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STARLIKENESS CRITERIA FOR ODD SYMMETRIC ANALYTIC FUNCTIONS

Abstract. We investigate some starlikeness conditions for odd symmetric analytic functions defined in the unit disc.

1. Introduction. Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$.

In [4], Sakaguchi defined the class of starlike functions with respect to symmetrical points as follows:

Let $f \in \mathcal{A}$. Then f is said to be *starlike with respect to symmetrical points* in E if

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0, \qquad z \in E.$$

Obviously, such functions form a subclass of close-to-convex functions and hence are univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin (see [4]).

We denote by $S_s(\alpha)$ the class of univalent starlike functions with respect to symmetrical points of order α; that is, $f \in S_s(\alpha)$ if and only if

$$\operatorname{Re} \frac{2zf'(z)}{f(z) - f(-z)} > \alpha$$

for some α, $0 \le \alpha < 1$. This class was first defined by Das and Singh [1] (see also [2]).

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Similarly a function $f \in \mathcal{A}$ is said to be *starlike of order* α $(0 \leq \alpha < 1)$ in E if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \qquad z \in E,$$

we denote by $S^*(\alpha)$ the class of all such functions.

LEMMA 1.1 (Jack [3]). *Suppose $w(z)$ is a nonconstant analytic function in E with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in E$ on the circle $|z| = r < 1$, then*

$$z_0 w'(z_0) = k w(z_0),$$

where $k \geq 1$ is some real number.

2. Main results

THEOREM 2.1. *Let $f \in \mathcal{A}$ and suppose that*

$$\phi(z) = \frac{f(z) - f(-z)}{2} \tag{2.1}$$

is an odd function. If

$$\left| \frac{z\phi'(z)}{\phi(z)} - 1 \right|^{\gamma} \left| \frac{z\phi''(z)}{\phi'(z)} \right|^{\beta} < \Psi(\alpha, \beta, \gamma), \qquad z \in E, \tag{2.2}$$

for some real numbers α, β and γ such that $0 \leq \alpha < 1$, $\beta \geq 0$, $\gamma \geq 0$, and $\beta + \gamma > 0$, where

$$\Psi(\alpha, \beta, \gamma) = \begin{cases} (1-\alpha)^{\gamma}(3/2 - \alpha)^{\beta}, & 0 \leq \alpha < 1/2, \\ (1-\alpha)^{\beta+\gamma} 2^{\beta}, & 1/2 \leq \alpha < 1, \end{cases}$$

then $\phi \in S^*(\alpha)$.

Proof. CASE (i). Let $0 \leq \alpha < 1/2$. Differentiating (2.1) logarithmically, we have

$$\begin{aligned} \frac{z\phi'(z)}{\phi(z)} &= \frac{zf'(z)}{f(z) - f(-z)} + \frac{zf'(-z)}{f(z) - f(-z)} \\ &= \frac{1}{2}[p_1(z) + p_2(z)], \qquad p_1(z), p_2(z) \in \mathcal{P}, \end{aligned}$$

where $\mathcal{P}$ is the well known class of functions with positive real part.

Set

$$\frac{z\phi'(z)}{\phi(z)} = \frac{1 + (1 - 2\alpha) w(z)}{1 - w(z)}, \qquad z \in E. \tag{2.3}$$

We note that w is analytic in E, $w(0) = 0$ and $w(z) \neq 1$ in E. Taking the logarithmic derivative of (2.3), we have

$$1+\frac{z\phi''(z)}{\phi'(z)}-\frac{z\phi'(z)}{\phi(z)}=\frac{(1-2\alpha)zw'(z)}{1+(1-2\alpha)w(z)}+\frac{zw'(z)}{1-w(z)},$$

$$1+\frac{z\phi''(z)}{\phi'(z)}=\frac{1+(1-2\alpha)w(z)}{1-w(z)}+\frac{2(1-\alpha)zw'(z)}{[1+(1-2\alpha)w(z)](1-w(z))}.$$

This implies that

$$\frac{z\phi''(z)}{\phi'(z)}=\frac{2(1-\alpha)w(z)}{1-w(z)}+\frac{2(1-\alpha)zw'(z)}{[1+(1-2\alpha)w(z)](1-w(z))}$$

and

$$\frac{z\phi'(z)}{\phi(z)}-1=\frac{2(1-\alpha)w(z)}{1-w(z)}.$$

Thus, we have

$$\begin{aligned}
&\left|\frac{z\phi'(z)}{\phi(z)}-1\right|^{\gamma}\left|\frac{z\phi''(z)}{\phi'(z)}\right|^{\beta}\\
&\quad=\left|\frac{2(1-\alpha)w(z)}{1-w(z)}\right|^{\gamma}\left|\frac{2(1-\alpha)w(z)}{1-w(z)}+\frac{2(1-\alpha)zw'(z)}{[1+(1-2\alpha)w(z)(1-w(z))}\right|^{\beta}\\
&\quad=\left|\frac{2(1-\alpha)w(z)}{1-w(z)}\right|^{\beta+\gamma}\left|1+\frac{zw'(z)}{[1+(1-2\alpha)w(z)](w(z))}\right|^{\beta}.
\end{aligned}$$

Suppose that there exists a point $z_0\in E$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1.$$

Then by using Lemma 1.1, we have $w(z_0)=e^{i\theta}$ for some $0<\theta\leq 2\pi$ and $z_0w'(z_0)=kw(z_0)$, $k\geq 1$. Therefore

$$\begin{aligned}
\left|\frac{z_0\phi'(z_0)}{\phi(z_0)}-1\right|^{\gamma}\left|\frac{z_0\phi''(z_0)}{\phi'(z_0)}\right|^{\beta}&=\left|\frac{2(1-\alpha)w(z_0)}{1-w(z_0)}\right|^{\beta+\gamma}\\
&\quad\times\left|1+\frac{kw'(z_0)}{[1+(1-2\alpha)w(z_0)](w(z_0))}\right|^{\beta}\\
&=\frac{2^{\{(\beta+\gamma)(1-\alpha)\}^{(\beta+\gamma)}}}{|1-e^{i\theta}|^{\beta+\gamma}}\left|1+\frac{k}{[1+(1-2\alpha)e^{i\theta}]}\right|^{\beta}\\
&\geq(1-\alpha)^{(\beta+\gamma)}\left(1+\frac{k}{[2(1-\alpha)]}\right)^{\beta}\\
&\geq(1-\alpha)^{(\beta+\gamma)}\left(1+\frac{1}{[2(1-\alpha)]}\right)^{\beta}\\
&=(1-\alpha)^{\gamma}\left(\frac{3}{2}-\alpha\right)^{\beta},
\end{aligned}$$

which contradicts (2.2) for $0 \leq \alpha < 1/2$. Therefore, we must have $|w(z)| < 1$ for all $z \in E$, and hence $\phi(z) = \frac{f(z)-f(-z)}{2} \in S^*(\alpha)$.

Case (ii). Suppose $1/2 < \alpha < 1$. Let $w(z)$ be defined by

$$\frac{z\phi'(z)}{\phi(z)} = \frac{\alpha}{\alpha-(1-\alpha)w(z)}, \qquad z \in E \quad \left[\text{with } \phi(z) = \frac{f(z)-f(-z)}{2}\right],$$

where $w(z) \neq \frac{\alpha}{1-\alpha}$ in E. Then $w(z)$ is analytic in E and $w(0) = 0$. Using the same arguments as in Case (i), we obtain

$$1 + \frac{z\phi''(z)}{\phi'(z)} - \frac{z\phi'(z)}{\phi(z)} = \frac{(1-\alpha)zw'(z)}{[\alpha-(1-\alpha)w(z)]}.$$

This implies that

$$1 + \frac{z\phi''(z)}{\phi'(z)} = \frac{\alpha}{\alpha-(1-\alpha)w(z)} + \frac{(1-\alpha)zw'(z)}{[\alpha-(1-\alpha)w(z)]}.$$

Thus, we have

$$\begin{aligned}
&\left|\frac{z\phi'(z)}{\phi(z)} - 1\right|^{\gamma}\left|\frac{z\phi''(z)}{\phi'(z)}\right|^{\beta} \\
&\quad = \left|\frac{(1-\alpha)w(z)}{\alpha-(1-\alpha)w(z)}\right|^{\gamma}\left|\frac{(1-\alpha)w(z)}{\alpha-(1-\alpha)w(z)} + \frac{(1-\alpha)zw'(z)}{[\alpha-(1-\alpha)w(z)]}\right|^{\beta} \\
&\quad = \left|\frac{(1-\alpha)w(z)}{\alpha-(1-\alpha)w(z)}\right|^{\beta+\gamma}\left|1 + \frac{zw'(z)}{w(z)}\right|^{\beta} \\
&\quad = \left|\frac{(1-\alpha)w(z)}{\alpha-(1-\alpha)w(z)}\right|^{\beta+\gamma} |w(z)|^{\gamma}|w(z)+zw'(z)|^{\beta}.
\end{aligned}$$

Suppose that there exists a point $z_0 \in E$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1.$$

Then by applying Lemma 1.1, we have $w(z_0) = e^{i\theta}$ and $z_0w'(z_0) = kw(z_0)$, $k \geq 1$. Therefore,

$$\begin{aligned}
\left|\frac{z_0\phi'(z_0)}{\phi(z_0)} - 1\right|^{\gamma}\left|\frac{z_0\phi''(z_0)}{\phi'(z_0)}\right|^{\beta} &= \left|\frac{(1-\alpha)w(z_0)}{\alpha-(1-\alpha)w(z_0)}\right|^{\beta+\gamma}\left|1 + \frac{kw(z_0)}{w(z_0)}\right|^{\beta} \\
&= \frac{(1-\alpha)^{\beta+\gamma}}{|\alpha-(1-\alpha)e^{i\theta}|^{\beta+\gamma}}(1+k)^{\beta} \\
&\geq (1-\alpha)^{\beta+\gamma}(1+1)^{\beta} = (1-\alpha)^{\beta+\gamma}2^{\beta},
\end{aligned}$$

which contradicts (2.2) for $1/2 < \alpha < 1$. Therefore, we must have $|w(z)| < 1$ for all $z \in E$, and hence $\phi(z) = \frac{f(z)-f(-z)}{2} \in S^*(\alpha)$. This completes the proof of Theorem 2.1. ∎

Corollary 2.2. *Let* $\beta = 1$, $\gamma = 0$ *and let* $\phi(z)$ *be defined by* (2.1). *If*

$$\left|\frac{z\phi''(z)}{\phi'(z)}\right| < \begin{cases} 3/2 - \alpha, & 0 \le \alpha < 1/2, \\ 2(1-\alpha), & 1/2 \le \alpha < 1, \end{cases} \qquad z \in E,$$

for some $0 \le \alpha < 1$, *then*

$$\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) > \alpha, \qquad z \in E.$$

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