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EXISTENCE OF SOLUTIONS TO THE NONSTATIONARY STOKES SYSTEM IN $H^{2,1}_{-\mu}$, $\mu \in (0,1)$, IN A DOMAIN WITH A DISTINGUISHED AXIS. PART 1. EXISTENCE NEAR THE AXIS IN 2d

Abstract. We consider the nonstationary Stokes system with slip boundary conditions in a bounded domain which contains some distinguished axis. We assume that the data functions belong to weighted Sobolev spaces with the weight equal to some power function of the distance to the axis. The aim is to prove the existence of solutions in corresponding weighted Sobolev spaces. The proof is divided into three parts. In the first, the existence in 2d in weighted spaces near the axis is shown. In the second, we show an estimate in 3d in weighted spaces near the axis. Finally, in the third, the existence in a bounded domain is proved. This paper contains the first part of the proof.

1. Introduction. We consider the problem

$$
\begin{aligned}
& v_{,t} - \nu \operatorname{div} \mathbb{D}(v) + \nabla p = f && \text{in } \Omega^T = \Omega \times (0,T),\\
& \operatorname{div} v = 0 && \text{in } \Omega^T,\\
& v \cdot \overline{n} = 0 && \text{on } S^T = S \times (0,T),\\
& \overline{n} \cdot \mathbb{D}(v) \cdot \overline{\tau}_\alpha + \gamma v \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\
& v|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

in a bounded domain $\Omega \subset \mathbb{R}^3$, where v is the velocity of the fluid, p the pressure, f the external force field, $\nu > 0$ the constant viscosity coefficient, $\gamma > 0$ the constant slip coefficient, $\overline{n}$ the unit outward vector normal to the boundary S, and $\overline{\tau}_1$, $\overline{\tau}_2$ unit tangent vectors to S. By $\mathbb{D}(v)$ we denote the

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dilatation tensor of the form

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}. \tag{1.2}$$

Finally, the dot denotes the scalar product in $\mathbb{R}^3$. Let us assume that the domain Ω contains a distinguished axis L.

To formulate the main result we introduce some weighted spaces. Let $\mu \in \mathbb{R}$ and $\varrho(x) = \operatorname{dist}(x, L)$. Then

$$\begin{aligned} H^{2,1}_\mu(\Omega^T) = \Big\{u : \|u\|_{H^{2,1}_\mu(\Omega^T)} = \Big[\int_{\Omega^T} [(u^2_{,xx} + u^2_{,t})\varrho^{2\mu}(x) \\ + u^2_{,x}\varrho^{2\mu-2}(x) + u^2\varrho^{2\mu-4}(x)]\,dx\,dt\Big]^{1/2} < \infty\Big\}, \end{aligned}$$

$$L_{2,\mu}(\Omega^T) = \Big\{u : \|u\|_{L_{2,\mu}(\Omega^T)} = \Big(\int_{\Omega^T} u^2\varrho^{2\mu}(x)\,dx\,dt\Big)^{1/2} < \infty\Big\},$$

$$H^1_\mu(\Omega) = \Big\{u : \|u\|_{H^1_\mu(\Omega)} = \Big[\int_{\Omega} [u^2_{,x}\varrho^{2\mu}(x) + u^2\varrho^{2\mu-2}(x)]\,dx\Big]^{1/2} < \infty\Big\}.$$

In the above definitions and throughout the paper we do not distinguish vector and scalar valued functions. Let $u = (u_1, \ldots, u_n)$, $x = (x_1, \ldots, x_n)$. Then $u^2 = u_1^2 + \cdots + u_n^2$ and $u^2_{,x} = \sum_{i,j=1}^n u^2_{i,x_j}$ where $u_{,x} = \partial_x u$.

The aim of this paper and of [5, 3] is to prove the following result:

Theorem A. *Assume that* $\mu \in (0,1)$, $f \in L_{2,-\mu}(\Omega^T)$, $v_0 \in H^1_{-\mu}(\Omega)$, $S \in C^2$. *Then there exists a solution to problem* (1.1) *such that* $v \in H^{2,1}_{-\mu}(\Omega^T)$, $p \in L_2(0,T; H^1_{-\mu}(\Omega))$ *and a constant* c_1 *depending only on* μ, ν, S *such that*

$$\|v\|_{H^{2,1}_{-\mu}(\Omega^T)} + \|p\|_{L_2(0,T;H^1_{-\mu}(\Omega))} \le c_1(\|f\|_{L_{2,-\mu}(\Omega^T)} + \|v_0\|_{H^1_{-\mu}(\Omega)}). \tag{1.3}$$

We prove the theorem by using regularization of weak solutions (see [2]). For this purpose we have to examine problem (1.1) locally. Hence, problem (1.1) is considered in four different subdomains of Ω: near the axis L, near points where L meets S, near S at a positive distance from L and in interior subdomains at a positive distance from L and S.

In this paper we examine the 2-dimensional stationary Stokes problem (1.1) near L. Introducing a local Cartesian coordinate system (x_1, x_2, x_3) such that L is the x_3-axis we reduce problem (1.1) to the two-dimensional problems

$$\begin{aligned} &-\nu(\partial^2_{x_1} + \partial^2_{x_2})v_i + \partial_{x_i}p = f_i - v_{i,t} + \nu\partial^2_{x_3}v_i \equiv g_i, \quad i = 1,2, \\ &v_{1,x_1} + v_{2,x_2} = h, \\ &v_i|_{\varphi=0} = v_i|_{\varphi=2\pi}, \quad i = 1,2, \\ &\begin{pmatrix} v_{1,x_2} + v_{2,x_1} \\ 2\nu v_{2,x_2} - p \end{pmatrix}\Big|_{\varphi=0} = \begin{pmatrix} v_{1,x_2} + v_{2,x_1} \\ 2\nu v_{2,x_2} - p \end{pmatrix}\Big|_{\varphi=2\pi}, \end{aligned} \tag{1.4}$$

where r,φ are the polar coordinates in $\mathbb{R}^2$, and

$$
\begin{aligned}
&-\nu(\partial_{x_1}^2+\partial_{x_2}^2)v_3 = f_3 - v_{3,t} + \nu\partial_{x_3}^2 v_3 \equiv g_3,\\
&v_3|_{\varphi=0} = v_3|_{\varphi=2\pi},\\
&\frac{\partial v_3}{\partial\varphi}\bigg|_{\varphi=0} = \frac{\partial v_3}{\partial\varphi}\bigg|_{\varphi=2\pi},
\end{aligned}
\tag{1.5}
$$

where the derivatives with respect to t and x_3 are treated as given.

The main results of this paper are the following:

THEOREM 1. *Let $g_i\in L_{2,\mu}(\mathbb{R}^2)$, $i=1,2$, $h\in H^1_\mu(\mathbb{R}^2)$, $\mu\in\mathbb{R}$, $\mu\notin\mathbb{Z}$. Then there exists a unique solution to problem* (1.4) *such that $v_i\in H^2_\mu(\mathbb{R}^2)$, $i=1,2$, $p\in H^1_\mu(\mathbb{R}^2)$ and*

$$
\sum_{i=1}^2 \|v_i\|_{H^2_\mu(\mathbb{R}^2)} + \|p\|_{H^1_\mu(\mathbb{R}^2)} \le c\Big(\sum_{i=1}^2 \|g_i\|_{L_{2,\mu}(\mathbb{R}^2)} + \|h\|_{H^1_\mu(\mathbb{R}^2)}\Big).
\tag{1.6}
$$

THEOREM 2. *Let $g_3\in L_{2,\mu}(\mathbb{R}^2)$, $\mu\in\mathbb{R}$, $\mu\notin\mathbb{Z}$. Then there exists a unique solution to problem* (1.5) *such that $v_3\in H^2_\mu(\mathbb{R}^2)$ and*

$$
\|v_3\|_{H^2_\mu(\mathbb{R}^2)} \le c\|g_3\|_{L_{2,\mu}(\mathbb{R}^2)}.
\tag{1.7}
$$

To prove Theorem A we need only the case $\mu\in(-1,0)$.

The above two theorems are proved in Section 2. They are crucial for deriving estimates in weighted Sobolev spaces near the distinguished axis L in 3d case. Then Theorem A follows because existence results near S and in an interior subdomain, both at a positive distance from L, are well known. In those cases we do not need weighted spaces. To prove Theorems 1 and 2 we use some techniques from [1], however, we have to prove the existence directly because the stationary 2d Stokes system in $\mathbb{R}^2$ has not been considered there.

Finally, we discuss in more detail the norm of the space $H^k_\mu(\mathbb{R}^2)$. We have

$$
\|u\|_{H^k_\mu(\mathbb{R}^2)} = \bigg(\sum_{|\alpha|\le k}\int_{\mathbb{R}^2} |D_x^\alpha u|^2 |x|^{2(\mu+|\alpha|-k)}\,dx\bigg)^{1/2},
\tag{1.8}
$$

where $k\in\mathbb{N}\cup\{0\}$, $\mu\in\mathbb{R}$, $\alpha=(\alpha_1,\alpha_2)$ is a multiindex and $|\alpha|=\alpha_1+\alpha_2$.

Passing to the polar coordinates (r,φ) (see Section 2) and next to the variables $\tau=-\log r$, φ we obtain

$$
\|u\|_{H^k_\mu(\mathbb{R}^2)} = \Big(\sum_{|\alpha|\le k}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\varphi\, |\partial_\tau^{\alpha_1}\partial_\varphi^{\alpha_2} u|^2 e^{-2(\mu+1-k)\tau}\Big)^{1/2}.
\tag{1.9}
$$

Passing to the Fourier transform

$$
u(\tau,\varphi) = \int_{-\infty}^{\infty} e^{i\lambda\tau}\widetilde{u}(\lambda,\varphi)\,d\lambda
$$

we deduce from the Parseval identity that the norm (1.9) is equivalent to the norm

$$\|u\|_{H^k_\mu(\mathbb{R}^2)} = \Big(\sum_{|\alpha|\le k} \int\limits_{-\infty+ih}^{+\infty+ih} d\lambda \int\limits_0^{2\pi} |\lambda|^{2\alpha_1} \Big|\frac{\partial^{\alpha_2}\widetilde{u}}{\partial\varphi^{\alpha_2}}\Big|^2 d\varphi\Big)^{1/2}, \tag{1.10}$$

where $h = \mu + 1 - k$.

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2. Stationary 2d Stokes system near L. In a neighbourhood of L we introduce local Cartesian coordinates such that L is the x_3-axis and x_1, x_2 are coordinates in the plane perpendicular to L. We introduce the cylindrical coordinates r, φ, z by $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$.

Let us introduce a local cylinder C_R with axis L such that

$$C_R = \{x \in \mathbb{R}^3 : r < R,\ z \in (-a, a),\ \varphi \in [0, 2\pi]\},$$

where R and a are given and $C_R \cap S = \emptyset$ (for more details see [3]).

To apply Kondrat'ev's methods [1] we localize problem (1.1), by multiplying it by a function from a partition of unity with support in C_R, and formulate it as a problem in the angle 2π. Therefore we consider problem (1.1) in the form (see [4])

$$\begin{aligned} & v_{,t} - \operatorname{div} \mathbb{T}(v,p) = f_1, \\ & \operatorname{div} v = h_1 \\ & v|_{\Gamma_0} = v|_{\Gamma_{2\pi}}, \\ & \overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_0} = -\overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_{2\pi}}, \\ & v|_{\partial C_R} = 0, \quad p|_{\partial C_R} = 0, \quad f_1|_{\partial C_R} = 0, \quad h_1|_{\partial C_R} = 0, \end{aligned} \tag{2.1}$$

where $\Gamma_0 = \Gamma_{2\pi} = \{x \in \mathbb{R}^3 : x_2 = 0\}$, $\overline{n}|_{\Gamma_0} = (0,-1,0)$, $\overline{n}|_{\Gamma_{2\pi}} = (0,1,0)$, and the last condition follows from localization of problem (1.1). Moreover, $\mathbb{T}(v,p)$ denotes the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - pI,$$

where I is the unit matrix.

In view of $(2.1)_5$ we can extend all functions by zero to $\mathbb{R}^3$. Then condition $(2.1)_5$ can be omitted. We denote the resulting problem by $(2.1)'$.

To prove Theorems 1 and 2 we examine problem $(2.1)'$ in the plane perpendicular to the axis L and for a fixed t. For this purpose we apply the Laplace–Fourier transform

$$u(x,t) = \int\limits_0^\infty ds \int\limits_{-\infty}^\infty d\xi\, e^{ix_3\xi+st}\widetilde{u}(x',\xi,s), \tag{2.2}$$

where $s = i\xi_0 + \gamma$, $\operatorname{re} s = \gamma > 0$, $\xi_0 \in \mathbb{R}$, to problem $(2.1)'$ separating it into two two-dimensional problems

$$
(2.3)\qquad
\begin{aligned}
&-\nu\Delta'\widetilde{v}_j + \widetilde{p}_{,x_j} = \widetilde{f}_{1j} - q\widetilde{v}_j \equiv \widetilde{g}_j, \quad j = 1,2, && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_{1,x_1} + \widetilde{v}_{2,x_2} = -\xi\widetilde{v}_3 + \widetilde{h}_1 \equiv \widetilde{k}, && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_j|_{\gamma_0} = \widetilde{v}_j|_{\gamma_{2\pi}}, \quad j = 1,2, && x_2 = 0,\\
&\begin{pmatrix} \widetilde{v}_{1,x_2} + \widetilde{v}_{2,x_1} \\ 2\nu\widetilde{v}_{2,x_2} - \widetilde{p} \end{pmatrix}\bigg|_{\gamma_0} = \begin{pmatrix} \widetilde{v}_{1,x_2} + \widetilde{v}_{2,x_1} \\ 2\nu\widetilde{v}_{2,x_2} - \widetilde{p} \end{pmatrix}\bigg|_{\gamma_{2\pi}}, && x_2 = 0,
\end{aligned}
$$

where $q = s + \nu\xi^2$, $\gamma_0 = \gamma_{2\pi} = \{x \in \mathbb{R}^2 : x_2 = 0\}$, $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$ and

$$
(2.4)\qquad
\begin{aligned}
&-\nu\Delta'\widetilde{v}_3 = \widetilde{f}_{13} - q\widetilde{v}_3 - i\xi\widetilde{p} \equiv \widetilde{g}_3 && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_3|_{\gamma_0} = \widetilde{v}_3|_{\gamma_{2\pi}}, && x_2 = 0,\\
&\widetilde{v}_{3,x_2}|_{\gamma_0} = \widetilde{v}_{3,x_2}|_{\gamma_{2\pi}}, && x_2 = 0.
\end{aligned}
$$

First we examine problem (2.3). Using the polar coordinates we introduce $u_r' = \widetilde{v} \cdot \overline{e}_r$, $u_\varphi' = \widetilde{v} \cdot \overline{e}_\varphi$, $d_r' = \widetilde{g} \cdot \overline{e}_r$, $d_\varphi' = \widetilde{g} \cdot \overline{e}_\varphi$, $h_0 = \widetilde{k}$, $\overline{e}_r = (\cos\varphi, \sin\varphi)$, $\overline{e}_\varphi = (-\sin\varphi, \cos\varphi)$; the dots denote the scalar product in $\mathbb{R}^2$. Then problem (2.3) takes the form

$$
(2.5)\qquad
\begin{aligned}
&-\nu\left[\frac{1}{r}\,\partial_r(ru_{r,r}') + \frac{1}{r^2}\,u_{r,\varphi\varphi}' - \frac{1}{r^2}\,u_r' - \frac{2}{r^2}\,u_{\varphi,\varphi}'\right] + p_{,r} = d_r',\\
&-\nu\left[\frac{1}{r}\,\partial_r(ru_{\varphi,r}') + \frac{1}{r^2}\,u_{\varphi,\varphi\varphi}' - \frac{1}{r^2}\,u_\varphi' + \frac{2}{r^2}\,u_{r,\varphi}'\right] + \frac{1}{r}\,p_{,\varphi} = d_\varphi',\\
&u_{r,r}' + \frac{1}{r}\,u_r' + \frac{1}{r}\,u_{\varphi,\varphi}' = h_0,\\
&\left(\frac{1}{r}\,u_{r,\varphi}' + u_{\varphi,r}' - \frac{1}{r}\,u_\varphi'\right)\bigg|_{\varphi=0} = \left(\frac{1}{r}\,u_{r,\varphi}' + u_{\varphi,r}' - \frac{1}{r}\,u_\varphi'\right)\bigg|_{\varphi=2\pi},\\
&\left[\frac{2\nu}{r}\,(u_{\varphi,\varphi}' + u_r') - p\right]\bigg|_{\varphi=0} = \left[\frac{2\nu}{r}\,(u_{\varphi,\varphi}' + u_r') - p\right]\bigg|_{\varphi=2\pi}.
\end{aligned}
$$

Introduce the new variable

$$
(2.6)\qquad q' = \frac{rp}{\nu}
$$

and the quantities

$$
(2.7)\qquad
\begin{aligned}
u_r(\tau,\varphi) &= u_r'(e^{-\tau},\varphi), & u_\varphi(\tau,\varphi) &= u_\varphi'(e^{-\tau},\varphi),\\
q(\tau,\varphi) &= q'(e^{-\tau},\varphi), & d_r(\tau,\varphi) &= e^{-2\tau}d_r'(e^{-\tau},\varphi),\\
d_\varphi(\tau,\varphi) &= e^{-2\tau}d_\varphi'(e^{-\tau},\varphi), & h(\tau,\varphi) &= e^{-\tau}h_0(e^{-\tau},\varphi),
\end{aligned}
$$

where τ is the variable introduced by the Fourier transform

$$u(\tau,\varphi)=\int\limits_{-\infty}^{\infty} e^{i\lambda\tau}\widetilde{u}(\lambda,\varphi)\,d\lambda,\qquad \tau=-\ln r,\ r=e^{-\tau}. \tag{2.8}$$

Then problem (2.5) takes the form

$$\begin{aligned}
&-[u_{r,\tau\tau}+u_{r,\varphi\varphi}-u_r-2u_{\varphi,\varphi}+q+q_{,r}]=d_r && \text{in } \mathbb{R}^1\times(0,2\pi),\\
&-[u_{\varphi,\tau\tau}+u_{\varphi,\varphi\varphi}-u_\varphi+2u_{r,\varphi}-q_{,\varphi}]=d_\varphi && \text{in } \mathbb{R}^1\times(0,2\pi),\\
&-u_{r,\tau}+u_{\varphi,\varphi}+u_r=h && \text{in } \mathbb{R}^1\times(0,2\pi),\\
&u_r|_{\varphi=0}=u_r|_{\varphi=2\pi} && \text{on } \mathbb{R}^1,\\
&u_\varphi|_{\varphi=0}=u_\varphi|_{\varphi=2\pi} && \text{on } \mathbb{R}^1,\\
&(-u_{\varphi,\tau}+u_{r,\varphi}-u_\varphi)|_{\varphi=0}=(-u_{\varphi,\tau}+u_{r,\varphi}-u_\varphi)|_{\varphi=2\pi} && \text{on } \mathbb{R}^1,\\
&[2(u_{\varphi,\varphi}+u_r)-q]|_{\varphi=0}=[2(u_{\varphi,\varphi}+u_r)-q]|_{\varphi=2\pi} && \text{on } \mathbb{R}^1.
\end{aligned}\tag{2.9}$$

Applying the Fourier transform (2.8) yields

$$\begin{aligned}
&-\widetilde{u}_{r,\varphi\varphi}+(1+\lambda^2)\widetilde{u}_r+2\widetilde{u}_{\varphi,\varphi}-(1+i\lambda)\widetilde{q}=\widetilde{d}_r && \text{in } [0,2\pi],\\
&-\widetilde{u}_{\varphi,\varphi\varphi}+(1+\lambda^2)\widetilde{u}_\varphi-2\widetilde{u}_{r,\varphi}+\widetilde{q}_{,\varphi}=\widetilde{d}_\varphi && \text{in } [0,2\pi],\\
&\widetilde{u}_{\varphi,\varphi}+(1-i\lambda)\widetilde{u}_r=\widetilde{h} && \text{in } [0,2\pi],\\
&\widetilde{u}_r|_{\varphi=0}=\widetilde{u}_r|_{\varphi=2\pi},\\
&\widetilde{u}_\varphi|_{\varphi=0}=\widetilde{u}_\varphi|_{\varphi=2\pi},\\
&\bigg(\frac{d\widetilde{u}_r}{d\varphi}-(1+i\lambda)\widetilde{u}_\varphi\bigg)\bigg|_{\varphi=0}=\bigg(\frac{d\widetilde{u}_r}{d\varphi}-(1+i\lambda)\widetilde{u}_\varphi\bigg)\bigg|_{\varphi=2\pi},\\
&\bigg[2\bigg(\frac{d\widetilde{u}_\varphi}{d\varphi}+\widetilde{u}_r\bigg)-\widetilde{q}\bigg]\bigg|_{\varphi=0}=\bigg[2\bigg(\frac{d\widetilde{u}_\varphi}{d\varphi}+\widetilde{u}_r\bigg)-\widetilde{q}\bigg]\bigg|_{\varphi=2\pi}.
\end{aligned}\tag{2.10}$$

Solutions of the homogeneous equations $(2.10)_{1,2,3}$ have the form (see [4])

$$\begin{aligned}
\widetilde{u}_r&=c_1\sin(1-i\lambda)\varphi+c_2\cos(1-i\lambda)\varphi+c_3(1+i\lambda)\sin(1+i\lambda)\varphi\\
&\quad+c_4(1+i\lambda)\cos(1+i\lambda)\varphi\equiv\widetilde{u}_r^{(g)},\\
\widetilde{u}_\varphi&=c_1\cos(1-i\lambda)\varphi-c_2\sin(1-i\lambda)\varphi+c_3(1-i\lambda)\cos(1+i\lambda)\varphi\\
&\quad-c_4(1-i\lambda)\sin(1+i\lambda)\varphi\equiv\widetilde{u}_\varphi^{(g)},\\
\widetilde{q}&=4i\lambda c_3\sin(1+i\lambda)\varphi+4i\lambda c_4\cos(1+i\lambda)\varphi\equiv\widetilde{q}^{(g)}.
\end{aligned}\tag{2.11}$$

LEMMA 2.1. *Let* $(1-i\lambda)\varphi=\varphi_1$, $(1+i\lambda)\varphi=\varphi_2$, $(1-i\lambda)\varphi'=\varphi_1'$, $(1+i\lambda)\varphi'=\varphi_2'$, $a_1=\widetilde{d}_\varphi+\widetilde{h}_{,\varphi}$, $a_2=-\widetilde{d}_r+(1+i\lambda)\widetilde{h}$, $a_3=\widetilde{h}$. *Then solutions of equations* $(2.10)_{1,2,3}$ *have the form*

$$\widetilde{u}_r=\widetilde{u}_r^{(g)}+\widetilde{u}_r^{(p)},\qquad \widetilde{u}_\varphi=\widetilde{u}_\varphi^{(g)}+\widetilde{u}_\varphi^{(p)},\qquad \widetilde{q}=\widetilde{q}^{(g)}+\widetilde{q}^{(p)},\tag{2.12}$$

where

$$\begin{aligned}\widetilde{u}_r^{(g)}&=c_1\sin\varphi_1+c_2\cos\varphi_1+c_1(1+i\lambda)\sin\varphi_2+c_4(1+i\lambda)\cos\varphi_2,\\ \widetilde{u}_\varphi^{(g)}&=c_1\cos\varphi_1-c_2\sin\varphi_1+c_3(1-i\lambda)\cos\varphi_2-c_4(1-i\lambda)\sin\varphi_2,\\ \widetilde{q}^{(g)}&=4i\lambda(c_3\sin\varphi_2+c_4\cos\varphi_2),\end{aligned}$$

where c_i, $i=1,\ldots,4$, *are arbitrary constants and* $\widetilde{u}_r^{(p)}$, $\widetilde{u}_\varphi^{(p)}$, $\widetilde{q}^{(p)}$ *are given in* (2.20).

Proof. To prove the lemma we write $(2.10)_{1,2,3}$ in the form

$$\begin{aligned}-\widetilde{u}_{r,\varphi\varphi}+(1+\lambda^2)\widetilde{u}_r-2(1-i\lambda)\widetilde{u}_r-(1+i\lambda)\widetilde{q}&=\widetilde{d}_r-2\widetilde{h},\\ \widetilde{q}_{,\varphi}+(1+\lambda^2)\widetilde{u}_\varphi-(1+i\lambda)\widetilde{u}_{r,\varphi}&=\widetilde{d}_\varphi+\widetilde{h}_{,\varphi},\\ \widetilde{u}_{\varphi,\varphi}+(1-i\lambda)\widetilde{u}_r&=\widetilde{h}.\end{aligned}\tag{2.13}$$

We show (2.12) by variation of constants. We obtain from (2.13) a linear system for the first derivatives $dc_i/d\varphi$, $i=1,2,3,4$. Inserting $(2.11)_1$ in $(2.13)_1$ and looking for the first derivatives of c_i, $i=1,\ldots,4$, we have

$$\begin{aligned}\frac{dc_1}{d\varphi}\sin{(1-i\lambda)\varphi}+\frac{dc_2}{d\varphi}\cos{(1-i\lambda)\varphi}+\frac{dc_3}{d\varphi}(1+i\lambda)\sin{(1+i\lambda)\varphi}&\\ +\frac{dc_4}{d\varphi}(1+i\lambda)\cos{(1+i\lambda)\varphi}&=0,\end{aligned}\tag{2.14}$$

and $(2.13)_1$ also implies

$$\begin{aligned}&\frac{dc_1}{d\varphi}(1-i\lambda)\cos{(1-i\lambda)\varphi}-\frac{dc_2}{d\varphi}(1-i\lambda)\sin{(1-i\lambda)\varphi}\\ &\quad+\frac{dc_3}{d\varphi}(1+i\lambda)^2\cos{(1+i\lambda)\varphi}-\frac{dc_4}{d\varphi}(1+i\lambda)^2\sin{(1+i\lambda)\varphi}\\ &=-\widetilde{d}_r+2\widetilde{h}.\end{aligned}\tag{2.15}$$

Next, $(2.13)_2$ gives

$$\frac{dc_3}{d\varphi}\sin{(1+i\lambda)\varphi}+\frac{dc_4}{d\varphi}\cos{(1+i\lambda)\varphi}=\frac{1}{4i\lambda}(\widetilde{d}_\varphi+\widetilde{h}_{,\varphi}).\tag{2.16}$$

Finally, $(2.13)_3$ yields

$$
\frac{dc_1}{d\varphi}\cos(1-i\lambda)\varphi-\frac{dc_2}{d\varphi}\sin(1-i\lambda)\varphi+\frac{dc_3}{d\varphi}(1-i\lambda)\cos(1+i\lambda)\varphi
-\frac{dc_4}{d\varphi}(1-i\lambda)\sin(1+i\lambda)\varphi=\widetilde{h}. \tag{2.17}
$$

Summarizing, (2.14)–(2.17) imply

$$
\begin{pmatrix}
\sin\varphi_1 & \cos\varphi_1 & (1+i\lambda)\sin\varphi_2 & (1+i\lambda)\cos\varphi_2\\
(1-i\lambda)\cos\varphi_1 & -(1-i\lambda)\sin\varphi_1 & (1+i\lambda)^2\cos\varphi_2 & -(1+i\lambda)^2\sin\varphi_2\\
0 & 0 & \sin\varphi_2 & \cos\varphi_2\\
\cos\varphi_1 & -\sin\varphi_1 & (1-i\lambda)\cos\varphi_2 & -(1-i\lambda)\sin\varphi_2
\end{pmatrix}
\begin{pmatrix} c_{1,\varphi}\\ c_{2,\varphi}\\ c_{3,\varphi}\\ c_{4,\varphi}\end{pmatrix}
=\begin{pmatrix} 0\\ -d_r+2h\\ \frac{1}{4i\lambda}(d_\varphi+h_{,\varphi})\\ h\end{pmatrix},
$$

where $\varphi_1=(1-i\lambda)\varphi$, $\varphi_2=(1+i\lambda)\varphi$. Simplifying, we get

$$
\begin{pmatrix}
\sin\varphi_1 & \cos\varphi_1 & 0 & 0\\
0 & 0 & \cos\varphi_2 & -\sin\varphi_2\\
0 & 0 & \sin\varphi_2 & \cos\varphi_2\\
\cos\varphi_1 & -\sin\varphi_1 & 0 & 0
\end{pmatrix}
\begin{pmatrix} c_{1,\varphi}\\ c_{2,\varphi}\\ c_{3,\varphi}\\ c_{4,\varphi}\end{pmatrix}
=\begin{pmatrix}
\frac{-(1+i\lambda)}{4i\lambda}(\widetilde{d}_\varphi+\widetilde{h}_{,\varphi})\\
\frac{-\widetilde{d}_r+(1+i\lambda)\widetilde{h}}{4i\lambda}\\
\frac{1}{4i\lambda}(\widetilde{d}_\varphi+\widetilde{h}_{,\varphi})\\
\widetilde{h}-\frac{1-i\lambda}{4i\lambda}(-\widetilde{d}_r+(1+i\lambda)\widetilde{h})
\end{pmatrix}
\equiv\begin{pmatrix}
\frac{-1+i\lambda}{4i\lambda}a_1\\
\frac{a_2}{4i\lambda}\\
\frac{a_1}{4i\lambda}\\
h-\frac{1-i\lambda}{4i\lambda}a_2
\end{pmatrix}. \tag{2.18}
$$

Solving (2.18) yields

$$
\begin{aligned}
c_1&=\int_0^\varphi\Big[-\frac{1+i\lambda}{4i\lambda}\,a_1\sin\varphi_1'+\Big(\widetilde{h}-\frac{1-i\lambda}{4i\lambda}\,a_2\Big)\cos\varphi_1'\Big]\,d\varphi',\\
c_2&=\int_0^\varphi\Big[-\frac{1+i\lambda}{4i\lambda}\,a_1\cos\varphi_1'-\Big(\widetilde{h}-\frac{1-i\lambda}{4i\lambda}\,a_2\Big)\sin\varphi_1'\Big]\,d\varphi',\\
c_3&=\frac{1}{4i\lambda}\int_0^\varphi(a_2\cos\varphi_2'+a_1\sin\varphi_2')\,d\varphi',\\
c_4&=\frac{1}{4i\lambda}\int_0^\varphi(-a_2\sin\varphi_2'+a_1\cos\varphi_2't)\,d\varphi',
\end{aligned} \tag{2.19}
$$

In view of (2.11) a particular solution to (2.13) has the form

$$
\begin{aligned}
\widetilde{u}_r^{(p)} &= \int\limits_0^{\varphi}\Big[-\frac{1+i\lambda}{4i\lambda}\,a_1\cos(\varphi_1-\varphi_1')\\
&\quad+\Big(h-\frac{1-i\lambda}{4i\lambda}\,a_2\Big)\sin(\varphi_1-\varphi_1')\Big]\,d\varphi'\\
&\quad+\frac{1+i\lambda}{4i\lambda}\int\limits_0^{\varphi}[a_2\sin(\varphi_2-\varphi_2')+a_1\cos(\varphi_2-\varphi_2')]\,d\varphi',\\
\widetilde{u}_\varphi^{(p)} &= \int\limits_0^{\varphi}\Big[\frac{1+i\lambda}{4i\lambda}\,a_1\sin(\varphi_1-\varphi_1')\\
&\quad+\Big(\widetilde{h}-\frac{1-i\lambda}{4i\lambda}\,a_2\Big)\cos(\varphi_1-\varphi_1')\Big]\,d\varphi'\\
&\quad+\frac{1-i\lambda}{4i\lambda}\int\limits_0^{\varphi}[a_2\cos(\varphi_2-\varphi_2')-a_1\sin(\varphi_2-\varphi_2')]\,d\varphi',\\
\widetilde{q}^{(p)} &= \int\limits_0^{\varphi}[a_2\sin(\varphi_2-\varphi_2')+a_1\cos(\varphi_2-\varphi_2')]\,d\varphi'.
\end{aligned}
\tag{2.20}
$$

From (2.11) and (2.20) we obtain (2.12). This concludes the proof.

Lemma 2.2. *Let* $\alpha_1=(1-i\lambda)2\pi$, $\alpha_2=(1+i\lambda)2\pi$. *Let* $\varphi_2=(1+i\lambda)\varphi$, $\varphi_2'=(1+i\lambda)\varphi'$, $\Delta_2=2(1-\cos\alpha_2)$. *Let* $a_1=\widetilde{d}_\varphi+\widetilde{h}_{,\varphi}$, $a_2=-d_r+(1+i\lambda)\widetilde{h}$. *Then*

$$
\begin{aligned}
\widetilde{q} = \frac{1}{\Delta_2}\int\limits_0^{2\pi}\{&a_2[\sin\varphi_2\,(\cos\varphi_2'+\cos(\alpha_2-\varphi_2'))\\
&\qquad+\cos\varphi_2\,(\sin\varphi_2'+\sin(\alpha_2-\varphi_2'))]\\
&+a_1[-\sin\varphi_2\,(\sin\varphi_2'+\sin(\alpha_2-\varphi_2'))\\
&\qquad+\cos\varphi_2\,(\cos(\alpha_2-\varphi_2')-\cos\varphi_2')]\}\,d\varphi'\\
&+\int\limits_0^{\varphi}[a_2\sin(\varphi_2-\varphi_2')+a_1\cos(\varphi_2-\varphi_2')]\,d\varphi'.
\end{aligned}
\tag{2.21}
$$

Proof. We prove the lemma by calculating the constants $c_1,\ldots,c_4$ in (2.12) from the boundary conditions $(2.10)_{4-7}$.

The boundary condition $(2.10)_4$ implies

$$
\begin{aligned}
c_2+(1+i\lambda)c_4 &= c_1\sin\alpha_1+c_2\cos\alpha_1+c_3(1+i\lambda)\sin\alpha_2\\
&\quad+c_4(1+i\lambda)\cos\alpha_2+f_1,
\end{aligned}
\tag{2.22}
$$

where

$$f_1=\int_0^{2\pi}\Big[-\frac{1+i\lambda}{4i\lambda}\,a_1\cos(\alpha_1-\varphi_1')+\Big(\widetilde{h}-\frac{1-i\lambda}{4i\lambda}\,a_2\Big)\sin(\alpha_1-\varphi_1')\Big]d\varphi' \tag{2.23}$$
$$+\frac{1+i\lambda}{4i\lambda}\int_0^{2\pi}[a_2\sin(\alpha_2-\varphi_2')+a_1\cos(\alpha_2-\varphi_2')]\,d\varphi'.$$

Condition $(2.10)_5$ yields

$$c_1+c_3(1-i\lambda)=c_1\cos\alpha_1-c_2\sin\alpha_1+c_3(1-i\lambda)\cos\alpha_2 \tag{2.24}$$
$$-c_4(1-i\lambda)\sin\alpha_2+f_2,$$

where

$$f_2=\int_0^{2\pi}\Big[\frac{1+i\lambda}{4i\lambda}\,a_1\sin(\alpha_1-\varphi_1')+\Big(\widetilde{h}-\frac{1-i\lambda}{4i\lambda}\,a_2\Big)\cos(\alpha_1-\varphi_1')\Big]d\varphi' \tag{2.25}$$
$$+\frac{1-i\lambda}{4i\lambda}\int_0^{2\pi}[a_2\cos(\alpha_2-\varphi_2')-a_1\sin(\alpha_2-\varphi_2')]\,d\varphi'.$$

Condition $(2.10)_6$ takes the form

$$\frac{d\widetilde{u}_r}{d\varphi}\Big|_{\varphi=0}=\frac{d\widetilde{u}_r}{d\varphi}\Big|_{\varphi=2\pi}$$

so it yields

$$c_1(1-i\lambda)+c_3(1+i\lambda)^2=c_1(1-i\lambda)\cos\alpha_1-c_2(1-i\lambda)\sin\alpha_1 \tag{2.26}$$
$$+c_3(1+i\lambda)^2\cos\alpha_2-c_4(1+i\lambda)^2\sin\alpha_2+f_3,$$

where

$$f_3=\int_0^{2\pi}\Big[\frac{1+\lambda^2}{4i\lambda}\,a_1\sin(\alpha_1-\varphi_1') \tag{2.27}$$
$$+\Big(\widetilde{h}-\frac{1-i\lambda}{4i\lambda}\,a_2\Big)(1-i\lambda)\cos(\alpha_1-\varphi_1')\Big]\,d\varphi'$$
$$+\frac{(1+i\lambda)^2}{4i\lambda}\int_0^{2\pi}[a_2\cos(\alpha_2-\varphi_2')-a_1\sin(\alpha_2-\varphi_2')]\,d\varphi'.$$

Finally, the boundary condition $(2.10)_7$ simplifies to

$$\Big(2\,\frac{d\widetilde{u}_\varphi}{d\varphi}-\widetilde{q}\Big)\Big|_{\varphi=0}=\Big(2\,\frac{d\widetilde{u}_\varphi}{d\varphi}-\widetilde{q}\Big)\Big|_{\varphi=2\pi},$$

so it assumes the form

$$c_2(1-i\lambda)+c_4(1+i\lambda)(1-i\lambda)+2i\lambda c_4 \tag{2.28}$$
$$=c_1(1-i\lambda)\sin\alpha_1+c_2(1-i\lambda)\cos\alpha_1+c_3(1+\lambda^2)\sin\alpha_2$$
$$+c_4(1+\lambda^2)\cos\alpha_2+2i\lambda(c_3\sin\alpha_2+c_4\cos\alpha_2)+f_4,$$

where

$$f_4 = \int_0^{2\pi} \Big[\frac{1+\lambda^2}{4i\lambda}\, a_1 \cos(\alpha_1 - \varphi_1') - \Big(\widetilde{h} - \frac{1-i\lambda}{4i\lambda}\, a_2\Big)(1-i\lambda)\sin(\alpha_1-\varphi_1')\Big]\, d\varphi' - \frac{(1-i\lambda)(1+i\lambda)}{4i\lambda}\int_0^{2\pi} [a_2 \sin(\alpha_2-\varphi_2') + a_1\cos(\alpha_2-\varphi_2')]\, d\varphi' - \frac{1}{2}\int_0^{2\pi} [a_2 \sin(\alpha_2-\varphi_2') + a_1\cos(\alpha_2-\varphi_2')]\, d\varphi'. \tag{2.29}$$

From (2.22), (2.24), (2.26) and (2.28) we obtain the following system of equations:

$$\begin{bmatrix} -\sin\alpha_1 & 1-\cos\alpha_1 & -(1+i\lambda)\sin\alpha_2 & -(1+i\lambda)\cos\alpha_2 \\ 1-\cos\alpha_1 & \sin\alpha_1 & (1-i\lambda)(1-\cos\alpha_2) & (1-i\lambda)\sin\alpha_2 \\ 1-\cos\alpha_1 & \sin\alpha_1 & \frac{(1+i\lambda)^2}{1-i\lambda}(1-\cos\alpha_2) & \frac{(1+i\lambda)^2}{1-i\lambda}\sin\alpha_2 \\ \sin\alpha_1 & -(1-\cos\alpha_1) & 1+i\lambda+\frac{2i\lambda}{1-i\lambda}\sin\alpha_2 & -\big(1+i\lambda+\frac{2i\lambda}{1-i\lambda}\big)(1-\cos\alpha_2) \end{bmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \frac{f_3}{1-i\lambda} \\ \frac{f_4}{1-i\lambda} \end{pmatrix}.$$

Simplifying, we have

$$\begin{bmatrix} -\sin\alpha_1 & 1-\cos\alpha_1 & -(1+i\lambda)\sin\alpha_2 & (1+i\lambda)(1-\cos\alpha_2) \\ 1-\cos\alpha_1 & \sin\alpha_1 & (1-i\lambda)(1-\cos\alpha_2) & (1-i\lambda)\sin\alpha_2 \\ 0 & 0 & 1-\cos\alpha_2 & \sin\alpha_2 \\ 0 & 0 & \sin\alpha_2 & -(1-\cos\alpha_2) \end{bmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \frac{1-i\lambda}{4i\lambda}\big(-f_2+\frac{1}{1-i\lambda}f_3\big) \\ \frac{1-i\lambda}{2i\lambda}\big(f_1+\frac{1}{1-i\lambda}f_4\big) \end{pmatrix} \equiv \begin{pmatrix} f_1 \\ f_2 \\ f_3' \\ f_4' \end{pmatrix}. \tag{2.30}$$

Using (2.25) and (2.27) we have

$$f_3' = \frac{1}{4i\lambda}\int_0^{2\pi} [a_2\cos(\alpha_2-\varphi_2') - a_1\sin(\alpha_2-\varphi_2')]\, d\varphi'.$$

Next, (2.23) and (2.29) imply

$$f_4' = -\frac{1}{4i\lambda}\int_0^{2\pi} [a_2\sin(\alpha_2-\varphi_2') + a_1\cos(\alpha_2-\varphi_2')]\, d\varphi'.$$

Solving the last two equations of (2.30) yields

$$
\begin{aligned}
c_3 &= \frac{1}{\varDelta_2}\,[f_3'(1-\cos\alpha_2)+f_4'\sin\alpha_2],\\
c_4 &= \frac{1}{\varDelta_2}\,[f_3'\sin\alpha_2-f_4'(1-\cos\alpha_2)],
\end{aligned}
\tag{2.31}
$$

where $\varDelta_i=\sin^2\alpha_i+(1-\cos\alpha_i)^2$, $i=1,2$.

Similarly, c_1 and c_2 satisfy the equations

$$
\begin{aligned}
-\sin\alpha_1\,c_1+(1-\cos\alpha_1)c_2 &= f_1',\\
(1-\cos\alpha_1)c_1+\sin\alpha_1\,c_2 &= f_2',
\end{aligned}
$$

where $f_1'=f_1+(1+i\lambda)f_4'$, $f_2'=f_2-(1-i\lambda)f_3'$. Hence

$$
\begin{aligned}
c_1 &= \frac{1}{\varDelta_1}\,[f_1'(1-\cos\alpha_1)+f_2'\sin\alpha_1],\\
c_2 &= \frac{1}{\varDelta_1}\,[-f_1'\sin\alpha_1+f_2'(1-\cos\alpha_1)].
\end{aligned}
\tag{2.32}
$$

Inserting the expressions for f_3' and f_4' in (2.31) yields

$$
c_3=\frac{1}{4i\lambda\varDelta_2}\int_0^{2\pi}[a_2(\cos\varphi_2'+\cos(\alpha_2-\varphi_2'))-a_1(\sin\varphi_2'+\sin(\alpha_2-\varphi_2'))]\,d\varphi',
$$

$$
c_4=\frac{1}{4i\lambda\varDelta_2}\int_0^{2\pi}[a_2(\sin\varphi_2'+\sin(\alpha_2-\varphi_2'))+a_1(\cos(\alpha_2-\varphi_2')-\cos\varphi_2')]\,d\varphi'.
$$

Using the above expressions in the formula for $\widetilde{q}$ gives

$$
\widetilde{q}=4i\lambda(c_3\sin\varphi_2+c_4\cos\varphi_2)+\int_0^{\varphi}[a_2\sin(\varphi_2-\varphi_2')+a_1\cos(\varphi_2-\varphi_2')]\,d\varphi',
$$

so we obtain (2.21). This concludes the proof.

Inserting $\lambda=i\sigma$ in (2.10) yields

$$
\begin{aligned}
&-\frac{d^2\widetilde{u}_r}{d\varphi^2}+(1-\sigma^2)\widetilde{u}_r+2\,\frac{d\widetilde{u}_\varphi}{d\varphi}+(\sigma-1)\widetilde{q}=\widetilde{d}_r && \text{in } (0,2\pi),\\
&-\frac{d^2\widetilde{u}_\varphi}{d\varphi^2}+(1-\sigma^2)\widetilde{u}_\varphi-2\,\frac{d\widetilde{u}_r}{d\varphi}+\frac{d\widetilde{q}}{d\varphi}=\widetilde{d}_\varphi && \text{in } (0,2\pi),\\
&\frac{d\widetilde{u}_\varphi}{d\varphi}+(\sigma+1)\widetilde{u}_r=\widetilde{h} && \text{in } (0,2\pi),\\
&\widetilde{u}_i|_{\varphi=0}=\widetilde{u}_i|_{\varphi=2\pi},\quad i=r,\varphi,\\
&\bigg(\frac{d\widetilde{u}_r}{d\varphi}+(\sigma-1)\widetilde{u}_\varphi\bigg)\bigg|_{\varphi=0}=\bigg(\frac{d\widetilde{u}_r}{d\varphi}+(\sigma-1)\widetilde{u}_\varphi\bigg)\bigg|_{\varphi=2\pi},\\
&\bigg[2\bigg(\frac{d\widetilde{u}_\varphi}{d\varphi}+\widetilde{u}_r\bigg)-\widetilde{q}\bigg]\bigg|_{\varphi=0}=\bigg[2\bigg(\frac{d\widetilde{u}_\varphi}{d\varphi}+\widetilde{u}_r\bigg)-\widetilde{q}\bigg]\bigg|_{\varphi=2\pi}.
\end{aligned}
\tag{2.33}
$$

A general solution to the homogeneous equations $(2.33)_{1,2,3}$ has the form (see [4])

$$\begin{pmatrix} \widetilde{u}_r \\ \widetilde{u}_\varphi \\ \widetilde{q} \end{pmatrix} = \begin{pmatrix} (\sigma-1)\gamma \\ (\sigma+1)\delta \\ 4\sigma\gamma \end{pmatrix} \cos(\sigma-1)\varphi + \begin{pmatrix} (\sigma-1)\delta \\ -(\sigma+1)\gamma \\ 4\sigma\delta \end{pmatrix} \sin(\sigma-1)\varphi$$
$$+ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \cos(\sigma+1)\varphi + \begin{pmatrix} \beta \\ -\alpha \\ 0 \end{pmatrix} \sin(\sigma+1)\varphi,$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters.

Lemma 2.3 ([4]). *The homogeneous problem* (2.33) *has only integer eigenvalues*, *so* $\sigma \in \mathbb{Z}$.

Lemma 2.4. *Let* $\mu \in (0,1)$ *and* $\widetilde{g} \in L_{2,-\mu}(\mathbb{R}^2)$, $\widetilde{k} \in H^1_{-\mu}(\mathbb{R}^2)$. *Then there exists a solution to problem* (2.3) *such that* $\widetilde{v} \in H^2_{-\mu}(\mathbb{R}^2)$, $\widetilde{p} \in H^1_{-\mu}(\mathbb{R}^2)$ *and*

$$\|\widetilde{v}\|_{H^2_{-\mu}(\mathbb{R}^2)} + \|\widetilde{p}\|_{H^1_{-\mu}(\mathbb{R}^2)} \le c(\|\widetilde{g}\|_{L_{2,-\mu}(\mathbb{R}^2)} + \|\widetilde{k}\|_{H^1_{-\mu}(\mathbb{R}^2)}). \tag{2.34}$$

Proof. First we examine problem (2.10). Having solutions to (2.10) we have to find an estimate.

Multiplying $(2.10)_1$ by $\overline{\widetilde{u}}_r$ (where the bar denotes complex conjugation) and integrating with respect to φ yields

$$\int_0^{2\pi} (\lambda^2|\widetilde{u}_r|^2 + |\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_r|^2)\, d\varphi + 2\int_0^{2\pi} \widetilde{u}_{\varphi,\varphi}\overline{\widetilde{u}}_r\, d\varphi - \int_0^{2\pi} (1+i\lambda)\widetilde{q}\,\overline{\widetilde{u}}_r\, d\varphi = \int_0^{2\pi} \widetilde{d}_r\overline{\widetilde{u}}_r\, d\varphi. \tag{2.35}$$

Multiplying $(2.10)_2$ by $\overline{\widetilde{u}}_\varphi$ and integrating with respect to φ gives

$$\int_0^{2\pi} (\lambda^2|\widetilde{u}_\varphi|^2 + |\widetilde{u}_\varphi|^2 + |\widetilde{u}_{\varphi,\varphi}|^2)\, d\varphi - 2\int_0^{2\pi} \widetilde{u}_{r,\varphi}\overline{\widetilde{u}}_\varphi\, d\varphi + \int_0^{2\pi} \widetilde{q}_{,\varphi}\overline{\widetilde{u}}_\varphi\, d\varphi = \int_0^{2\pi} \widetilde{d}_\varphi\overline{\widetilde{u}}_\varphi\, d\varphi. \tag{2.36}$$

Adding (2.35) and (2.36) we obtain

$$\begin{aligned} &\int_0^{2\pi} [\lambda^2(|\widetilde{u}_r|^2 + |\widetilde{u}_\varphi|^2) + |\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2 + |\widetilde{u}_r|^2 + |\widetilde{u}_\varphi|^2]\, d\varphi \\ &\quad + 2\int_0^{2\pi} (\widetilde{u}_{\varphi,\varphi}\overline{\widetilde{u}}_r + \widetilde{u}_r\overline{\widetilde{u}}_{\varphi,\varphi})\, d\varphi - \int_0^{2\pi} [(1+i\lambda)\widetilde{q}\,\overline{\widetilde{u}}_r + \widetilde{q}\,\overline{\widetilde{u}}_{\varphi,\varphi}]\, d\varphi \\ &= \int_0^{2\pi} (\widetilde{d}_r\overline{\widetilde{u}}_r + \widetilde{d}_\varphi\overline{\widetilde{u}}_\varphi)\, d\varphi. \end{aligned} \tag{2.37}$$

By the equation of continuity $(2.10)_3$, the second term on the l.h.s. of (2.37) equals

$$-4(1+\operatorname{im}\lambda)\int_0^{2\pi}|\widetilde{u}_r|^2\,d\varphi+2\int_0^{2\pi}(\widetilde{h}\overline{\widetilde{u}}_r+\widetilde{u}_r\overline{\widetilde{h}})\,d\varphi.$$

Similarly, the last term on the l.h.s. of (2.37) takes the form

$$-\int_0^{2\pi}\widetilde{q}\overline{\widetilde{h}}\,d\varphi+2\operatorname{im}\lambda\int_0^{2\pi}\widetilde{q}\overline{\widetilde{u}}_r\,d\varphi.$$

Inserting the above expressions in (2.37), taking the modulus, using the fact that $|\operatorname{re}\lambda|\ge|\operatorname{im}\lambda|$ and applying the Hölder and Young inequalities yields

$$\begin{aligned}(2.38)\qquad &\int_0^{2\pi}[|\lambda|^2(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+|\widetilde{u}_{r,\varphi}|^2+|\widetilde{u}_{\varphi,\varphi}|^2+|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2]\,d\varphi\\ &\le\frac{\varepsilon_1}{2}\int_0^{2\pi}(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)\,d\varphi+\frac{1}{2\varepsilon_1}\int_0^{2\pi}(|\widetilde{d}_r|^2+|\widetilde{d}_\varphi|^2)\,d\varphi\\ &\quad+\frac{\varepsilon_2}{2}\int_0^{2\pi}(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)\,d\varphi+\frac{4}{\varepsilon_2}\int_0^{2\pi}|\widetilde{h}|^2\,d\varphi+\frac12\int_0^{2\pi}|\widetilde{q}|^2\,d\varphi\\ &\quad+\frac12\int_0^{2\pi}|\widetilde{h}|^2\,d\varphi+\frac{\varepsilon_3}{2}\int_0^{2\pi}|\widetilde{u}_r|^2\,d\varphi+\frac{2|\operatorname{im}\lambda|^2}{\varepsilon_3}\int_0^{2\pi}|\widetilde{q}|^2\,d\varphi\\ &\quad+4(1+|\operatorname{im}\lambda|)\int_0^{2\pi}|\widetilde{u}_r|^2\,d\varphi.\end{aligned}$$

Choosing $\varepsilon_1=\varepsilon_2=\varepsilon_3=1/3$ we obtain from (2.38) the inequality

$$\begin{aligned}(2.39)\qquad &\int_0^{2\pi}\Big[|\lambda|^2(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+|\widetilde{u}_{r,\varphi}|^2+|\widetilde{u}_{\varphi,\varphi}|^2+\frac12(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)\Big]\,d\varphi\\ &\le\frac32\int_0^{2\pi}(|\widetilde{d}_r|^2+|\widetilde{d}_\varphi|^2)\,d\varphi+13\int_0^{2\pi}|\widetilde{h}|^2\,d\varphi+\Big(\frac12+6|\operatorname{im}\lambda|^2\Big)\int_0^{2\pi}|\widetilde{q}|^2\,d\varphi\\ &\quad+4(1+|\operatorname{im}\lambda|)\int_0^{2\pi}|\widetilde{u}_r|^2\,d\varphi.\end{aligned}$$

Let us consider (2.39) in the region $|\operatorname{im}\lambda|\le c_1$ and $|\operatorname{re}\lambda|\ge c_2$. Assuming that $c_2\ge 8(1+c_1)$ we obtain from (2.39) the inequality

$$\begin{aligned}(2.40)\qquad &\int_0^{2\pi}[|\lambda|^2(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+|\widetilde{u}_{r,\varphi}|^2+|\widetilde{u}_{\varphi,\varphi}|^2+|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2]\,d\varphi\\ &\le(1+12|\operatorname{im}\lambda|^2)\int_0^{2\pi}|\widetilde{q}|^2\,d\varphi+c\int_0^{2\pi}(|\widetilde{d}_r|^2+|\widetilde{d}_\varphi|^2+|\widetilde{h}|^2)\,d\varphi.\end{aligned}$$

For $|\mathrm{re}\,\lambda| \le c_2$ we have

$$\int_0^{2\pi} |\lambda|^2(|\widetilde{u}_r|^2 + |\widetilde{u}_\varphi|^2)\,d\varphi \le c_2^2 \int_0^{2\pi} (|\widetilde{u}_r|^2 + |\widetilde{u}_\varphi|^2)\,d\varphi$$

and the r.h.s. can be directly estimated from the explicit form of the solution to (2.10) by the second integral on the r.h.s. of (2.40). Therefore (2.40) is also valid for λ such that $|\mathrm{re}\,\lambda| \le c_2$.

Differentiating $(2.10)_1$ with respect to φ, multiplying by $\overline{\widetilde{u}}_{r,\varphi}$ and integrating with respect to φ yields

$$\begin{aligned}(2.41)\qquad &\int_0^{2\pi} (\lambda^2|\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{r,\varphi\varphi}|^2 + |\widetilde{u}_{r,\varphi}|^2)\,d\varphi + 2\int_0^{2\pi} \widetilde{u}_{\varphi,\varphi\varphi}\overline{\widetilde{u}}_{r,\varphi}\,d\varphi \\ &- \int_0^{2\pi} (1+i\lambda)\widetilde{q}_{,\varphi}\overline{\widetilde{u}}_{r,\varphi}\,d\varphi = \int_0^{2\pi} \widetilde{d}_{r,\varphi}\overline{\widetilde{u}}_{r,\varphi}\,d\varphi = -\int_0^{2\pi} \widetilde{d}_r\overline{\widetilde{u}}_{r,\varphi\varphi}\,d\varphi,\end{aligned}$$

where in the last equality we used the fact that $\widetilde{d}_r|_{\varphi=0} = \widetilde{d}_r|_{\varphi=2\pi}$.

Differentiating $(2.10)_2$ with respect to φ, multiplying the result by $\overline{\widetilde{u}}_{\varphi,\varphi}$ and integrating with respect to φ implies

$$\begin{aligned}(2.42)\qquad &\int_0^{2\pi} (\lambda^2|\widetilde{u}_{\varphi,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2)\,d\varphi - 2\int_0^{2\pi} \widetilde{u}_{r,\varphi\varphi}\overline{\widetilde{u}}_{\varphi,\varphi}\,d\varphi + \int_0^{2\pi} \widetilde{q}_{,\varphi\varphi}\overline{\widetilde{u}}_{\varphi,\varphi}\,d\varphi \\ &= \int_0^{2\pi} \widetilde{d}_{\varphi,\varphi}\overline{\widetilde{u}}_{\varphi,\varphi}\,d\varphi = -\int_0^{2\pi} \widetilde{d}_\varphi\overline{\widetilde{u}}_{\varphi,\varphi\varphi}\,d\varphi,\end{aligned}$$

where the last equality holds because $\widetilde{d}_\varphi|_{\varphi=0} = \widetilde{d}_\varphi|_{\varphi=2\pi}$.

Adding (2.41) and (2.42) and using the fact that

$$\begin{aligned}\int_0^{2\pi} (\widetilde{u}_{\varphi,\varphi\varphi}\overline{\widetilde{u}}_{r,\varphi} - \widetilde{u}_{r,\varphi\varphi}\overline{\widetilde{u}}_{\varphi,\varphi})\,d\varphi &= \int_0^{2\pi} (\widetilde{u}_{\varphi,\varphi\varphi}\overline{\widetilde{u}}_{r,\varphi} + \widetilde{u}_{r,\varphi}\overline{\widetilde{u}}_{\varphi,\varphi\varphi})\,d\varphi \\ &= \int_0^{2\pi} [(\widetilde{h}_{,\varphi}\overline{\widetilde{u}}_{r,\varphi} + \widetilde{u}_{r,\varphi}\overline{\widetilde{h}}_{,\varphi}) - 2(1+\mathrm{im}\,\lambda)|\widetilde{u}_{r,\varphi}|^2]\,d\varphi\end{aligned}$$

and

$$\begin{aligned}\int_0^{2\pi} [-(1+i\lambda)\widetilde{q}_{,\varphi}\overline{\widetilde{u}}_{r,\varphi} + \widetilde{q}_{,\varphi\varphi}\overline{\widetilde{u}}_{\varphi,\varphi}]\,d\varphi &= -\int_0^{2\pi} [(1+i\lambda)\widetilde{q}_{,\varphi}\overline{\widetilde{u}}_{r,\varphi} + \widetilde{q}_{,\varphi}\overline{\widetilde{u}}_{\varphi,\varphi\varphi}]\,d\varphi \\ &= -\int_0^{2\pi} \widetilde{q}_{,\varphi}[(1+i\lambda)\overline{\widetilde{u}}_{r,\varphi} + \overline{\widetilde{h}}_{,\varphi} - (1+i\overline{\lambda})\overline{\widetilde{u}}_{r,\varphi}]\,d\varphi \\ &= \int_0^{2\pi} [2\,\mathrm{im}\,\lambda\widetilde{q}_{,\varphi}\overline{\widetilde{u}}_{r,\varphi} - \widetilde{q}_{,\varphi}\overline{\widetilde{h}}_{,\varphi}]\,d\varphi = -\int_0^{2\pi} [2\,\mathrm{im}\,\lambda\widetilde{q}\,\overline{\widetilde{u}}_{r,\varphi\varphi} + \widetilde{q}_{,\varphi}\overline{\widetilde{h}}_{,\varphi}]\,d\varphi,\end{aligned}$$

where we employed the continuity equation $(2.10)_3$, we obtain, for $|\mathrm{re}\,\lambda| \ge |\mathrm{im}\,\lambda|$,

$$
\begin{aligned}
(2.43)\qquad &\int_0^{2\pi} [|\lambda|^2(|\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2) + |\widetilde{u}_{r,\varphi\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi\varphi}|^2 + |\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2]\,d\varphi \\
&\le \frac{\varepsilon_1}{2}\int_0^{2\pi} (|\widetilde{u}_{r,\varphi\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi\varphi}|^2)\,d\varphi + \frac{1}{2\varepsilon_1}\int_0^{2\pi} (|\widetilde{d}_r|^2 + |\widetilde{d}_\varphi|^2)\,d\varphi \\
&\quad + \frac{\varepsilon_2}{2}\int_0^{2\pi} |\widetilde{u}_{r,\varphi}|^2\,d\varphi + \frac{c}{\varepsilon_2}\int_0^{2\pi} |\widetilde{h}_{,\varphi}|^2\,d\varphi + \frac{\varepsilon_3}{2}\int_0^{2\pi} |\widetilde{u}_{r,\varphi\varphi}|^2\,d\varphi \\
&\quad + \frac{2}{\varepsilon_3}|\mathrm{im}\,\lambda|^2\int_0^{2\pi} |\widetilde{q}|^2\,d\varphi + \frac{\varepsilon_4}{2}\int_0^{2\pi} |\widetilde{q}_{,\varphi}|^2\,d\varphi + \frac{1}{2\varepsilon_4}\int_0^{2\pi} |\widetilde{h}_{,\varphi}|^2\,d\varphi \\
&\quad + 2(1 + |\mathrm{im}\,\lambda|)\int_0^{2\pi} |\widetilde{u}_{r,\varphi}|^2\,d\varphi.
\end{aligned}
$$

Choosing $\varepsilon_1 = \varepsilon_3 = 1/2$, $\varepsilon_2 = 1$ and $c_2 \ge 4(1 + c_1)$ we obtain from (2.43) the inequality

$$
\begin{aligned}
(2.44)\qquad &\int_0^{2\pi} [|\lambda|^2(|\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2) + |\widetilde{u}_{r,\varphi\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi\varphi}|^2 + |\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2]\,d\varphi \\
&\le 8|\mathrm{im}\,\lambda|^2\int_0^{2\pi} |\widetilde{q}|^2\,d\varphi + \overline{\varepsilon}_1\int_0^{2\pi} |q_{,\varphi}|^2\,d\varphi + \frac{c}{\overline{\varepsilon}_1}\int_0^{2\pi} |\widetilde{h}_{,\varphi}|^2\,d\varphi + c\int_0^{2\pi} (|\widetilde{d}_r|^2 + |\widetilde{d}_\varphi|^2)\,d\varphi.
\end{aligned}
$$

Multiplying $(2.10)_1$ by $\overline{\lambda}^2\overline{\widetilde{u}}_r$, $(2.10)_2$ by $\overline{\lambda}^2\overline{\widetilde{u}}_\varphi$, adding the results and integrating with respect to φ yields

$$
\begin{aligned}
(2.45)\qquad &\int_0^{2\pi} [|\lambda|^4(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+(1+\overline{\lambda}^2)(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+\overline{\lambda}^2(|\widetilde{u}_{r,\varphi}|^2+|\widetilde{u}_{\varphi,\varphi}|^2)]\,d\varphi \\
&= -2\int_0^{2\pi} \overline{\lambda}^2(\widetilde{u}_{\varphi,\varphi}\overline{\widetilde{u}}_r - \widetilde{u}_{r,\varphi}\overline{\widetilde{u}}_\varphi)\,d\varphi - \int_0^{2\pi} \overline{\lambda}^2\widetilde{q}[-(1+i\lambda)\overline{\widetilde{u}}_r - \overline{\widetilde{u}}_{\varphi,\varphi}]\,d\varphi \\
&\quad + \int_0^{2\pi} (\widetilde{d}_r\overline{\lambda}^2\overline{\widetilde{u}}_r + \widetilde{d}_\varphi\overline{\lambda}^2\overline{\widetilde{u}}_\varphi)\,d\varphi,
\end{aligned}
$$

The first term on the r.h.s. of (2.45) is estimated by

$$
\frac{\varepsilon_1}{2}\int_0^{2\pi} |\lambda|^4(|\widetilde{u}_r|^2 + |\widetilde{u}_\varphi|^2)\,d\varphi + \frac{2}{\varepsilon_1}\int_0^{2\pi} (|\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2)\,d\varphi.
$$

By the equation of continuity $(2.10)_3$ the second term on the r.h.s. of (2.45) takes the form

$$
-\int_0^{2\pi} \overline{\lambda}^2\widetilde{q}[-(1+i\lambda)\overline{\widetilde{u}}_r - \overline{\widetilde{h}} + (1+i\lambda)\overline{\widetilde{u}}_r]\,d\varphi = \int_0^{2\pi} \overline{\lambda}^2\widetilde{q}(\overline{\widetilde{h}} - 2\,\mathrm{im}\,\lambda\overline{\widetilde{u}}_r)\,d\varphi \equiv J,
$$

and

$$|J| \le \frac{\varepsilon_2}{2}\int_0^{2\pi} |\lambda|^2|\widetilde{q}|^2\,d\varphi + \frac{1}{2\varepsilon_2}\int_0^{2\pi} |\lambda|^2|\widetilde{h}|^2\,d\varphi + \frac{2|\operatorname{im}\lambda|^2}{\varepsilon_2}\int_0^{2\pi} |\lambda|^2|\widetilde{u}_r|^2\,d\varphi.$$

Finally, we estimate the last term on the r.h.s. of (2.45) by

$$\frac{\varepsilon_3}{2}\int_0^{2\pi} |\lambda|^4(|\widetilde{u}_r|^2 + |\widetilde{u}_\varphi|^2)\,d\varphi + \frac{1}{2\varepsilon_3}\int_0^{2\pi} (|\widetilde{d}_r|^2 + |\widetilde{d}_\varphi|^2)\,d\varphi.$$

Assuming $\varepsilon_1 = \varepsilon_3 = 1/2$, $c_2 > 8(1+c_1)$ we obtain from (2.45) the inequality

$$\begin{aligned}(2.46)\qquad &\int_0^{2\pi} \bigg[\frac{1}{2}|\lambda|^4(|\widetilde{u}_r|^2 + |\widetilde{u}_\varphi|^2) \\ &\qquad + (1+|\lambda|^2)(|\widetilde{u}_r|^2 + |\widetilde{u}_\varphi|^2) + |\lambda|^2(|\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2)\bigg]\,d\varphi \\ &\le \frac{\overline{\varepsilon}_2}{2}\int_0^{2\pi} |\lambda|^2|\widetilde{q}|^2\,d\varphi + \frac{c}{2\overline{\varepsilon}_2}\int_0^{2\pi} |\lambda|^2|\widetilde{h}|^2\,d\varphi \\ &\quad + \frac{4|\operatorname{im}\lambda|^2}{\overline{\varepsilon}_2}\int_0^{2\pi} |\lambda|^2|\widetilde{u}_r|^2\,d\varphi + \int_0^{2\pi} (|\widetilde{d}_r|^2 + |\widetilde{d}_\varphi|^2)\,d\varphi.\end{aligned}$$

From $(2.10)_1$ we have

$$\begin{aligned}(2.47)\qquad &\int_0^{2\pi} |\lambda|^2|\widetilde{q}|^2\,d\varphi \\ &\le c\int_0^{2\pi} (|\lambda|^4|\widetilde{u}_r|^2 + |\widetilde{u}_{r,\varphi\varphi}|^2 + |\widetilde{u}_r|^2 + |\widetilde{u}_{\varphi,\varphi}|^2)\,d\varphi + c\int_0^{2\pi} |\widetilde{d}_r|^2\,d\varphi,\end{aligned}$$

and $(2.10)_2$ gives

$$\begin{aligned}(2.48)\qquad &\int_0^{2\pi} |\widetilde{q}_{,\varphi}|^2\,d\varphi \\ &\le c\int_0^{2\pi} (|\lambda|^4|\widetilde{u}_\varphi|^2 + |\widetilde{u}_{\varphi,\varphi\varphi}|^2 + |\widetilde{u}_\varphi|^2 + |\widetilde{u}_{r,\varphi}|^2)\,d\varphi + c\int_0^{2\pi} |\widetilde{d}_\varphi|^2\,d\varphi.\end{aligned}$$

Since $|\lambda| \ge c_2$ and c_2 is sufficiently large, from (2.44) and (2.47) we obtain

$$\begin{aligned}(2.49)\qquad &\int_0^{2\pi} (|\lambda|^2(|\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2) + |\widetilde{u}_{r,\varphi\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi\varphi}|^2 + |\widetilde{u}_{r,\varphi}|^2 + |\widetilde{u}_{\varphi,\varphi}|^2)\,d\varphi \\ &\le \overline{\varepsilon}_1\int_0^{2\pi} |\widetilde{q}_{,\varphi}|^2\,d\varphi + \frac{8|\operatorname{im}\lambda|^2}{c_2^2}\int_0^{2\pi} |\lambda|^4|\widetilde{u}_r|^2\,d\varphi \\ &\quad + \frac{c}{\overline{\varepsilon}_1}\int_0^{2\pi} |\widetilde{h}_{,\varphi}|^2\,d\varphi + c\int_0^{2\pi} (|\widetilde{d}_r|^2 + |\widetilde{d}_\varphi|^2)\,d\varphi.\end{aligned}$$

Inserting (2.47) in (2.46) yields

$$
\begin{aligned}
(2.50)\quad & \int_0^{2\pi} [|\lambda|^4(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+|\lambda|^2(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)\\
&\qquad + |\lambda|^2(|\widetilde{u}_{r,\varphi}|^2+|\widetilde{u}_{\varphi,\varphi}|^2)]\,d\varphi\\
&\le \overline{\varepsilon}_2\int_0^{2\pi}|\widetilde{u}_{r,\varphi\varphi}|^2\,d\varphi+\frac{c}{\overline{\varepsilon}_2}\int_0^{2\pi}|\lambda|^2|\widetilde{h}|^2\,d\varphi+\frac{4|\mathrm{im}\,\lambda|^2}{\overline{\varepsilon}_2}\int_0^{2\pi}|\lambda|^2|\widetilde{u}_r|^2\,d\varphi\\
&\qquad + c\int_0^{2\pi}(|\widetilde{d}_r|^2+|\widetilde{d}_\varphi|^2)\,d\varphi.
\end{aligned}
$$

From (2.49) and (2.50) we obtain the following inequality for $\overline{\varepsilon}_2=1/2$ and $c_2\ge 16c_1$:

$$
\begin{aligned}
(2.51)\quad & \int_0^{2\pi} [|\lambda|^4(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+|\lambda|^2(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+|\lambda|^2(|\widetilde{u}_{r,\varphi}|^2+|\widetilde{u}_{\varphi,\varphi}|^2)\\
&\qquad + |\widetilde{u}_{r,\varphi\varphi}|^2+|\widetilde{u}_{\varphi,\varphi\varphi}|^2+|\widetilde{u}_{r,\varphi}|^2+|\widetilde{u}_{\varphi,\varphi}|^2]\,d\varphi\\
&\le \overline{\varepsilon}_1\int_0^{2\pi}|\widetilde{q}_{,\varphi}|^2\,d\varphi+c\int_0^{2\pi}\Big(|\widetilde{d}_r|^2+|\widetilde{d}_\varphi|^2+|\lambda|^2|\widetilde{h}|^2+\frac{1}{\overline{\varepsilon}_1}|\widetilde{h}_{,\varphi}|^2\Big)\,d\varphi.
\end{aligned}
$$

Employing (2.48) in (2.51) and using (2.47) and (2.48) we have

$$
\begin{aligned}
(2.52)\quad & \int_0^{2\pi}\Big[\sum_{i=0}^{2}|\lambda|^{2i}(|\widetilde{u}_r|^2+|\widetilde{u}_\varphi|^2)+\sum_{i=0}^{1}|\lambda|^{2i}(|\widetilde{u}_{r,\varphi}|^2+|\widetilde{u}_{\varphi,\varphi}|^2)\\
&\qquad + |\widetilde{u}_{r,\varphi\varphi}|^2+|\widetilde{u}_{\varphi,\varphi\varphi}|^2+|\widetilde{q}|^2+|\lambda|^2|\widetilde{q}|^2+|\widetilde{q}_{,\varphi}|^2\Big]\,d\varphi\\
&\le \int_0^{2\pi}(|\widetilde{d}_r|^2+|\widetilde{d}_\varphi|^2+|\widetilde{h}|^2+|\lambda|^2|\widetilde{h}|^2+|\widetilde{h}_{,\varphi}|^2)\,d\varphi.
\end{aligned}
$$

For $|\mathrm{re}\,\lambda|\le c_2$ the above estimate follows from the explicit form of solution to (2.10). Integrating (2.52) with respect to λ along the line $\mathrm{im}\,\lambda=1+\mu$ and using (2.6), (2.7) we obtain (2.34). This concludes the proof.

Remark 2.5. We can generalize the result of Lemma 2.4 by integrating (2.52) with respect to λ from $-\infty+ih$ to $+\infty+ih$, $h=1-\mu$, $\mu\in\mathbb{R}$, $\mu\notin\mathbb{Z}$. Then there exists a solution to problem (2.3) such that $\widetilde{v}\in H^2_\mu(\mathbb{R}^2)$, $\widetilde{p}\in H^1_\mu(\mathbb{R}^2)$ and

$$
\|\widetilde{v}\|_{H^2_\mu(\mathbb{R}^2)}+\|\widetilde{p}\|_{H^1_\mu(\mathbb{R}^2)}\le c(\|\widetilde{g}\|_{L_{2,\mu}(\mathbb{R}^2)}+\|\widetilde{k}\|_{H^1_\mu(\mathbb{R}^2)}),
$$

where the r.h.s. is finite.

Now we consider problem (2.4) rewritten in the form

$$\begin{aligned} &\Delta\psi = g,\\ &\psi|_{\gamma_0} = \psi|_{\gamma_{2\pi}},\\ &\psi_{,\varphi}|_{\gamma_0} = \psi_{,\varphi}|_{\gamma_{2\pi}}. \end{aligned} \tag{2.53}$$

LEMMA 2.6. *Assume that $g \in L_{2,-\mu}(\mathbb{R}^2)$, $\mu \in (0,1)$. Then there exists a solution to problem* (2.53) *such that $\psi \in H^2_{-\mu}(\mathbb{R}^2)$ and*

$$\|\psi\|_{H^2_{-\mu}(\mathbb{R}^2)} \le c\|g\|_{L_{2,-\mu}(\mathbb{R}^2)}. \tag{2.54}$$

Proof. Passing to polar coordinates yields

$$\begin{aligned} &r\partial_r(r\partial_r\psi) + \psi_{,\varphi\varphi} = r^2 g \equiv k,\\ &\psi|_{\varphi=0} = \psi|_{\varphi=2\pi},\\ &\frac{\partial\psi}{\partial\varphi}\Big|_{\varphi=0} = \frac{\partial\psi}{\partial\varphi}\Big|_{\varphi=2\pi}. \end{aligned} \tag{2.55}$$

Applying the Fourier transform (2.8) yields

$$\begin{aligned} &-\lambda^2\widetilde{\psi} + \widetilde{\psi}_{,\varphi\varphi} = \widetilde{k},\\ &\widetilde{\psi}|_{\varphi=0} = \widetilde{\psi}|_{\varphi=2\pi},\\ &\frac{\partial\widetilde{\psi}}{\partial\varphi}\Big|_{\varphi=0} = \frac{\partial\widetilde{\psi}}{\partial\varphi}\Big|_{\varphi=2\pi}. \end{aligned} \tag{2.56}$$

The homogeneous problem (2.56) has integer eigenvalues $\sigma = -i\lambda \in \mathbb{Z}$ and corresponding eigenfunctions $\sin(\sigma\varphi)$, $\cos(\sigma\varphi)$, $\sigma \in \mathbb{Z}$ (see [4]).

We are looking for solutions to (2.56) with $\lambda = i\sigma$ in the form

$$\widetilde{\psi} = \alpha\sin(\sigma\varphi) + \beta\cos(\sigma\varphi).$$

By variation of constants we calculate α and β from the equations

$$\begin{aligned} &\frac{d\alpha}{d\varphi}\sin(\sigma\varphi) + \frac{d\beta}{d\varphi}\cos(\sigma\varphi) = 0,\\ &\frac{d\alpha}{d\varphi}\cos(\sigma\varphi) - \frac{d\beta}{d\varphi}\sin(\sigma\varphi) = \frac{1}{\sigma}\widetilde{k}. \end{aligned}$$

Solving the equations we obtain

$$\frac{d\alpha}{d\varphi} = \frac{1}{\sigma}\cos(\sigma\varphi)\widetilde{k}, \qquad \frac{d\beta}{d\varphi} = -\frac{1}{\sigma}\sin(\sigma\varphi)\widetilde{k}.$$

Integrating with respect to φ yields

$$\alpha = \frac{1}{\sigma}\int_0^{\varphi}\cos(\sigma\varphi')\widetilde{k}\,d\varphi', \qquad \beta = -\frac{1}{\sigma}\int_0^{\varphi}\sin(\sigma\varphi')\widetilde{k}\,d\varphi'.$$

Hence a general solution of (2.56) has the form

$$\widetilde{\psi}=\alpha\sin(\sigma\varphi)+\beta\cos(\sigma\varphi)+\frac{\sin(\sigma\varphi)}{\sigma}\int_0^\varphi\cos(\sigma\varphi')\widetilde{k}(\varphi')\,d\varphi' - \frac{\cos(\sigma\varphi)}{\sigma}\int_0^\varphi\sin(\sigma\varphi')\widetilde{k}(\varphi')\,d\varphi'. \tag{2.57}$$

The boundary conditions $(2.56)_{2,3}$ imply the relations

$$\begin{aligned}-\sin(2\pi\sigma)\,\alpha+(1-\cos(2\pi\sigma))\beta&=\frac{\sin(2\pi\sigma)}{\sigma}\int_0^{2\pi}\cos(\sigma\varphi')\widetilde{k}(\varphi')\,d\varphi'\\&\quad-\frac{\cos(2\pi\sigma)}{\sigma}\int_0^{2\pi}\sin(\sigma\varphi')\widetilde{k}(\varphi')\,d\varphi'\equiv A_1,\\(1-\cos(2\pi\sigma))\alpha+\sin(2\pi\sigma)\,\beta&=\frac{\cos(2\pi\sigma)}{\sigma}\int_0^{2\pi}\cos(\sigma\varphi')\widetilde{k}(\varphi')\,d\varphi'\\&\quad+\frac{\sin(2\pi\sigma)}{\sigma}\int_0^{2\pi}\sin(\sigma\varphi')\widetilde{k}(\varphi')\,d\varphi'\equiv A_2.\end{aligned} \tag{4.58}$$

Solving (2.58) yields

$$\begin{aligned}\alpha&=\frac{-A_1\sin(2\pi\sigma)+A_2(1-\cos(2\pi\sigma))}{2(1-\cos(2\pi\sigma))},\\\beta&=\frac{A_1(1-\cos(2\pi\sigma))+A_2\sin(2\pi\sigma)}{2(1-\cos(2\pi\sigma))}.\end{aligned} \tag{2.59}$$

Hence, our solution (2.57) is determined.

Assuming that $\widetilde{k}$ is sufficiently regular we obtain estimates for solutions to problem (2.56) for $\sigma\notin\mathbb{Z}$.

Multiplying (2.56) by $\overline{\widetilde{\psi}}$, integrating with respect to φ, assuming that $\int_0^{2\pi}\widetilde{\psi}\,d\varphi=0$ and $|\mathrm{re}\,\lambda|\ge|\mathrm{im}\,\lambda|$ we obtain

$$\int_0^{2\pi}(|\lambda|^2|\widetilde{\psi}|^2+|\widetilde{\psi}_{,\varphi}|^2)\,d\varphi\le c\int_0^{2\pi}|\widetilde{k}|^2\,d\varphi. \tag{2.60}$$

Differentiating $(2.56)_1$ with respect to φ, multiplying the result by $\overline{\widetilde{\psi}}_{,\varphi}$ and using $|\mathrm{re}\,\lambda|\ge|\mathrm{im}\,\lambda|$ we obtain

$$\int_0^{2\pi}(|\lambda|^2|\widetilde{\psi}_{,\varphi}|^2+|\widetilde{\psi}_{,\varphi\varphi}|^2)\,d\varphi\le c\int_0^{2\pi}|\widetilde{k}|^2\,d\varphi. \tag{2.61}$$

From $(2.56)_1$ we have

$$(2.62)\qquad \int_0^{2\pi} |\lambda|^4 |\widetilde{\psi}|^2\,d\varphi \le c\int_0^{2\pi} |\widetilde{\psi}_{,\varphi\varphi}|^2\,d\varphi + c\int_0^{2\pi} |\widetilde{k}|^2\,d\varphi.$$

From (2.60)–(2.62) we obtain

$$(2.63)\qquad \int_0^{2\pi} (|\lambda|^4|\widetilde{\psi}|^2 + |\lambda|^2|\widetilde{\psi}_{,\varphi}|^2 + |\widetilde{\psi}_{,\varphi\varphi}|^2)\,d\varphi \le c\int_0^{2\pi} |\widetilde{k}|^2\,d\varphi.$$

Hence there exists a solution to (2.56) which satisfies (2.63).

Integrating (2.63) with respect to λ from $-\infty + ih$ to $+\infty + ih$, $h = 1+\mu$ we find that $\psi \in H^2_{-\mu}(\mathbb{R}^2)$ and (2.54) holds. This concludes the proof.

Remark 2.7. We generalize the result of Lemma 2.5 by integrating (2.63) with respect to λ from $-\infty + ih$ to $+\infty + ih$, $h = 1-\mu$, $\mu \in \mathbb{R}$ and $\mu \notin \mathbb{Z}$. Then we have existence of solutions to problem (2.53) in $H^2_\mu(\mathbb{R}^2)$ and the estimate

$$(2.64)\qquad \|\psi\|_{H^2_\mu(\mathbb{R}^2)} \le c\|g\|_{L_{2,\mu}(\mathbb{R}^2)},$$

where we assume that the r.h.s. is finite.

Remark 2.7. Uniqueness of solutions to problems (1.4) and (1.5) is evident.

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