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UNBIASED ESTIMATION OF RELIABILITY FOR TWO-PARAMETER EXPONENTIAL DISTRIBUTION UNDER TIME CENSORED SAMPLING

Abstract. The problem considered is that of unbiased estimation of reliability for a two-parameter exponential distribution under time censored sampling. We give necessary and sufficient conditions for the existence of uniformly minimum variance unbiased estimator and also provide a characterization of a complete class of unbiased estimators in situations where unbiased estimators exist.

1. Introduction. Let the life-length X of an item follow a two-parameter exponential distribution with unknown real parameters μ and λ (> 0), to be denoted hereafter as $\exp(\mu, \lambda)$ distribution, defined by the probability function (p. f.)

$$f(x \mid \mu, \lambda) = \frac{1}{\lambda} e^{-(x-\mu)/\lambda}, \qquad x > \mu. \tag{1}$$

An important characteristic of the life distribution is its reliability function viz.

$$R(t) = P(X > t) = \begin{cases} 1, & t \le \mu, \\ e^{-(t-\mu)/\lambda}, & t > \mu, \end{cases} \tag{2}$$

and a problem of interest in reliability theory is to estimate $R(t)$ at a given finite time point t (> 0) through a life testing experiment.

In this paper we consider the problem of unbiased estimation of $R(t)$ under a time censored sampling plan wherein a random collection of n identical items are put on test and the experiment is terminated after a pre-assigned finite time T (> 0). For $t \le T$, an unbiased estimator of $R(t)$

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under time censored sampling based on a sufficient statistic was obtained in Bartoszewicz [1] through Rao–Blackwellization of a simple unbiased estimator

$$\hat{R}(t) = 1 - D_0/n \tag{3}$$

where D_0 is the number of items failed up to time t. Sengupta [3] showed that the condition $t \le T$ is necessary as well for the existence of an unbiased estimator and also obtained an alternative unbiased estimator of $R(t)$ based on the sufficient statistic for $t \le T$. Bartoszewicz [1] showed that the sufficient statistic is not, however, complete except for $n = 1, 2$ (see also Section 4) and as such the well known Lehmann–Scheffe theorem can not generally be applied to obtain the uniformly minimum variance unbiased estimator (UMVUE) of R (t) in situations where unbiased estimators exist.

Our main purpose in this article is to study the existence of the UMVUE of $R(t)$ under time censored sampling for an exp (μ, λ) distribution. It is proved that for $n > 2$, there does not exist UMVUE of $R(t)$ for $t \le T$. We also provide a characterization of a complete class of unbiased estimators of $R(t)$ for values of t for which $R(t)$ is unbiasedly estimable.

2. Preliminaries. For a time censored sample, the data consist of D and $X_{(0)}, X_{(1)}, \dots, X_{(D)}$, where D is the number of items failed up to preassigned time T (> 0) out of n test items, $X_{(i)}$ being the life-length of the ith failed item, $1 \le i \le D$ and $X_{(0)} = 0$. Let $p = R(T)$ and note that D follows a binomial distribution with mean nq, $q = 1 - p$. The joint p.f. of D and $X_{(0)}, X_{(1)}, \dots, X_{(D)}$ is (see Bartoszewicz [1])

$$\begin{aligned} &p(d, x_{(0)}, x_{(1)}, \dots, x_{(d)}) \\ &\quad = \begin{cases} p^n, \quad d = 0, \\ d! \dbinom{n}{d} p^{n-d} I(T > \mu)\, \dfrac{1}{\lambda^d}\, e^{-\sum_{i=1}^{d} (x_{(i)} - \mu)/\lambda} I(x_{(1)} > \mu), \\ \qquad\qquad\qquad 1 \le d \le n,\ x_{(1)} < \dots < x_{(d)} \le T, \end{cases} \end{aligned} \tag{4}$$

where $I(A)$ is the indicator function of the set A. Clearly a sufficient statistic is $V = (D, Z_D)$ where

$$Z_d = \begin{cases} X_{(d)}, & d = 0, 1, \\ \Big(X_{(1)}, S_d = \displaystyle\sum_{i=2}^{d} (X_{(i)} - X_{(1)})\Big), & 2 \le d \le n. \end{cases} \tag{5}$$

The p.f. of V is given by (see Bartoszewicz [1])

$$(6)\quad p(d,z_d)=\begin{cases} p^n, & d=0,\\ np^{n-1}qI(T>\mu)p(x_{(1)}\,|\,d), & d=1,\\ \binom{n}{d}p^{n-d}q^dI(T>\mu)p(x_{(1)}\,|\,d)p(s_d\,|\,d,x_{(1)}), & 2\le d\le n,\end{cases}$$

where for $d\ge 1$,

$$(7)\quad \begin{aligned} p(x(1)\,|\,d) &= \text{ the conditional p.f. of } X_{(1)} \text{ given } D=d\\ &= \frac{d}{\lambda q^d}\, e^{-(x_{(1)}-\mu)/\lambda}[e^{-(x_{(1)}-\mu)/\lambda}-e^{-(T-\mu)/\lambda}]^{d-1}, \qquad \mu<x_{(1)}\le T,\end{aligned}$$

and for $d\ge 2$,

$$(8)\quad p(s_d\,|\,d,x_{(1)}) = \text{ the conditional p.f. of } S_d \text{ given } D=d \text{ and } X_{(1)}=x_{(1)}$$
$$=\frac{1}{\lambda^{d-1}(1-e^{-(T-x_{(1)})/\lambda})^{d-1}}\, e^{-s_d/\lambda}f_d(s_d,T-x_{(1)}),\quad 0<s_d\le(d-1)(T-x(1)),$$

with

$$(9)\qquad f_d(u,w)=\frac{1}{\Gamma(d-1)}\sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}(u-jw)^{d-2}I(u>jw).$$

In particular, for $d=2,3$,

$$(10)\qquad f_2(u,w)=1,\qquad 0<u\le w,$$

$$(11)\qquad f_3(u,w)=\begin{cases} u, & 0<u\le w,\\ 2w-u, & w<u\le 2w.\end{cases}$$

In view of sufficiency of V it is enough to restrict to estimators based on V to study unbiased estimation of $R(t)$. In the following lemma we obtain a representation of the expectation of an estimator based on V which plays an important role in the derivation of the results in the subsequent sections.

LEMMA 1. *Let $g(V)$ be an estimator based on the sufficient statistic V with $g(0,0)=g(0)$. Then*

$$(12)\quad E[g(V)]$$
$$=\begin{cases} g(0) & \textit{for } \mu\ge T,\\ g(0)p^n+\dfrac{n}{\lambda}\displaystyle\int_{\mu}^{\infty} e^{-n(x_{(1)}-\mu)/\lambda}g^*(x_{(1)},\lambda)I(x_{(1)}\le T)\,dx_{(1)} & \textit{for } \mu<T,\end{cases}$$

where

$$(13)\quad g^*(x_{(1)},\lambda)$$
$$= \frac{1}{\lambda^{n-1}}\int_0^\infty e^{-s/\lambda}\Bigg\{\frac{g(1,x_{(1)})(s-(n-1)(T-x_{(1)}))^{n-2}I(s>(n-1)(T-x_{(1)}))}{\Gamma(n-1)}$$
$$+\sum_{d=2}^{n-1}\frac{\binom{n-1}{d-1}u_d(x_{(1)},s)I(s>(n-d)(T-x_{(1)}))}{\Gamma(n-d)}$$
$$+ g(n,x_{(1)},s)f_n(s,T-x_{(1)})I(s\le (n-1)(T-x_{(1)}))\Bigg\}\,ds$$

and for $2\le d\le n-1$,

$$(14)\quad u_d(x_{(1)},s)=$$
$$\begin{cases}\displaystyle\int_0^{s-(n-d)(T-x_{(1)})} g(d,x_{(1)},s_d)(s-s_d-(n-d)(T-x_{(1)}))^{n-d-1}f_d(s_d,T-x_{(1)})\,ds_d, \\ \qquad\qquad (n-d)(T-x(1))<s\le(n-1)(T-x(1)),\\ \displaystyle\sum_{c=0}^{n-d-1}a_{cd}s^c, \qquad (n-1)(T-x(1))<s<\infty.\end{cases}$$

with

$$(15)\quad a_{cd}=(-1)^{n-d-c-1}\binom{n-d-1}{c}$$
$$\times\int_0^{(d-1)(T-x_{(1)})} g(d,x_{(1)},s_d)(s_d+(n-d)(T-x_{(1)}))^{n-d-c-1}f_d(s_d,T-x_{(1)})\,ds_d.$$

Proof. For $\mu\ge T$, we have $D=0$ with probability 1 and hence (12) is obvious. For $\mu<T$, it can be readily verified using (6)–(8) that $E[g(V)]$ is of the form (12) where

$$(16)\quad g^*(x(1),\lambda)=e^{-(n-1)(T-x_{(1)})/\lambda}g(1,x_{(1)})$$
$$+\sum_{d=2}^{n}\binom{n-1}{d-1}\frac{1}{\lambda^d-1}\,e^{-(n-d)(T-x_{(1)})/\lambda}$$
$$\times\int_0^{(d-1)(T-x_{(1)})} g(d,x_{(1)},s_d)e^{-s_d/\lambda}f_d(s_d,T-x_{(1)})\,ds_d.$$

Now for $1 \le d \le n-1$,

$$(17)\qquad e^{-(n-d)(T-x_{(1)})/\lambda}$$
$$= \frac{1}{\Gamma(n-d)\lambda^{n-d}} \int_0^\infty e^{-y/\lambda}(y-(n-d)(T-x_{(1)}))^{n-d-1} I(y > (n-d)(T-x_{(1)}))\, dy$$

so that for $2 \le d \le n-1$, the dth term in the sum on the RHS of (16), on substituting $s = y + s_d$, can be expressed as

$$(18)\qquad \binom{n-1}{d-1} \frac{1}{\Gamma(n-d)\lambda^{n-1}} \int_0^\infty e^{-s/\lambda} u_d(x_{(1)}, s) I(s > (n-d)(T-x_{(1)}))\, ds$$

where

(19)
$$u_d(x_{(1)}, s) = \int_0^\infty g(d, x_{(1)}, s_d)(s - s_d - (n-d)(T-x_{(1)}))^{n-d-1} f_d(s_d, T-x_{(1)})$$
$$\times I(s_d \le (d-1)(T-x_{(1)})) I(s - s_d > (n-d)(T-x(1)))\, ds_d.$$

Since

$$I(s_d \le (d-1)(T-x(1))) I(s - s_d > (n-d)(T-x(1)))$$
$$= \begin{cases} I(s_d \le (d-1)(T-x_{(1)})) & \text{for } (n-1)(T-x_{(1)}) < s < \infty, \\ I(s - s_d > (n-d)(T-x_{(1)})) & \\ & \text{for } (n-d)(T-x_{(1)}) < s \le (n-1)(T-x_{(1)}), \end{cases}$$

it readily follows that the RHS of (19) is equal to the RHS of (14). Thus for $\mu < T$, (12) follows from (16)–(18).

3. Complete class of unbiased estimators. We recall that under time censored sampling, $R(t)$ is unbiasedly estimable if and only if $t \le T$. For $t \le T$, we obtain a characterization of a complete class of unbiased estimators of $R(t)$ in the following theorem.

THEOREM 1. *For $t \le T$, $g(V)$ is an unbiased estimator of $R(t)$ based on the sufficient statistic V if and only if it satisfies the following:*

$$(20)\qquad g(0) = g(0,0) = 1,$$

$$(21)\qquad g(1, x(1)) = \begin{cases} 1 & \textit{if } x_{(1)} \ge t, \\ \dfrac{n-1}{n} & \textit{if } x_{(1)} < t, \end{cases}$$

$$\sum_{c=2}^{d} \frac{(-1)^{d-c}\binom{n-1}{c-1}\binom{n-c-1}{d-c}}{\Gamma(n-c)} \tag{22}$$
$$\times \int_0^{(c-1)(T-x_{(1)})} g(c,x_{(1)},s_c)(s_c+(n-c)(T-x_{(1)}))^{d-c} f_c(s_c,T-x_{(1)})\,ds_c$$
$$= \begin{cases} \dfrac{(-1)^{d-2}\binom{n-2}{d-1}(n-1)^{d-1}(T-x_{(1)})^{d-1}}{\Gamma(n-1)} & \textit{if } x_{(1)} \ge t, \\ \dfrac{n-1}{n}\dfrac{(-1)^{d-2}\binom{n-2}{d-1}\{(n-1)^{d-1}(T-x_{(1)})^{d}-1-(t-x_{(1)})^{d-1}\}}{\Gamma(n-1)} & \\ & \textit{if } x_{(1)} < t, \end{cases}$$

for $2 \le d \le n-1$. *For* $x_{(1)} \ge t$, $m(T-x(1)) < s_n \le (m+1)(T-x(1))$, $m = 0,1,\dots,n-2$,

(23)

$$g(n,x(1),s_n)f_n(s_n,T-x(1)) + \sum_{d=n-m}^{n-1} \frac{\binom{n-1}{d-1}}{\Gamma(n-d)} \int_0^{s_n-(n-d)(T-x_{(1)})} g(d,x_{(1)},s_d)$$
$$\times (s_n-s_d-(n-d)(T-x_{(1)}))^{n-d-1} f_d(s_d,T-x_{(1)})\,ds_d = \frac{s_n^{n-2}}{\Gamma(n-1)}.$$

For $x(1) < t$, $0 < s_n \le t - x_{(1)}$,

$$g(n,x_{(1)},s_n) = 0. \tag{24}$$

For $x_{(1)} < t$, $\max(t-x(1), m(T-x(1))) < s_n \le (m+1)(T-x_{(1)})$, $m = 0,1,\dots,n-2$,

(25)

$$g(n,x(1),s_n)f_n(s_n,T-x(1)) + \sum_{d=n-m}^{n-1} \frac{\binom{n-1}{d-1}}{\Gamma(n-d)} \int_0^{s_n-(n-d)(T-x_{(1)})} g(d,x_{(1)},s_d)$$
$$\times (s_n-s_d-(n-d)(T-x_{(1)}))^{n-d-1} f_d(s_d,T-x_{(1)})\,ds_d$$
$$= \frac{n-1}{n}\,\frac{1}{\Gamma(n-1)}\,(s_n-(t-x_{(1)}))^{n-2}.$$

Further, a subclass of unbiased estimators $g(V)$ *with*

$$g(V) = 1 \quad \textit{if } X_{(1)} \ge t \tag{26}$$

is a complete class of unbiased estimators of $R(t)$.

Proof. Let $t \le T$ and $g(V)$ be an unbiased estimator of $R(t)$. Since $R(t) = 1$ for $\mu \ge T$, (12) implies (20). Also since for $\mu < T$,

$$R(t) = p^n + \frac{n}{\lambda}\int_{\mu}^{\infty} g_0(x_{(1)},\lambda)e^{-n(x_{(1)}-\mu)/\lambda} I(x_{(1)} \le T)\,dx_{(1)} \tag{27}$$

with

$$g_0(x_{(1)},\lambda) = \begin{cases} 1 & \text{for } x_{(1)} \ge t, \\ \dfrac{n-1}{n} e^{-(t-x_{(1)})/\lambda} & \text{for } x_{(1)} < t, \end{cases} \tag{28}$$

and the distribution of the smallest order statistic is complete for a complete random sample of size n from an $\exp(\mu,\lambda)$ distribution with known λ, we have, by (12) and (20), $g^*(x_{(1)},\lambda) = g_0(x_{(1)},\lambda)$ for all $\lambda > 0$, where $g^*(x_{(1)},\lambda)$ is given by (13). Note that (28) can also be expressed as

$$\begin{aligned} &g_0(x_{(1)},\lambda) \\ &= \begin{cases} \dfrac{1}{\Gamma(n-1)\lambda^{n-1}}\displaystyle\int_0^\infty e^{-s/\lambda}s^{n-2}\,ds \quad \text{for } x_{(1)} \ge t, \\ \dfrac{n-1}{n}\dfrac{1}{\Gamma(n-1)\lambda^{n-1}}\displaystyle\int_0^\infty e^{-s/\lambda}(s-(t-x_{(1)}))^{n-2}I(s > t-x_{(1)})\,ds \\ \hfill \text{for } x_{(1)} < t. \end{cases} \end{aligned} \tag{29}$$

Hence, by the completeness of the exponential distribution, (13) and (29) imply for $0 < s < \infty$,

$$\begin{aligned} &\sum_{d=2}^{n-1} \frac{\binom{n-1}{d-1} u_d(x_{(1)},s) I(s > (n-d)(T-x_{(1)}))}{\Gamma(n-d)} \\ &\quad + g(n,x_{(1)},s) f_n(s,T-x_{(1)}) I(s \le (n-1)(T-x_{(1)})) \\ &= \frac{1}{\Gamma(n-1)}\{w(x_{(1)},s) \\ &\quad - g(1,x_{(1)})(s-(n-1)(T-x_{(1)})^{n-2} I(s > (n-1)(T-x_{(1)})))\} \end{aligned} \tag{30}$$

with

$$w(x_{(1)},s) = \begin{cases} s^{n-2} & \text{for } x_{(1)} \ge t, \\ \dfrac{n-1}{n}(s-(t-x_{(1)}))^{n-2} I(s > t-x_{(1)}) & \text{for } x_{(1)} < t, \end{cases} \tag{31}$$

and also for $(n-1)(T-x_{(1)}) < s < \infty$,

$$(32)\qquad \sum_{d=2}^{n-1} \frac{\binom{n-1}{d-1}}{\Gamma(n-d)} \sum_{c=0}^{n-d-1} a_{cd} s^c = \sum_{c=0}^{n-3} s^c \sum_{d=2}^{n-c-1} \frac{\binom{n-1}{d-1} a_{cd}}{\Gamma(n-d)}$$

$$= \begin{cases} \dfrac{1}{\Gamma(n-1)} \{s^{n-2} - g(1, x_{(1)})(s-(n-1)(T-x_{(1)}))^{n-2}\} & \text{for } x_{(1)} \ge t, \\ \dfrac{1}{\Gamma(n-1)} \left\{ \dfrac{n-1}{n} (s-(t-x_{(1)}))^{n-2} - g(1, x_{(1)})(s-(n-1)(T-x_{(1)}))^{n-2} \right\} & \\ & \text{for } x_{(1)} < t, \end{cases}$$

where $u_d(x_{(1)}, s)$ and a_{cd} are defined respectively by (14) and (15). Equating the coefficients of s^{n-c-2}, $c = 0, 1, \ldots, n-2$, on both sides of (32) we get (21) and

$$(33)\qquad \sum_{d=2}^{c+1} \frac{\binom{n-1}{d-1} a_{(n-c-2)d}}{\Gamma(n-d)}$$

$$= \begin{cases} \dfrac{(-1)^{c-1}\binom{n-2}{c}(n-1)^c (T-x_{(1)})^c}{\Gamma(n-1)} & \text{for } x_{(1)} \ge t, \\ \dfrac{n-1}{n} \, \dfrac{(-1)^{c-1}\binom{n-2}{c}\{(n-1)^c (T-x_{(1)})^c - (t-x_{(1)})^c\}}{\Gamma(n-1)} & \text{for } x_{(1)} < t, \end{cases}$$

which yields (22). Finally, (23)–(25) are obtained from (30). This completes the proof of the first part of the theorem.

Now let $g(V)$ be an unbiased estimator of $R(t)$ not satisfying (26) and let

$$g'(V) = \begin{cases} 1 & \text{if } X_{(1)} \ge t, \\ g(V) & \text{if } X_{(1)} < t. \end{cases}$$

It can then be verified using (12) and (16) that $g'(V)$ is an unbiased estimator of $R(t)$ satisfying (26) and further $E[g(V)]^2 - E[g'(V)]^2 = 0$ for $\mu \ge T$, while for $\mu < T$,

$$E[g(V)]^2 - E[g'(V)]^2 = \frac{n}{\lambda} \int_{\max(\mu,t)}^{T} g^{**}(x_{(1)}, \lambda) e^{-n(x_{(1)}-\mu)/\lambda} \, dx_{(1)} \ge 0$$

with strict inequality for $\mu < t$ where

$$g^{**}(x(1), \lambda) = \sum_{d=2}^{n} \binom{n-1}{d-1} \frac{1}{\lambda^d - 1} e^{-(n-d)(T-x_{(1)})/\lambda}$$
$$\times \int_0^{(d-1)(T-x_{(1)})} [g(d, x_{(1)}, s_d) - 1]^2 e^{-s_d/\lambda} f_d(s_d, T-x_{(1)}) \, ds_d.$$

Thus given any unbiased estimator of $R(t)$ based on V not satisfying (26), there exists a better unbiased estimator based on V satisfying (26), and this proves the second part of the theorem.

4. Existence of UMVUE. We finally study the existence of UMVUE of $R(t)$ for $t \le T$. The following characterization of the class of unbiased estimators of zero based on the sufficient statistic V is useful for this purpose.

THEOREM 2. *An estimator $h(V)$ based on the sufficient statistic V is an unbiased estimator of zero if and only if it satisfies the following:*

$$h(0) = h(0,0) = 0, \tag{34}$$

$$h(1, x(1)) = 0, \tag{35}$$

$$\sum_{c=2}^{d} \frac{(-1)^{d-c}\binom{n-1}{c-1}\binom{n-c-1}{d-c}}{\Gamma(n-c)} \times \int_0^{(c-1)(T-x_{(1)})} h(c, x_{(1)}, s_c)(s_c + (n-c)(T - x_{(1)}))^{d-c} f_c(s_c, T - x_{(1)})\, ds_c = 0 \tag{36}$$

for $2 \le d \le n-1$,

$$h(n, x_{(1)}, s_n) = 0 \quad \textit{for } 0 < s_n \le T - x_{(1)}, \tag{37}$$

$$\begin{aligned} &h(n, x_{(1)}, s_n) f_n(s_n, T - x(1)) \\ &+ \sum_{d=n-m}^{n-1} \frac{\binom{n-1}{d-1}}{\Gamma(n-d)} \\ &\times \int_0^{s_n-(n-d)(T-x_{(1)})} h(d, x_{(1)}, s_d)(s_n - s_d - (n-d)(T - x_{(1)}))^{n-d-1} \\ &\qquad\qquad \times f_d(s_d, T - x_{(1)})\, ds_d = 0 \end{aligned} \tag{38}$$

for $m(T - x_{(1)}) < s_n \le (m+1)(T - x(1))$, $m = 1, 2, \ldots, n-2$.

The proof of the theorem is similar to that of Theorem 1. As an immediate corollary we have the following result also obtained by Bartoszewicz [1].

COROLLARY 1. *The sufficient statistic V is complete if and only if $n = 1, 2$.*

Proof. For $n = 1, 2$, the corollary follows trivially from (34), (35) and (37). For $n > 2$, a non-trivial unbiased estimator of zero is $h_0(V)$ satisfying (34)–(38), with

$$h_0(2, x_{(1)}, s_2) = \begin{cases} -c, & 0 < s_2 \le (T - x_{(1)})/2, \\ c, & (T - x_{(1)})/2 < s_2 \le T - x_{(1)}, \end{cases} \tag{39}$$

where c is a non-zero real constant.

Thus for $n = 1, 2$ and $t \le T$, $\hat{R}(t)$ defined in (3) is the unique unbiased estimator based on V and is the UMVUE of $R(t)$. To study the existence of UMVUE for $n > 2$, we make use of the following result given in Rao ([2], p. 317).

THEOREM 3. *An unbiased estimator $g(V)$ is the UMVUE of $R(t)$ if and only if*

$$E[g(V)h(V)] = 0 \quad \text{for every } \mu, \lambda \tag{40}$$

for every unbiased estimator $h(V)$ of zero.

In fact, in what follows we prove that for $n > 2$ and $t \le T$ there does not exist UMVUE of $R(t)$. Not to obscure the essential steps of the reasoning, we first prove some necessary results in the following lemmas.

LEMMA 2. *For $t \le T$, $g(V)$ is an unbiased estimator of $R(t)$ satisfying (40) for every unbiased estimator $h(V)$ of zero only if for $x_{(1)} < t$,*

$$g(2, x_{(1)}, s_2) = \frac{n-k-1}{n} \tag{41}$$

for $n > 2$ and further

$$g(3, x_{(1)}, s_3) = \frac{(n-k-1)(n-k-2)}{n(n-2)} \tag{42}$$

for $n > 3$, where $k = k(x_{(1)}) = (t - x_{(1)})/(T - x_{(1)})$.

Proof. Let $t \le T$, $n > 2$ and $g(V)$ be an unbiased estimator of $R(t)$ such that $h^*(V) = g(V)h(V)$ satisfies (34)–(38) with h replaced by h^* for every $h(V)$ satisfying (34)–(38). Then for $x_{(1)} < t$ and $d = 2$, (22) and (36) imply

$$\int_0^{T-x_{(1)}} g(2, x_{(1)}, s_2) f_2(s_2, T - x_{(1)})\, ds_2 = \frac{(n-k-1)(T - x_{(1)})}{n}. \tag{43}$$

and

$$\int_0^{T-x_{(1)}} h^*(2, x_{(1)}, s_2) f_2(s_2, T - x_{(1)})\, ds_2 = 0 \tag{44}$$

for every h satisfying

$$\int_0^{T-x_{(1)}} h(2, x_{(1)}, s_2) f_2(s_2, T - x_{(1)})\, ds_2 = 0. \tag{45}$$

By the same arguments used to prove Theorem 3, (43)–(45) and (10) imply

$$g(2, x_{(1)}, s_2) = \frac{(n-k-1)(T - x_{(1)})}{n \int_0^{T-x_{(1)}} f_2(s_2, T - x_{(1)})\, ds_2} = \frac{n-k-1}{n},$$

and this proves (41). If further $n > 3$, then for $t < x_{(1)}$, $d = 3$ and $h(2, x_{(1)}, s_2) = 0$, (22) and (36) along with (41) imply

$$(46)\qquad \int\limits_0^{2(T-x_{(1)})} g(3, x_{(1)}, s_3) f_3(s_3, T - x_{(1)})\, ds_3 = \frac{(n-k-1)(n-k-2)(T-x_{(1)})^2}{n(n-2)}$$

and

$$(47)\qquad \int\limits_0^{2(T-x_{(1)})} h^*(3, x_{(1)}, s_3) f_3(s_3, T - x_{(1)})\, ds_3 = 0$$

for every h satisfying

$$(48)\qquad \int\limits_0^{2(T-x_{(1)})} h(3, x_{(1)}, s_3) f_3(s_3, T - x_{(1)})\, ds_3 = 0.$$

As before, (46)–(48) and (11) imply

$$\begin{aligned} g(3, x_{(1)}, s_3) &= \frac{(n-k-1)(n-k-2)(T-x_{(1)})^2}{n(n-2)\int_0^{2(T-x_{(1)})} f_3(s_3, T - x_{(1)})\, ds_3} \\ &= \frac{(n-k-1)(n-k-2)}{n(n-2)}, \end{aligned}$$

which proves (42).

Lemma 3. *For $n > 2$ and $t \le T$, an unbiased estimator $g(V)$ of $R(t)$ satisfying* (41) *and* (42) *does not satisfy* (40) *for $h(V) = h_0(V)$, where $h_0(V)$ satisfies* (34)–(39).

Proof. Let $t \le T$ and suppose that an unbiased estimator $g(V)$ of $R(t)$ satisfying (41) and (42) also satisfies (40) for $h(V) = h_0(V)$.

Consider first $n = 3$. By (10), (11), (25) and (38), it then follows that for $x_{(1)} < t$ and $T - x_{(1)} < s_3 < 2(T - x(1))$,

$$(49)\qquad h_0(3, x_{(1)}, s_3) \ne 0,$$

$$(50)\qquad g(3, x_{(1)}, s_3) h_0(3, x_{(1)}, s_3) f_3(s_3, T - x_{(1)}) = \frac{(2-k)(2(T-x_{(1)}) - s_3)}{3} h_0(3, x_{(1)}, s_3),$$

$$(51)\qquad g(3, x_{(1)}, s_3) f_3(s_3, T - x_{(1)}) = \frac{2(1-k)(2(T-x_{(1)}) - s_3)}{3}.$$

Clearly (49) and (50) contradict (51) and hence $g(V)$ cannot satisfy (40) for $h(V) = h_0(V)$.

Consider now $n > 3$ and assume $x_{(1)} < t$. For $d = 3$, (36) then implies

$$\sum_{c=2}^{3} \frac{(-1)^{3-c}\binom{n-1}{c-1}\binom{n-c-1}{3-c}}{\Gamma(n-c)} \int_0^{(c-1)(T-x_{(1)})} h_0(c, x_{(1)}, s_c)$$
$$\times\, g(c, x_{(1)}, s_c)(s_c + (n-c)(T - x_{(1)}))^{3-c} f_c(s_c, T - x_{(1)})\, ds_c$$
$$= \frac{(n-k-1)(n-k-2)}{n(n-2)} \sum_{c=2}^{3} \frac{(-1)^{3-c}\binom{n-1}{c-1}\binom{n-c-1}{3-c}}{\Gamma(n-c)}$$
$$\times \int_0^{(c-1)(T-x_{(1)})} h_0(c, x_{(1)}, s_c) g(c, x_{(1)}, s_c)$$
$$\times\, (s_c + (n-c)(T - x_{(1)}))^{3-c} f_c(s_c, T - x_{(1)})\, ds_c$$
$$- \frac{k(n-1)(n-3)(n-k-1)}{n(n-2)\Gamma(n-2)}$$
$$\times \int_0^{T-x_{(1)}} h_0(c, x_{(1)}, s_c)(s_2 + (n-2)(T - x_{(1)})) f_2(s_2, T - x_{(1)})\, ds_2$$
$$= - \frac{k(n-1)(n-3)(n-k-1)}{n(n-2)\Gamma(n-2)}$$
$$\times \int_0^{T-x_{(1)}} h_0(c, x_{(1)}, s_c)(s_2 + (n-2)(T - x_{(1)}))\, ds_2,$$

which is not zero, and this contradicts (36) with $h(V)$ replaced by $g(V)h_0(V)$ for $d = 3$. Hence, $g(V)$ can not satisfy (40) for $h(V) = h_0(V)$. This completes the proof of the lemma.

It follows from Lemmas 2 and 3 that for $n > 2$ and $t \le T$, there does not exist any unbiased estimator $g(V)$ of $R(t)$ satisfying (40) for every unbiased estimator $h(V)$ of zero, and hence by Theorem 3, there does not exist UMVUE of $R(t)$. The results obtained above are summarized in the following theorem.

THEOREM 4. *For* $\exp(\mu, \lambda)$ *distribution, there exists UMVUE of* $R(t)$ *under time censored sampling if and only if* $n = 1, 2$ *and* $t \le T$. *Also for* $n = 1, 2$ *and* $t \le T$, $\hat{R}(t)$ *defined in* (3) *is the UMVUE of* $R(t)$.

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