0^p(\Omega))$ it is possible to prove that $S''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon \to S''(u)\phi(x,t,u)\nabla u$ in $L^1(Q_T)$. Finally, by (3.11) we deduce that $f_\epsilon S'(u_\epsilon)$ converges to $fS'(u)$ in $L^1(Q_T)$.

It remains to prove that $B_S(u)$ satisfies the initial condition $B_S(t=0) = B_S(u_0)$ in Ω. To this end, first note that S being bounded, $B_S^\epsilon(u_\epsilon)$ is bounded in $L^\infty(Q)$. Secondly the above consideration of the behavior of the terms of this equation shows that $\partial B_S^\epsilon(u_\epsilon)/\partial t$ is bounded in $L^1(Q_T) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin type lemma (see e.g. [13]) implies that $B_S^\epsilon(u_\epsilon)$ lies in a compact subset of $C^0([0,T];L^1(\Omega))$. Finally, the smoothness of S implies that $B_S(t=0) = B_S(u_0)$ in Ω. ∎

References

[1] A. Aberqi, J. Bennouna, M. Mekkour and H. Redwane, *Renormalized solution for a nonlinear parabolic equation with lower order terms*, Austral. J. Math. Anal. Appl. 10 (2013), no. 1, art. 15, 15 pp.

[2] Y. Akdim, J. Bennouna and M. Mekkour, *Solvability of degenerated parabolic equations without sign condition and three unbounded nonlinearities*, Electron. J. Differetial Equations 2011, no. 3, 26 pp.

[3] D. Blanchard and F. Murat, *Renormalised solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness*, Proc. Roy. Soc. Edinburgh Sect. A Math. 127 (1997), 1137–1152.

[4] D. Blanchard, F. Murat and H. Redwane, *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems*, J. Differential Equations 177 (2001), 331–374.

[5] D. Blanchard and H. Redwane, *Renormalized solutions for a class of nonlinear evolution problems*, J. Math. Pures Appl. 77 (1998), 117–151.

[6] L. Boccardo, D. Giachetti, J.-I. Diaz and F. Murat, *Existence and regularity of renormalized solutions of some elliptic problems involving derivatives of nonlinear terms*, J. Differential Equations 106 (1993), 215–237.

[7] R. Di Nardo, *Nonlinear parabolic equations with a lower order term*, preprint N. 60-2008, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II".

[8] R. Di Nardo, F. Feo and O. Guibé, *Existence result for nonlinear parabolic equations with lower order terms*, Anal. Appl. (Singap.) 9 (2011), 161–186.

[9] R. DiPerna and P.-L. Lions, *Ordinary differential equations, Sobolev spaces and transport theory*, Invent. Math. 98 (1989), 511–547.

[10] R. Landes, *On the existence of weak solutions for quasilinear parabolic initial-boundary problems*, Proc. Roy. Soc. Edinburgh Sect. A 89, 1981, 217–237.

[11] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod and Gauthier-Villars, Paris, 1969.

[12] J.-M. Rakotoson, *Resolution of the critical cases for the problems with L^1 data, when right hand side is in $L^1(\Omega)$*, Asymptotic Anal. 6 (1993), 285–293.

[13] J. Simon, *Compact sets in the space $L^p(0,T,B)$*, Ann. Mat. Pura Appl. 146 (1987), 65–96.

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