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Arkadiusz Szymaniec (Warszawa)

L^p-L^q TIME DECAY ESTIMATES FOR THE SOLUTION OF THE LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THERMODIFFUSION

Abstract. We consider the initial-value problem for a linear hyperbolic parabolic system of three coupled partial differential equations of second order describing the process of thermodiffusion in a solid body (in one-dimensional space). We prove L^p-L^q time decay estimates for the solution of the associated linear Cauchy problem.

1. Introduction. In this paper we consider the differential equations of thermodiffusion given by W. Nowacki (cf. [18], [19]). He considered the displacement u, the temperature θ_1 and the chemical potential θ_2 as independent fields. These fields depend on the space variable x and the time variable t and satisfy the following system of equations:

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - (\lambda + 2\mu) \dfrac{\partial^2 u}{\partial x^2} + \gamma_1 \dfrac{\partial \theta_1}{\partial x} + \gamma_2 \dfrac{\partial \theta_2}{\partial x} = 0, \\ c \dfrac{\partial \theta_1}{\partial t} - k \dfrac{\partial^2 \theta_1}{\partial x^2} + \gamma_1 \dfrac{\partial^2 u}{\partial t \partial x} + d \dfrac{\partial \theta_2}{\partial t} = 0, \\ n \dfrac{\partial \theta_2}{\partial t} - D \dfrac{\partial^2 \theta_2}{\partial x^2} + \gamma_2 \dfrac{\partial^2 u}{\partial t \partial x} + d \dfrac{\partial \theta_1}{\partial t} = 0. \end{cases} \tag{1.1}$$

with the following initial data:

$$\begin{aligned} u(0,x) &= u_0(x), \qquad & \theta_1(0,x) &= \theta_{10}(x), \\ \partial_t u(0,x) &= u_1(x), \qquad & \theta_2(0,x) &= \theta_{20}(x), \end{aligned} \tag{1.2}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}$. In equations (1.1) we denoted by:

- λ, μ the material coefficients,
- ρ the density,

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- γ_1, γ_2 the coefficients of thermal and diffusion dilatation,
- k the coefficient of thermal conductivity,
- D the coefficient of diffusion,
- n, c, d the coefficients of thermodiffusion.

All the above constants are positive and satisfy

$$nc - d^2 > 0. \tag{1.3}$$

The condition (1.3) implies that (1.1) is a hyperbolic-parabolic system of partial differential equations. The initial-boundary value problem for the linear system of thermodiffusion was investigated by W. Nowacki (cf. [19]), Ya. S. Podstrigach [21] and G. Fichera [4] using the methods of integral transformations and integral equations. J. Gawinecki [9] proved the existence, uniqueness and regularity of the solution to the initial-boundary value problems for the linear system of thermodiffusion in a solid body. The matrix of fundamental solutions (cf. [8], [10]) was constructed using the Fourier transform for three cases:

- for the linear system of thermodiffusion,
- in the quasi-static case of the thermal stresses theory,
- for the whole system of equations.

In this paper we use the method of Sobolev spaces and L^p-L^q time decay estimates for the solution of the linearized system of equation (1.1) associated with the nonlinear system.

The paper is organized as follows (cf. [25]). In the introduction we present the equations of linear thermodiffusion in a solid body in one-dimensional space and formulate the main theorem. In Section 2 some basic notation is presented. Section 3 is devoted to the investigation of the behaviour of the roots of the characteristic equation of the system (1.1)–(1.2). In Section 4 we study the asymptotic behaviour of the solution of the Cauchy problem (1.1)–(1.2). In Section 5 we prove L^∞-L^1 and L^2-L^2 time decay estimates for the initial-value problem for the system (1.1)–(1.2). Section 6 is devoted to the L^p-L^q time decay estimate for the solution of the Cauchy problem for the system (1.1)–(1.2). First, we rewrite (1.1)–(1.2) in the form

$$\begin{cases} \partial_t U(t,x) + A(t,x,\partial_x,\partial_x^2)U = 0, \\ U(0,x) = U_0(x), \end{cases} \tag{1.4}$$

where $U = [\partial_x u, \partial_t u, \theta_1, \theta_2]^T$ and

$$A(t,x,\partial_x,\partial_x^2) = \begin{bmatrix} 0 & -\partial_x & 0 & 0 \\ -\frac{\lambda+2\mu}{\rho}\partial_x & 0 & \frac{\gamma_1}{\rho}\partial_x & \frac{\gamma_2}{\rho}\partial_x \\ 0 & \frac{n\gamma_1-d\gamma_2}{nc-d^2}\partial_x & -\frac{kn}{nc-d^2}\partial_x^2 & \frac{dD}{nc-d^2}\partial_x^2 \\ 0 & -\frac{d\gamma_1-c\gamma_2}{nc-d^2}\partial_x & \frac{kd}{nc-d^2}\partial_x^2 & -\frac{cD}{nc-d^2}\partial_x^2 \end{bmatrix}. \tag{1.5}$$

We now formulate the main result of the paper.

THEOREM 1.1 (Main Theorem). *Let p, q satisfy the conditions $1 < p < 2 < q < \infty$, $1/p+1/q = 1$, and $N \in \mathbb{N}$. If the initial data* (1.2) *are sufficiently smooth, namely $U_0 \in W^{N,p}(\mathbb{R})$ with $N > (2/p-1)(s-1)+1 \geq 0$, $s > 2$, then the solution of the initial-value problem* (1.1)–(1.2) *has the following properties:*

$$\|U(t,\cdot)\|_{L^q} \leq C(1+t)^{1/2-1/p}\|U_0\|_{W^{N,p}} \qquad \forall t > 0,$$

where C is independent of U_0 and t.

The proof of Theorem 1.1 will be divided into three steps:

- investigation of the behaviour of the roots of the characteristic equation of system (1.1),
- description of the behaviour of the solution of (1.1)–(1.2),
- the proof of (1.9).

2. Basic notation. Below we will give the main notation and recall some theorems which will be used in our paper. We denote the points of $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and equip $\mathbb{R}^n$ with the canonical metric

$$|x-y| = \Big[\sum_{i=1}^{n}(x_i - y_i)^2\Big]^{1/2}.$$

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\partial_i = \partial/\partial x_i$ for x_i, $i = 1, \ldots, n$, then $\partial_x^\alpha = \partial_1^{\alpha_1} \cdot \ldots \cdot \partial_n^{\alpha_n}$. Let X be a Banach space and $I \subset \mathbb{R}$ a compact interval. Then $C^k(I,X)$ ($k \geq 0$, an integer) denotes the space of k-times continuously differentiable functions f on I with values in X (cf. [12]) with the norm

$$\|f\|_{C^k(I,X)} = \sup_{t\in I}\sum_{i=0}^{k}\|\partial_t^i f(t)\|_X.$$

The Fourier transformation of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is defined as follows (cf. [11], [13]):

$$\mathcal{S}(\mathbb{R}^n) \ni \mathcal{F}f(\xi) \equiv \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx.$$

The inverse Fourier transformation of a rapidly decreasing function is defined as follows:

$$\mathcal{S}(\mathbb{R}^n) \ni \mathcal{F}^{-1}(\mathcal{F}f)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}(\mathcal{F}f)(\xi)\,d\xi.$$

The direct and inverse Fourier transformation are extended to $\mathcal{S}'(\mathbb{R}^n)$ by

$$\mathcal{F}T(\varphi) \equiv T(\mathcal{F}\varphi), \qquad \mathcal{F}^{-1}T(\varphi) \equiv T(\mathcal{F}^{-1}\varphi).$$

Theorem 2.1. *Let $1 \le p \le 2$. The Fourier transformation is a linear and continuous map from $L^p(\mathbb{R}^n)$ to $L^{p/(p-1)}(\mathbb{R})$ and*

$$\|\mathcal{F}f\|_{L^{p/(p-1)}(\mathbb{R}^n)} \le (2\pi)^{n(1/2-1/p)}\|f\|_{L^p(\mathbb{R}^n)}.$$

The Sobolev space $W^{m,p}(\mathbb{R}^n)$, $1 \le p < \infty$, $m \in \mathbb{N}$, consists of all functions $u \in L^p(\mathbb{R}^n)$ for which the weak partial derivatives $\partial^\alpha u$ of order α ($|\alpha| \le m$) belong to $L^p(\mathbb{R}^n)$ (cf. [1], [2]), i.e.

$$W^{m,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \partial^\alpha u \in L^p(\mathbb{R}^n),\ |\alpha| \le m\},$$

with the norm

$$\|u\|_{W^{m,p}(\mathbb{R}^n)} = \Big(\sum_{|\alpha|\le m} \|\partial^\alpha u\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p}.$$

For $s \in \mathbb{R}$ and $1 \le p \le \infty$ let $L^{s,p}(\mathbb{R}^n)$ denote the image of $L^p(\mathbb{R}^n)$ under the linear mapping $J^s u = \mathcal{F}^{-1}((1+|\cdot|^2)^{-s/2}\mathcal{F}u)$, with the norm

$$\|f\|_{L^{s,p}(\mathbb{R}^n)} \equiv \|\mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\mathcal{F}f)\|_{L^p(\mathbb{R})}$$

(cf. [2]). It is a Banach space with this norm. We denote by $B^s_{pq}(\mathbb{R}^n)$ the Besov space (cf. [2]), i.e.

$$B^s_{pq}(\mathbb{R}^n) \equiv \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{pq}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{B^s_{pq}(\mathbb{R}^n)} \equiv \Big\{\|\psi \star f\|^q_{L^p(\mathbb{R})} + \sum_{k=1}^{\infty}(2^{sk}\|\varphi_k \star f\|_{L^p(\mathbb{R}^n)})^q\Big\}^{1/q}.$$

Here

- $\star$ denotes convolution,
- $1 \le p \le \infty,\ s \ge 0$,
- $\varphi \in C_0^\infty(\mathbb{R}^n)$ is a nonnegative function with

$$\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}, \qquad \sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1 \text{ for } \xi \ne 0,$$

- $\mathcal{F}\psi(\xi) = 1 - \sum_{k=1}^{\infty}\varphi(2^{-k}\xi),\ \mathcal{F}\varphi_k(\xi) = \varphi(2^{-k}\xi),\ k \ge 1.$

If $s \ge 0$ is an integer, then $B^s_{22}(\mathbb{R}^n) = L^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$.

Theorem 2.2. *Let "$\hookrightarrow$" denote a continuous imbedding, and $1 < p \le q < \infty$.*

(a) *If $s > k + n/p$, then $W^{s,p}(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$.*
(b) *If $s > t$ and $1/q \ge 1/p - (s-t)/n$, then $W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{t,q}(\mathbb{R}^n)$.*

Theorem 2.3.

(a) *If $s \ge 0$ is an integer, $1 \le p \le \infty$, then $B^s_{p1}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n)$.*
(b) *If $s \ge 0$, $1 < p < \infty$, $1 \le q \le \infty$, $\varepsilon \ge 0$, then $L^{s+\varepsilon,p}(\mathbb{R}^n) \hookrightarrow B^s_{pq}(\mathbb{R}^n)$.*

Definition 2.1.

(a) (X_0, X_1) is called an *interpolation couple* if X_0, X_1 are Banach spaces both continuously imbedded into a topological Hausdorff space.

(b) X is called an *intermediate space* between X_0 and X_1 if

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1 \quad \text{(continuous imbeddings)}.$$

(c) $[X_0, X_1]$, $[Y_0, Y_1]$ are called *interpolation spaces* for (X_0, X_1), (Y_0, Y_1) if and only if $[X_0, X_1]$ is an intermediate space between X_0 and X_1, $[Y_0, Y_1]$ is an intermediate space between Y_0 and Y_1 and

$$\text{if } T : X_j \to Y_j \text{ is continuous for } j = 0, 1, \text{ then}$$
$$T : [X_0, X_1] \to [Y_0, Y_1] \text{ is continuous.}$$

Theorem 2.4 (Riesz & Thorin). *Let* (L^{p_0}, L^{p_1}), (L^{q_0}, L^{q_1}), $1 \le p_0, q_0, p_1, q_1 \le \infty$, *be interpolation couples, let* $T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$ *be linear with*

$$T_{L^{p_0}} : L^{p_0}(\mathbb{R}^n) \to L^{q_0}(\mathbb{R}^n) \ \text{ bounded with norm } M_0$$
$$T_{L^{p_1}} : L^{p_1}(\mathbb{R}^n) \to L^{q_1}(\mathbb{R}^n) \ \text{ bounded with norm } M_1.$$

Then for all $\theta \in [0, 1]$,

$$T_{L^{p_\theta}} : L^{p_\theta}(\mathbb{R}^n) \to L^{q_\theta}(\mathbb{R}^n) \text{ is bounded with norm } M_0^{1-\theta} M_1^{\theta},$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

Theorem 2.5. *If* $s_0 \neq s_1$, $1 \le p_0, q_0, p_1, q_1 \le \infty$, $\Theta \in (0, 1)$, *then*

$$[B^{s_0}_{p_0 q_0}, B^{s_1}_{p_1 q_1}]_\Theta = B^{s_\Theta}_{p_\Theta q_\Theta}$$

where

$$s_\Theta = (1-\Theta)s_0 + \Theta s_1, \qquad \frac{1}{p_\Theta} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \qquad \frac{1}{q_\Theta} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}.$$

All theorems mentioned above can found in [1], [2], [3], [7], [11], [15], [16].

3. Behaviour of the roots of the characteristic equation of the system (1.1)**.** First, we investigate the behaviour of the roots of the characteristic equation of the system (1.1). Applying the Fourier transformation with respect to x, we can write the system (1.4) in the form

$$\begin{cases} \dfrac{d}{dt}\hat{U} + A(t, \xi, -i\xi, -\xi^2)\hat{U} = 0, \\ \hat{U}(0, \xi) = \hat{U}_0(\xi), \end{cases}$$

where

$$\hat{U} = (\hat{w}, \hat{v}, \hat{\theta}_1, \hat{\theta}_2)^T = (\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{U}_4)^T.$$

The Fourier transform of the matrix (1.5) can be written in the form

$$A(t,\xi,-i\xi,-\xi^2)=\begin{bmatrix} 0 & i\xi & 0 & 0\\ i\frac{\lambda+2\mu}{\rho} & 0 & -i\frac{\gamma_1}{\rho}\xi & -i\frac{\gamma_2}{\rho}\xi\\ 0 & -i\frac{n\gamma_1-d\gamma_2}{nc-d^2}\xi & \frac{kn}{nc-d^2}\xi^2 & -\frac{dD}{nc-d^2}\xi^2\\ 0 & i\frac{d\gamma_1-c\gamma_2}{nc-d^2}\xi & -\frac{kd}{nc-d^2}\xi^2 & \frac{cD}{nc-d^2}\xi^2 \end{bmatrix}.$$

After some calculations we find that the characteristic equation of the system (1.4) has the form

$$\det(A-\lambda I)=\lambda^4-\zeta\xi^2\lambda^3+(\eta\xi^2+\varphi)\xi^2\lambda^2-\psi\xi^4\lambda+c_1^2\eta\xi^6=0, \tag{3.1}$$

where

$$\begin{aligned} &\zeta=\tilde{\alpha}\rho(kn+cD),\qquad \eta=kD\tilde{\alpha}\rho,\\ &\psi=\tilde{\alpha}[D\gamma_1^2+k\gamma_2^2+c_1^2\rho(kn+cD)],\\ &\varphi=\tilde{\alpha}\gamma_1(\gamma_1 n-\gamma_2 d)+\tilde{\alpha}\gamma_2(c\gamma_2-d\gamma_1)+c_1^2,\\ &c_1^2=\frac{\lambda+2\mu}{\rho},\qquad \tilde{\alpha}=\frac{1}{\rho(nc-d^2)}. \end{aligned} \tag{3.2}$$

In order to find the roots of equation (3.1), we put

$$\lambda=y+\frac{1}{4}\,\zeta\xi^2,$$

so (3.1) takes the form

$$y^4+py^2+qy+r=0, \tag{3.3}$$

where

$$\begin{aligned} p&=\left[\left(\eta-\frac{3}{8}\,\zeta^2\right)\xi^2+\varphi\right]\zeta^2,\\ q&=\left[\frac{1}{2}\left(\eta-\frac{1}{4}\,\zeta^2\right)\zeta\xi^2+\frac{1}{2}\,\zeta\varphi-\psi\right]\xi^4,\\ r&=\frac{1}{16}\left(\eta-\frac{3}{16}\,\zeta^2\right)\zeta^2\xi^8+\frac{1}{16}\,\zeta^2\varphi\xi^6-\frac{1}{4}\,\zeta\psi\xi^6+c_1^2\eta\xi^6. \end{aligned}$$

The resolvent of equation (3.3) is

$$z^3+2pz^2+(p^2-4r)z-q^2=0. \tag{3.4}$$

Applying the Cardano method (cf. [17]) we can represent the roots of (3.4) in the form

$$\begin{aligned} z_1&=\alpha+\beta-2\delta,\\ z_2&=-\frac{1}{2}\,(\alpha+\beta)+i\,\frac{\sqrt{3}}{2}\,(\alpha-\beta)-2\delta,\\ z_3&=-\frac{1}{2}\,(\alpha+\beta)-i\,\frac{\sqrt{3}}{2}\,(\alpha-\beta)-2\delta, \end{aligned} \tag{3.5}$$

where

$$\alpha = \sqrt[3]{\frac{1}{27}(\eta\xi^2+\varphi)^3\xi^6 - \frac{1}{6}(M\xi^2+N)\xi^8 + \sqrt{\Delta(\xi)}},$$

$$\beta = \sqrt[3]{\frac{1}{27}(\eta\xi^2+\varphi)^3\xi^6 - \frac{1}{6}(M\xi^2+N)\xi^8 - \sqrt{\Delta(\xi)}}, \tag{3.6}$$

$$\delta = \frac{1}{3}\left[\left(\eta - \frac{3}{8}\zeta^2\right)\xi^2 + \varphi\right]\xi^2,$$

with

$$\Delta(\xi) = -\frac{1}{81}T(\eta\xi^2+\varphi)^4\xi^{14} - \frac{1}{81}(\eta\xi^2+\varphi)^3(M\xi^2+N)\xi^{14} + \frac{1}{36}(M\xi^2+N)^2\xi^{16} - \frac{1}{27}T^2(\eta\xi^2+\varphi)^2\xi^{16} - \frac{1}{27}T^3\xi^{18} \tag{3.7}$$

and

$$M = (8c_1^2\eta + \zeta\psi - 3c_1^2\zeta^2)\eta, \quad N = 8c_1^2\eta\varphi + \zeta\varphi\psi - 3\psi^2,$$
$$T = 4c_1^2\eta - \zeta\psi. \tag{3.8}$$

Now, from the real root z_1 of equation (3.4) we construct the roots of (3.1). We consider the linear system associated with (3.3) in the following form:

$$y^2 + \sqrt{z_1}\,y + \frac{1}{2}(z_1+p) - \frac{q}{2\sqrt{z_1}} = 0,$$
$$y^2 - \sqrt{z_1}\,y + \frac{1}{2}(z_1+p) - \frac{q}{2\sqrt{z_1}} = 0. \tag{3.9}$$

The roots of the system (3.9) can be written in the following form:

$$y_1 = -\frac{1}{2}\sqrt{z_1} - \frac{1}{2}\sqrt{-z_1 - 6\delta + 2q/\sqrt{z_1}},$$
$$y_2 = -\frac{1}{2}\sqrt{z_1} + \frac{1}{2}\sqrt{-z_1 - 6\delta + 2q/\sqrt{z_1}},$$
$$y_3 = \frac{1}{2}\sqrt{z_1} - \frac{1}{2}\sqrt{-z_1 - 6\delta - 2q/\sqrt{z_1}},$$
$$y_4 = \frac{1}{2}\sqrt{z_1} + \frac{1}{2}\sqrt{-z_1 - 6\delta - 2q/\sqrt{z_1}}.$$

So, the roots of (3.1) have the following form:

$$\lambda_{1,2} = \frac{1}{4}\zeta\xi^2 - \frac{1}{2}\sqrt{z_1} \mp \sqrt{-z_1 - 6\delta + 2q/\sqrt{z_1}},$$
$$\lambda_{3,4} = \frac{1}{4}\zeta\xi^2 + \frac{1}{2}\sqrt{z_1} \mp \sqrt{-z_1 - 6\delta - 2q/\sqrt{z_1}}. \tag{3.10}$$

We can now formulate the following theorem:

Theorem 3.1. *The characteristic roots have the following asymptotics:*

(i) *For* $\xi \to 0$ *we have:*

$$\begin{cases} \lambda_1(\xi) = \dfrac{1}{2\varphi}\,(\psi - \sqrt{\psi^2 - 4c_1^2\eta\varphi})\xi^2 + O(\xi^4), \\ \lambda_2(\xi) = \dfrac{1}{2\varphi}\,(\psi + \sqrt{\psi^2 - 4c_1^2\eta\varphi})\xi^2 + O(\xi^4), \\ \lambda_3(\xi) = -i\sqrt{\varphi}|\xi| + \dfrac{\zeta\varphi - \psi}{2\varphi}\,\xi^2 + O(|\xi|^3), \\ \lambda_4(\xi) = i\sqrt{\varphi}|\xi| + \dfrac{\zeta\varphi - \psi}{2\varphi}\,\xi^2 + O(|\xi|^3), \end{cases} \tag{3.11}$$

with

$$\begin{aligned} &\psi^2 - 4c_1^2\eta\varphi > 0, \\ &\psi - \sqrt{\psi^2 - 4c_1^2\eta\varphi} > 0, \\ &\zeta\varphi - \psi > 0, \\ &\varphi > 0. \end{aligned} \tag{3.12}$$

(ii) *For* $\xi \to \pm\infty$ *we have:*

$$\begin{cases} \lambda_1(\xi) = -ic_1|\xi| + \dfrac{\psi - c_1^2\zeta}{2\eta} + O(|\xi|^{-1}), \\ \lambda_2(\xi) = ic_1|\xi| + \dfrac{\psi - c_1^2\zeta}{2\eta} + O(|\xi|^{-1}), \\ \lambda_3(\xi) = \dfrac{1}{2}\,(\zeta - \sqrt{\zeta^2 - 4\eta})\xi^2 \\ \qquad\qquad - \dfrac{(\psi - c_1^2\zeta)(\zeta + \sqrt{\zeta^2 - 4\eta}) - 2\eta(\varphi - c_1^2)}{2\eta\sqrt{\zeta^2 - 4\eta}} + O(\xi^{-2}), \\ \lambda_4(\xi) = \dfrac{1}{2}\,(\zeta + \sqrt{\zeta^2 - 4\eta})\xi^2 \\ \qquad\qquad + \dfrac{(\psi - c_1^2\zeta)(\zeta - \sqrt{\zeta^2 - 4\eta}) - 2\eta(\varphi - c_1^2)}{2\eta\sqrt{\zeta^2 - 4\eta}} + O(\xi^{-2}), \end{cases} \tag{3.13}$$

with

$$\begin{aligned} &\psi - c_1^2\zeta > 0, \\ &\zeta^2 - 4\eta > 0, \\ &\zeta \pm \sqrt{\zeta^2 - 4\eta} > 0. \end{aligned} \tag{3.14}$$

Proof. The properties (3.11) and (3.13) come as a result of tedious calculations. First, we find the asymptotics of the root z_1 of equation (3.4). Since

$$z_1 = \alpha + \beta - 2\delta$$

where α, β, δ are given by (3.6) and $\alpha(\xi) = \bar{\alpha}(\xi)\xi^2$, $\beta(\xi) = \overline{\beta}\xi^2$, $\Delta(\xi) = \bar{\Delta}(\xi)\xi^{14}$ and

$$\begin{aligned}
\bar{\alpha}(\xi) &= \sqrt[3]{\frac{1}{27}\,(\eta\xi^2+\varphi)^3 - \frac{1}{6}\,(M\xi^2+N)\xi^2 + \sqrt{\bar{\Delta}(\xi)}\,|\xi|}\\
\bar{\beta}(\xi) &= \sqrt[3]{\frac{1}{27}\,(\eta\xi^2+\varphi)^3 - \frac{1}{6}\,(M\xi^2+N)\xi^2 - \sqrt{\bar{\Delta}(\xi)}\,|\xi|}\\
\bar{\Delta}(\xi) &= -\frac{1}{81}\,T(\eta\xi^2+\varphi)^4 - \frac{1}{81}\,(\eta\xi^2+\varphi)^3(M\xi^2+N)\\
&\quad + \frac{1}{36}\,(M\xi^2+N)^2\xi^2 - \frac{1}{27}\,T^2(\eta\xi^2+\varphi)^2\xi^2 - \frac{1}{27}\,T^3\xi^4
\end{aligned}$$

we have

$$\bar{\alpha}(0) = \frac{\varphi}{3}, \qquad \bar{\Delta}(0) = -\frac{\varphi^3}{81}\,(T\varphi+N).$$

Now, we can calculate the following limits:

$$\lim_{\xi\to 0}\frac{\bar{\alpha}(\xi)-\frac{1}{3}\varphi}{|\xi|} = -\frac{\sqrt{-T\varphi^2-N\varphi}}{3\varphi},$$

$$\lim_{\xi\to 0}\frac{1}{\xi^2}\left(\bar{\alpha}(\xi) - \frac{1}{3}\,\varphi + \frac{\sqrt{-T\varphi^2-N\varphi}}{3\varphi}\right) = \frac{1}{3\varphi^2}\left(\varphi^2\eta - \frac{1}{2}\,N + T\varphi\right). \tag{3.15}$$

After some calculations, we obtain the asymptotic expansion for $\bar{\alpha}(\xi)$ in the neighbourhood of the origin:

$$\bar{\alpha}(\xi) = \frac{1}{3}\,\varphi - \frac{i\sqrt{T\varphi^2+N\varphi}}{3\varphi}\,|\xi| + \frac{1}{3\varphi^2}\left(\varphi^2\eta + T\varphi - \frac{1}{2}\,N\right)\xi^2 + O(|\xi|^3)$$

Similarly, we can obtain the asymptotics for β. Finally, we have

$$\begin{aligned}
\alpha(\xi) &= \frac{1}{3}\,\varphi\xi^2 - \frac{i\sqrt{T\varphi^2+N\varphi}}{3\varphi}\,|\xi|^3 + \frac{1}{3\varphi^2}\left(\varphi^2\eta + T\varphi - \frac{1}{2}\,N\right)\xi^4 + O(|\xi|^5),\\
\beta(\xi) &= \frac{1}{3}\,\varphi\xi^2 + \frac{i\sqrt{T\varphi^2+N\varphi}}{3\varphi}\,|\xi|^3 + \frac{1}{3\varphi^2}\left(\varphi^2\eta + T\varphi - \frac{1}{2}\,N\right)\xi^4 + O(|\xi|^5).
\end{aligned}$$

Remark 3.1 (see (3.12)). We have

$$-T\varphi - N = 3(\psi^2 - 4c_1^2\eta\varphi) > 0.$$

Using (3.5), (3.6) we obtain

$$z_1 = \frac{(\psi - \frac{1}{2}\zeta\varphi)^2}{\varphi^2}\,\xi^4 + A\xi^6 + O(\xi^8) \tag{3.16}$$

where

$$A = \frac{1}{\varphi^5}\Big\{2\psi\Big(\frac{1}{2}\,\zeta\varphi - \psi\Big)^3 + 4c_1^2\eta\varphi\Big(\frac{1}{2}\,\zeta\varphi - \psi\Big)^2 + \frac{1}{2}\,\varphi^2\psi(4\eta - \zeta^2)\Big(\frac{1}{2}\,\zeta\varphi - \psi\Big)\Big\}$$

Moreover, from (3.16) we get the following asymptotic expansions in the neighbourhood of the origin:

$$\sqrt{z_1} = \frac{1}{\varphi}\Big|\frac{1}{2}\,\zeta\varphi - \psi\Big|\xi^2 + \frac{A\varphi}{2(\frac{1}{2}\zeta\varphi)}\,\xi^4 + O(\xi^6),$$

$$\sqrt{-z_1 - 6\delta + \frac{2q}{\sqrt{z_1}}} = \begin{cases} \dfrac{\sqrt{\psi^2 - 4c_1^2\eta\varphi}}{\varphi}\,\xi^2 + O(\xi^2), & \dfrac{1}{2}\,\zeta\varphi - \psi > 0,\\[2mm] 2i\sqrt{\varphi}|\xi| + O(|\xi|^3), & \dfrac{1}{2}\,\zeta\varphi - \psi \le 0,\end{cases}$$

$$\sqrt{-z_1 - 6\delta - \frac{2q}{\sqrt{z_1}}} = \begin{cases} 2i\sqrt{\varphi}|\xi| + O(|\xi|^3), & \dfrac{1}{2}\,\zeta\varphi - \psi > 0,\\[2mm] \sqrt{\dfrac{\psi^2 - 4c_1^2\eta\varphi}{\varphi}}\,\xi^2 + O(\xi^2), & \dfrac{1}{2}\,\zeta\varphi - \psi \le 0.\end{cases}$$

In view of (3.5), (3.6), (3.8), (3.10), we have

$$\lambda_i(\xi) = 0 \quad \text{ for } \xi = 0,\ i = 1, 2, 3, 4.$$

Putting together all the above considerations we obtain

$$\lambda_1(\xi) = \begin{cases} \dfrac{1}{2\varphi}\,(\psi - \sqrt{\psi^2 - 4c_1^2\eta\varphi})\xi^2 + O(\xi^4), & \dfrac{1}{2}\,\zeta\varphi - \psi > 0,\\[2mm] -i\sqrt{\varphi}|\xi| + \dfrac{\zeta\varphi - \psi}{2\varphi}\,\xi^2 + O(|\xi|^3), & \dfrac{1}{2}\,\zeta\varphi - \psi \le 0,\end{cases}$$

$$\lambda_2(\xi) = \begin{cases} \dfrac{1}{2\varphi}\,(\psi + \sqrt{\psi^2 - 4c_1^2\eta\varphi})\xi^2 + O(\xi^4), & \dfrac{1}{2}\,\zeta\varphi - \psi > 0,\\[2mm] i\sqrt{\varphi}|\xi| + \dfrac{\zeta\varphi - \psi}{2\varphi}\,\xi^2 + O(|\xi|^3), & \dfrac{1}{2}\,\zeta\varphi - \psi \le 0,\end{cases}$$

$$\lambda_3(\xi) = \begin{cases} -i\sqrt{\varphi}|\xi| + \dfrac{\zeta\varphi - \psi}{2\varphi}\,\xi^2 + O(|\xi|^3), & \dfrac{1}{2}\,\zeta\varphi - \psi > 0,\\[2mm] \dfrac{1}{2\varphi}\,(\psi - \sqrt{\psi^2 - 4c_1^2\eta\varphi})\xi^2 + O(\xi^4), & \dfrac{1}{2}\,\zeta\varphi - \psi \le 0,\end{cases}$$

$$\lambda_4(\xi) = \begin{cases} i\sqrt{\varphi}|\xi| + \dfrac{\zeta\varphi - \psi}{2\varphi}\,\xi^2 + O(|\xi|^3), & \dfrac{1}{2}\,\zeta\varphi - \psi > 0, \\ \dfrac{1}{2\varphi}\,(\psi - \sqrt{\psi^2 + 4c_1^2\eta\varphi})\xi^2 + O(\xi^4), & \dfrac{1}{2}\,\zeta\varphi - \psi \le 0. \end{cases}$$

Independently of the sign of $\frac{1}{2}\zeta\varphi - \psi$ we have the same asymptotic properties for the roots of equation (3.1) as in (3.11). Similarly, we can analyze the asymptotics of the roots in the neighbourhood of $\pm\infty$. The asymptotic expansion of z_1 in the neighbourhood of $\pm\infty$ is

$$z_1(\xi) = \frac{1}{4}\,\zeta^2\xi^4 - \frac{(\psi - c_1^2\zeta)\zeta}{\eta}\,\xi^2 + \frac{2}{3\eta^5}\,B + O(\xi^{-2})$$

where

$$B = \eta\varphi(M - \eta T) - \frac{1}{3}\,(M - \eta T)^2 + \frac{1}{2}\,\eta\Big(\frac{1}{3}\,MT - \eta N\Big).$$

The asymptotic expansion of the terms of the roots λ_i, $i = 1, 2, 3, 4$, can be obtained in the following form:

$$\begin{aligned} \frac{2q}{\sqrt{z_1}} &= \Big(2\eta - \frac{1}{2}\,\zeta^2\Big)\xi^4 + \Big[2(\varphi - 2c_1^2) - \frac{(\psi - c_1^2\zeta)\zeta}{\eta}\Big]\xi^2 \\ &\quad + \Big[\Big(2\eta - \frac{1}{2}\,\zeta^2\Big)\Big(-\frac{4B}{3\eta^5\zeta^2} + 6\,\frac{(\psi - c_1^2\zeta)^2}{\eta^2\zeta^2}\Big) + 4\,\frac{(\zeta\varphi - 2\psi)(\psi - c_1^2\zeta)}{\eta\zeta^2}\Big] \\ &\quad + O(\xi^{-2}) \end{aligned}$$

and

$$\sqrt{z_1} = \frac{1}{2}\,\zeta\xi^2 - \frac{\psi - c_1^2\zeta}{\eta} + O(\xi^{-2}),$$
$$\sqrt{-z_1 - 6\delta + 2q/\sqrt{z_1}} = ic_1|\xi| + O(|\xi|^{-1}),$$
$$\sqrt{-z_1 - 6\delta - 2q/\sqrt{z_1}} = \sqrt{\zeta^2 - 4\eta}\,\xi^2 + \frac{(\psi - c_1^2\zeta)\zeta - 2\eta(\varphi - c_1^2)}{\eta\sqrt{\zeta^2 - 4\eta}} + O(\xi^{-2}).$$

Using (3.10) we obtain (3.13). To prove the first relation in (3.12), using (3.2) after easy calculations, we have

$$\begin{aligned} \psi^2 - 4c_1^2\eta\varphi &= \tilde{\alpha}^2[-D\gamma_1^2 + k\gamma_2^2 + c_1^2(kn - cD)]^2 \\ &\qquad + 4\tilde{\alpha}^2 kD\gamma_1\gamma_2(\gamma_1\gamma_2 + 2c_1^2\rho d) + 4c_1^4\tilde{\alpha}^2\rho^2 kDd^2 > 0. \end{aligned}$$

From this and (1.3) we obtain the second relation of (3.12). To prove the third relation of (3.12), we note that by (1.3),

$$\begin{aligned} \zeta\varphi - \psi &= \tilde{\alpha}^2\rho[(kn^2 + Dd^2)\gamma_1^2 + (c^2D + kd^2)\gamma_2^2 - 2(kn + cD)d\gamma_1\gamma_2] \\ &= \tilde{\alpha}^2\rho[k(n\gamma_1 - d\gamma_2)^2 + D(d\gamma_1 - c\gamma_2)^2] > 0 \end{aligned}$$

and

$$n\gamma_1 - d\gamma_2 \neq 0 \ \vee \ d\gamma_1 - c\gamma_2 \neq 0.$$

For the last relation in (3.12), observe that

$$\varphi = \tilde{\alpha}(n\gamma_1^2 + c\gamma_2^2 - 2d\gamma_1\gamma_2) + c_1^2$$

and in view of (1.3) the discriminant of the bracketed expression is equal to $4(d^2 - nc) < 0$.

Now, the first relation in (3.14) is a consequence of (3.2) and Remark 3.1. So, we have

$$\psi - c_1^2\zeta = \tilde{\alpha}(D\gamma_1^2 + k\gamma_2^2) > 0.$$

The second inequality of (3.14) is a consequence of

$$\zeta^2 - 4\eta = c_1^2\tilde{\alpha}^2\rho^2[4kDd^2 + (kn - cD)^2] > 0.$$

The last inequality in (3.14) is a consequence of the above considerations. ∎

Remark 3.2. The characteristic equation (3.1) has multiple roots for at most eight values of $\xi \neq 0$, and we have the following possibilities:

(a) For $\xi = 0$ the characteristic equation has one fourfold root.
(b) If $\xi \neq 0$ the characteristic equation has one double root or two double roots.

Proof. The characteristic equation has multiple roots if and only if $\Delta(\xi)$ given by (3.7) is equal to zero. We can write

$$\Delta(\xi) = \Big\{-\frac{1}{81}T(\eta\xi^2 + \varphi)^4 - \frac{1}{81}(\eta\xi^2 + \varphi)^3(M\xi^2 + N) + \frac{1}{36}(M\xi^2 + N)^2\xi^2 - \frac{1}{27}T^2(\eta\xi^2 + \varphi)^2\xi^2 - \frac{1}{27}T^3\xi^4\Big\}\xi^{14}. \tag{3.17}$$

In brackets in (3.17) we have a polynomial of degree eight. We consider the following cases:

(a) If $\xi = 0$, then $\lambda_i(0) = 0$ for $i = 1, 2, 3, 4$ (cf. (3.1)).

(b) Let $\xi \neq 0$. The characteristic equation does not have a fourfold root, because in this case all coefficients of the equation (3.3) are zero, which is impossible. For example:

$$\begin{cases} \Big(\eta - \frac{3}{8}\zeta^2\Big)\xi^2 + \varphi = 0, \\ \frac{1}{2}\Big(\eta - \frac{1}{4}\zeta^2\Big)\zeta\xi^2 + \frac{1}{2}\zeta\varphi - \psi = 0. \end{cases}$$

Solving the first equation and substituting ξ in the second equation we have

$$\zeta^3\varphi - 2(8\eta - 3\zeta^2)\psi > 0.$$

If the characteristic equation has one real triple root, then all coefficients of the equation

$$x^3 - \left[\frac{1}{3}(\eta\xi^2+\varphi)^2 + T\xi^2\right]\xi^4 x - \frac{2}{27}(\eta\xi^2+\varphi)^3\xi^6 + \frac{1}{3}M\xi^{10} + \frac{1}{3}N\xi^8 = 0$$

are zero. This is not possible, because $4\eta\varphi + 3T > 0$ and $\xi \in \mathbb{R}$. ■

LEMMA 3.1.

(a) *For all* $|\xi| > 0$, $\operatorname{Re}\lambda_j(\xi) > 0$, $j = 1,2,3,4$.
(b) *There are positive constants* r_1, r_2 *and* C_j, $j = 1,2,3,4,5$, *that depend on* r_1, r_2, *for which:*

$$\text{if } |\xi| < r_1 \text{ then } C_1|\xi|^2 < \operatorname{Re}\lambda_j(\xi) < C_2|\xi|^2 \quad \text{for } j = 1,2,3,4, \tag{3.18}$$

$$\text{if } |\xi| > r_2 \text{ then } \begin{cases} \operatorname{Re}\lambda_j(\xi) > C_3 & \text{for } j = 1,2, \\ C_4\xi^2 < \operatorname{Re}\lambda_j(\xi) < C_5\xi^2 & \text{for } j = 3,4. \end{cases} \tag{3.19}$$

Proof. This is a consequence of the asymptotic expansion and the continuity of roots.

(b) Inequalities (3.18) and (3.19) follow from the asymptotic expansion of roots (3.11) and (3.13) of the characteristic equation.

(a) In view of (b) we know that the real parts of the roots of the characteristic equation are positive in a sufficiently small neighbourhood of the points 0 and $\pm\infty$. Since the roots given by (3.10) are continuous functions, the real parts of the roots can be negative when there exist purely imaginary roots of (3.1), i.e. $\operatorname{Re}\lambda_j = 0$ and

$$\begin{aligned} \det(A - iyI) = y^4 - (\eta\xi^2+\varphi)\xi^2 y^2 + c_1^2\eta\xi^6 \\ + i(\zeta\xi^2 y^2 - \psi\xi^4)y = 0, \quad y \in \mathbb{R}. \end{aligned} \tag{3.20}$$

(3.20) is equivalent to the system

$$\begin{cases} y^4 - (\eta\xi^2+\varphi)\xi^2 y^2 + c_1^2\eta\xi^6 = 0, \\ (\zeta y^2 - \psi\xi^2)\xi^2 y = 0. \end{cases} \tag{3.21}$$

The second equation has two solutions

$$y = 0 \quad \text{or} \quad y = \pm\sqrt{\psi\xi^2/\zeta}.$$

The solution $y = 0$ does not satisfy the first equation in (3.21). Substituting the second solution $y = \pm\sqrt{\psi\xi^2/\zeta}$ to the first equation (3.21) we get

$$\frac{1}{\zeta^2}\left[\eta\zeta(c_1^2\zeta - \psi)\xi^2 + \psi(\psi - \zeta\varphi)\right] = 0. \tag{3.22}$$

Basing on Remark 3.1 and the formula (3.12) we can see that the solution of equation (3.22) has no real part for $|y| > 0$.

4. The asymptotic behaviour of the solution of the Cauchy problem (1.1)–(1.2)**.** In this subsection we prove the asymptotic behaviour of the solution to the Cauchy problem (1.1)–(1.2). After some calculations, we get the modal matrix $\mathbb{M}$ of the eigenvectors in the form

$$\mathbb{M}= \tag{4.1}$$
$$\begin{bmatrix} \frac{i\xi}{\lambda_1} & \frac{i\xi}{\lambda_2} & \frac{i\xi}{\lambda_3} & \frac{i\xi}{\lambda_4} \\ 1 & 1 & 1 & 1 \\ -\frac{i\xi(\gamma_1\lambda_1-B_3)}{(\lambda_1-B_1)(\lambda_1-B_2)} & -\frac{i\xi(\gamma_1\lambda_2-B_3)}{(\lambda_2-B_1)(\lambda_2-B_2)} & -\frac{i\xi(\gamma_1\lambda_3-B_3)}{(\lambda_3-B_1)(\lambda_3-B_2)} & -\frac{i\xi(\gamma_1\lambda_4-B_3)}{(\lambda_4-B_1)(\lambda_4-B_2)} \\ \frac{i\xi(\gamma_1\lambda_1-B_4)}{(\lambda_1-B_1)(\lambda_1-B_2)} & \frac{i\xi(\gamma_1\lambda_2-B_4)}{(\lambda_2-B_1)(\lambda_2-B_2)} & \frac{i\xi(\gamma_1\lambda_3-B_4)}{(\lambda_3-B_1)(\lambda_3-B_2)} & \frac{i\xi(\gamma_1\lambda_4-B_4)}{(\lambda_4-B_1)(\lambda_4-B_2)} \end{bmatrix}$$

where

$$\begin{aligned} B_1 &= \frac{1}{2}\,\xi^2(\zeta+\sqrt{\zeta^2-4\eta}),\\ B_2 &= \frac{1}{2}\,\xi^2(\zeta-\sqrt{\zeta^2-4\eta}),\\ B_3 &= \tilde{\alpha}\rho\xi^2(kd\gamma_2+cD\gamma_1),\\ B_4 &= \tilde{\alpha}\rho\xi^2(kn\gamma_2+dD\gamma_1). \end{aligned} \tag{4.2}$$

Using the standard calculations of the modal matrix, we can write the inverse of the matrix (4.1) in the following form:

(4.3)

$$(\mathbb{M}^{-1})_{11} = -\frac{(\lambda_1-B_1)(\lambda_1-B_2)\prod_{i=1}^4\lambda_i}{i\xi B_1B_2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$

$$(\mathbb{M}^{-1})_{12} = \frac{\lambda_1(\lambda_1-B_1)(\lambda_1-B_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$

$$\begin{aligned}(\mathbb{M}^{-1})_{13} = {}&-\frac{i\lambda_1(\lambda_1-B_1)(\lambda_1-B_2)}{\xi B_1B_2(B_3\gamma_2-B_4\gamma_1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\\ &\times\{\lambda_2\lambda_3\lambda_4(B_4-\gamma_2B_2-\gamma_2B_1)+\gamma_2B_1B_2(\lambda_2\lambda_4+\lambda_2\lambda_3+\lambda_3\lambda_4)\\ &-B_1B_2B_4(\lambda_2+\lambda_3+\lambda_4)+B_1B_2(B_1B_4+B_2B_4-\gamma_2B_1B_2)\},\end{aligned}$$

$$\begin{aligned}(\mathbb{M}^{-1})_{14} = {}&\frac{i\lambda_1(\lambda_1-B_1)(\lambda_1-B_2)}{\xi B_1B_2(B_3\gamma_2-B_4\gamma_1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\\ &\times\{\lambda_2\lambda_3\lambda_4(B_3-\gamma_1B_2-\gamma_1B_1)+\gamma_1B_1B_2(\lambda_2\lambda_4+\lambda_2\lambda_3+\lambda_3\lambda_4)\\ &-B_1B_2B_3(\lambda_2+\lambda_3+\lambda_4)+B_1B_2(B_1B_3+B_2B_3-\gamma_1B_1B_2)\},\end{aligned}$$

$$(\mathbb{M}^{-1})_{21} = -\frac{(\lambda_2-B_1)(\lambda_2-B_2)\prod_{i=1}^4\lambda_i}{i\xi B_1B_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$

$$(\mathbb{M}^{-1})_{22} = \frac{\lambda_2(\lambda_2 - B_1)(\lambda_2 - B_2)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$

$$\begin{aligned}(\mathbb{M}^{-1})_{23} = &- \frac{i\lambda_2(\lambda_2 - B_1)(\lambda_2 - B_2)}{\xi B_1 B_2(B_3\gamma_2 - B_4\gamma_1)(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ &\times \{\lambda_1\lambda_3\lambda_4(B_4 - \gamma_2 B_2 - \gamma_2 B_1) + \gamma_2 B_1 B_2(\lambda_1\lambda_4 + \lambda_1\lambda_3 + \lambda_3\lambda_4) \\ &- B_1B_2B_4(\lambda_1 + \lambda_3 + \lambda_4) + B_1B_2(B_1B_4 + B_2B_4 - \gamma_2 B_1 B_2)\},\end{aligned}$$

$$\begin{aligned}(\mathbb{M}^{-1})_{24} = &\frac{i\lambda_2(\lambda_2 - B_1)(\lambda_2 - B_2)}{\xi B_1 B_2(B_3\gamma_2 - B_4\gamma_1)(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ &\times \{\lambda_1\lambda_3\lambda_4(B_3 - \gamma_1 B_2 - \gamma_1 B_1) + \gamma_1 B_1 B_2(\lambda_1\lambda_4 + \lambda_1\lambda_3 + \lambda_3\lambda_4) \\ &- B_1B_2B_3(\lambda_1 + \lambda_3 + \lambda_4) + B_1B_2(B_1B_3 + B_2B_3 - \gamma_1 B_1 B_2)\},\end{aligned}$$

$$(\mathbb{M}^{-1})_{31} = \frac{(\lambda_3 - B_1)(\lambda_3 - B_2)\prod_{i=1}^4 \lambda_i}{i\xi B_1B_2(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)},$$

$$(\mathbb{M}^{-1})_{32} = \frac{\lambda_3(\lambda_3 - B_1)(\lambda_3 - B_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)},$$

$$\begin{aligned}(\mathbb{M}^{-1})_{33} = &- \frac{i\lambda_3(\lambda_3 - B_1)(\lambda_3 - B_2)}{\xi B_1 B_2(B_3\gamma_2 - B_4\gamma_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} \\ &\times \{\lambda_1\lambda_2\lambda_4(B_4 - \gamma_2 B_2 - \gamma_2 B_1) + \gamma_2 B_1 B_2(\lambda_1\lambda_4 + \lambda_1\lambda_2 + \lambda_2\lambda_4) \\ &- B_1B_2B_4(\lambda_1 + \lambda_2 + \lambda_4) + B_1B_2(B_1B_4 + B_2B_4 - \gamma_2 B_1 B_2)\},\end{aligned}$$

$$\begin{aligned}(\mathbb{M}^{-1})_{34} = &\frac{i\lambda_3(\lambda_3 - B_1)(\lambda_3 - B_2)}{\xi B_1 B_2(B_3\gamma_2 - B_4\gamma_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} \\ &\times \{\lambda_1\lambda_2\lambda_4(B_3 - \gamma_1 B_2 - \gamma_1 B_1) + \gamma_1 B_1 B_2(\lambda_1\lambda_4 + \lambda_1\lambda_2 + \lambda_2\lambda_4) \\ &- B_1B_2B_3(\lambda_1 + \lambda_2 + \lambda_4) + B_1B_2(B_1B_3 + B_2B_3 - \gamma_1 B_1 B_2)\},\end{aligned}$$

$$(\mathbb{M}^{-1})_{41} = - \frac{(\lambda_4 - B_1)(\lambda_4 - B_2)\prod_{i=1}^4 \lambda_i}{i\xi B_1B_2(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)},$$

$$(\mathbb{M}^{-1})_{42} = \frac{\lambda_4(\lambda_4 - B_1)(\lambda_4 - B_2)}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)},$$

$$\begin{aligned}(\mathbb{M}^{-1})_{43} = &- \frac{i\lambda_4(\lambda_4 - B_1)(\lambda_4 - B_2)}{\xi B_1 B_2(B_3\gamma_2 - B_4\gamma_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \\ &\times \{\lambda_1\lambda_2\lambda_3(B_4 - \gamma_2 B_2 - \gamma_2 B_1) + \gamma_2 B_1 B_2(\lambda_1\lambda_3 + \lambda_1\lambda_2 + \lambda_2\lambda_3) \\ &- B_1B_2B_4(\lambda_1 + \lambda_2 + \lambda_3) + B_1B_2(B_1B_4 + B_2B_4 - \gamma_2 B_1 B_2)\},\end{aligned}$$

$$\begin{aligned}(\mathbb{M}^{-1})_{44} = &\frac{i\lambda_4(\lambda_4 - B_1)(\lambda_4 - B_2)}{\xi B_1 B_2(B_3\gamma_2 - B_4\gamma_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \\ &\times \{\lambda_1\lambda_2\lambda_3(B_3 - \gamma_1 B_2 - \gamma_1 B_1) + \gamma_1 B_1 B_2(\lambda_1\lambda_3 + \lambda_1\lambda_2 + \lambda_2\lambda_3) \\ &- B_1B_2B_3(\lambda_1 + \lambda_2 + \lambda_3) + B_1B_2(B_1B_3 + B_2B_3 - \gamma_1 B_1 B_2)\}.\end{aligned}$$

Taking into account (4.3) we can write the inverse matrix in the following form:

$$(\mathbb{M}^{-1})_{mn} = \frac{\lambda_m(\lambda_m - B_1)(\lambda_m - B_2)}{\xi B_1 B_2 (B_3\gamma_2 - B_4\gamma_1)\prod_{\substack{k=1\\k\neq m}}^{4}(\lambda_m - \lambda_k)}\Big\{\xi(B_3\gamma_2 - B_4\gamma_1)$$
$$\times\Big[-\prod_{\substack{k=1\\k\neq m}}^{4}\frac{\lambda_k\delta_{1n}}{i\xi} + B_1B_2\delta_{2n}\Big] + (-1)^n\Big[\prod_{\substack{k=1\\k\neq m}}^{4}\lambda_k(S_n - P_n(B_1+B_2))$$
$$+ P_nB_1B_2\sum_{k,l=1}^{4}(1-\delta_{kl})(1-\delta_{km})(1-\delta_{lm})\lambda_k\lambda_l$$
$$-B_1B_2S_n\sum_{\substack{k=1\\k\neq m}}^{4}\lambda_k + B_1B_2(S_n(B_1+B_2) - B_1B_2P_n)\Big]\Big\}$$

where

$$S_n = B_4\delta_{3n} + B_3\delta_{4n},$$
$$P_n = h_2\delta_{3n} + h_1\delta_{4n},$$
$$h_1 = \alpha\rho(n_1\gamma_1 - d\gamma_2),$$
$$h_2 = \alpha\rho(d_1\gamma_1 - c\gamma_2).$$

After some calculations on the Fourier transform of the solution to the Cauchy problem (1.1)–(1.2) we can write it in the following form:

$$\hat{U}_p = (\mathbb{M}e^{\Lambda t}\mathbb{M}^{-1})_{pq}\hat{U}_{0q} \tag{4.4}$$
$$= \sum_{q=1}^{4}\hat{U}_{0q}\sum_{\substack{n=1\\r=1}}^{4}\frac{e^{-\lambda_n t}}{\prod_{\substack{i=1\\i\neq n}}^{4}(\lambda_n - \lambda_i)}\Big(\frac{(-1)^r W_r(\lambda_n)}{B_3\gamma_2 - B_4\gamma_1} - i\xi\lambda_n\delta_{2q} + ic_1^2\delta_{1q}\Big)$$
$$\times\Big\{(\lambda_n - B_1)(\lambda_n - B_2)\Big(\frac{-i\xi\delta_{1p}}{\lambda_n} + \frac{i\delta_{2p}}{\xi}\Big) + i\xi\delta_{3p}(\gamma_1\lambda_n - B_3)$$
$$+ i\xi\delta_{4p}(\gamma_2\lambda_n - B_4)\Big\}$$

where

$$W_k(\lambda_n) = (\gamma_2\delta_{3k} + \gamma_1\delta_{4k})\lambda_n(\lambda_n^2 + \varphi\xi^2)$$
$$+ (\lambda_n^2 + c_1^2\xi^2)[B_4\delta_{3k} + B_3\delta_{4k} - (\gamma_2\delta_{3k} + \gamma_1\delta_{4k})\zeta\xi^2]$$

and δ_{ij} is the Kronecker symbol. After some calculations, we get

$$(Me^{\Lambda t}M^{-1})_{ij} = \hat{G}_{ij}(t,\xi) \tag{4.5}$$

where

$$\hat{G}_{ij}(t,\xi) \equiv a_{ij}e^{-\lambda_1 t} + b_{ij}e^{-\lambda_2 t} + c_{ij}e^{-\lambda_3 t} + d_{ij}e^{-\lambda_4 t}, \quad i,j = 1,2,3,4, \tag{4.6}$$

Our aim is to find the asymptotic expansion for $\hat{G}_{ij}(t,\xi)$. First we consider $\hat{G}_{ij}(t,\xi)$ in two intervals. For example we consider $\hat{G}_{11}(t,\xi)$ in the neighbourhood of 0 and $\pm\infty$. We expand the coefficients $a_{11}, b_{11}, c_{11}, d_{11}$ of (4.6) as follows:

$$\begin{aligned}
a_{11}(\xi) &= -\frac{\lambda_2\lambda_3\lambda_4(\lambda_1 - B_1)(\lambda_1 - B_2)}{B_1B_2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} \\
&= \frac{c_1^2(\psi-\zeta\varphi) + (\psi - c_1^2\zeta)\varphi + (\varphi - c_1^2)\sqrt{\psi^2 - 4c_1^2\eta\varphi}}{2\varphi\sqrt{\psi^2-4c_1^2\eta\varphi}} + O(\xi^2) \\
&= \bar{a}_{11} + O(\xi^2), \\
b_{11}(\xi) &= -\frac{\lambda_1\lambda_3\lambda_4(\lambda_2 - B_1)(\lambda_2 - B_2)}{B_1B_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\
&= \frac{c_1^2(\psi-\zeta\varphi) + (\psi - c_1^2\zeta)\varphi - (\varphi - c_1^2)\sqrt{\psi^2 - 4c_1^2\eta\varphi}}{2\varphi\sqrt{\psi^2-4c_1^2\eta\varphi}} + O(\xi^2) \\
&= \bar{b}_{11} + O(\xi^2), \\
c_{11}(\xi) &= -\frac{\lambda_1\lambda_2\lambda_4(\lambda_3 - B_1)(\lambda_3 - B_2)}{B_1B_2(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} \\
&= \frac{c_1^2}{2\varphi} + i\,\frac{3c_1^2(\zeta\varphi-\psi)}{4\varphi^2\sqrt{\varphi}}\,|\xi| + O(\xi^2) = \bar{c}_{11} + O(|\xi|), \\
d_{11}(\xi) &= -\frac{\lambda_1\lambda_2\lambda_3(\lambda_4 - B_1)(\lambda_4 - B_2)}{B_1B_2(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \\
&= \frac{c_1^2}{2\varphi} - i\,\frac{3c_1^2(\zeta\varphi-\psi)}{4\varphi^2\sqrt{\varphi}}\,|\xi| + O(\xi^2) = \bar{d}_{11} + O(|\xi|),
\end{aligned}$$

and similarly in the neighbourhood of $\pm\infty$:

$$\begin{aligned}
a_{11}(\xi) &= \frac{1}{2} - i\,\frac{\psi - c_1^2}{4c_1}\,|\xi|^{-1} + O(\xi^{-2}), \\
b_{11}(\xi) &= \frac{1}{2} + i\,\frac{\psi - c_1^2}{4c_1}\,|\xi|^{-1} + O(\xi^{-2}), \\
c_{11}(\xi) &= O(\xi^{-2}), \quad d_{11}(\xi) = O(\xi^{-2}).
\end{aligned}$$

By the above, we have for $|\xi| < r_1$ (where r_1 is sufficiently small), with accuracy up to terms with powers of highest exponents of ξ,

$$\begin{aligned}
\hat{G}_{11}(t,\xi) &= (a_{11}(\xi) - \bar{a}_{11})(e^{-\lambda_1 t} - e^{-\alpha_1\xi^2 t}) + \bar{a}_{11}(e^{-\lambda_1 t} - e^{-\alpha_1\xi^2 t}) \\
&\quad + (a_{11}(\xi) - \bar{a}_{11})e^{-\alpha_1\xi^2 t} + \bar{a}_{11}e^{-\alpha_1\xi^2 t} + (b_{11}(\xi) - \bar{b}_{11})(e^{-\lambda_2 t} - e^{-\alpha_2\xi^2 t})
\end{aligned}$$

$$
\begin{aligned}
&+ \bar{b}_{11}(e^{-\lambda_2 t} - e^{-\alpha_2\xi^2 t}) + (b_{11}(\xi) - \bar{b}_{11})e^{-\alpha_2\xi^2 t} + \bar{b}_{11}e^{-\alpha_2\xi^2 t}\\
&+ (c_{11}(\xi) - \bar{c}_{11})(e^{-\lambda_3 t} - e^{-i\sqrt{\varphi}|\xi|t-\alpha_3\xi^2 t})\\
&+ (c_{11}(\xi) - \bar{c}_{11})e^{-i\sqrt{\varphi}|\xi|t-\alpha_3\xi^2 t} + \bar{c}_{11}(e^{-\lambda_3 t} - e^{-i\sqrt{\varphi}|\xi|t-\alpha_3\xi^2 t})\\
&+ \bar{c}_{11}e^{-i\sqrt{\varphi}|\xi|t-\alpha_3\xi^2 t} + (d_{11}(\xi) - \bar{d}_{11})(e^{-\lambda_4 t} - e^{i\sqrt{\varphi}|\xi|t-\alpha_3\xi^2 t})\\
&+ (d_{11}(\xi) - \bar{d}_{11})e^{i\sqrt{\varphi}|\xi|t-\alpha_3\xi^2 t} + \bar{d}_{11}(e^{-\lambda_4 t} - e^{i\sqrt{\varphi}|\xi|t-\alpha_3\xi^2 t})\\
&+ \bar{d}_{11}e^{i\sqrt{\varphi}|\xi|t-\alpha_3\xi^2 t}
\end{aligned}
$$

where

$$
\alpha_1 = \frac{1}{2\varphi}(\psi - \sqrt{\psi^2 - 4c_1^2\eta\varphi}), \qquad \alpha_2 = \frac{1}{2\varphi}(\psi + \sqrt{\psi^2 - 4c_1^2\eta\varphi}),
$$
$$
\alpha_3 = \frac{\zeta\varphi - \psi}{2\varphi}.
$$

Moreover,

$$
\begin{aligned}
e^{-\lambda_1 t} - e^{-\alpha_1\xi^2 t} &= O(\xi^4)te^{-\alpha_1\xi^2 t},\\
e^{-\lambda_2 t} - e^{-\alpha_2\xi^2 t} &= O(\xi^4)te^{-\alpha_2\xi^2 t},\\
e^{-\lambda_3 t} - e^{-i\sqrt{\varphi}|\xi|-\alpha_3\xi^2 t} &= O(|\xi|^3)te^{-i\sqrt{\varphi}|\xi|-\alpha_3\xi^2 t},\\
e^{-\lambda_4 t} - e^{-i\sqrt{\varphi}|\xi|-\alpha_3\xi^2 t} &= O(|\xi|^3)te^{-i\sqrt{\varphi}|\xi|-\alpha_3\xi^2 t}.
\end{aligned}
$$

Then

$$
\hat{G}_{11}(t,\xi) = O(\xi)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}.
$$

Similarly we have for $|\xi| > r_2$ (for r_2 large enough), with accuracy up to terms with powers of highest exponents of ξ,

$$
\begin{aligned}
\hat{G}_{11}(t,\xi) = {}&\left(a_{11} - \frac{1}{2}\right)(e^{-\lambda_1 t} - e^{ic_1|\xi|t-\beta_1 t}) + \frac{1}{2}(e^{-\lambda_1 t} - e^{ic_1|\xi|t-\beta_1 t})\\
&+ \left(a_{11} - \frac{1}{2}\right)e^{ic_1|\xi|t-\beta_1 t} + \frac{1}{2}e^{ic_1|\xi|t-\beta_1 t} + \left(b_{11} - \frac{1}{2}\right)(e^{-\lambda_2 t} - e^{-ic_1|\xi|t-\beta_1 t})\\
&+ \frac{1}{2}(e^{-\lambda_2 t} - e^{-ic_1|\xi|t-\beta_1 t}) + \left(b_{11} - \frac{1}{2}\right)e^{-ic_1|\xi|t-\beta_1 t} + \frac{1}{2}e^{-ic_1|\xi|t-\beta_1 t}\\
&+ c_{11}(\xi)(e^{-\lambda_3 t} - e^{-\bar{B}_2\xi^2 t-\beta_2 t}) + c_{11}e^{-\bar{B}_2\xi^2 t-\beta_2 t}\\
&+ d_{11}(\xi)(e^{-\lambda_4 t} - e^{-\bar{B}_1\xi^2 t+\beta_2 t}) + c_{11}e^{-\bar{B}_1\xi^2 t+\beta_2 t},
\end{aligned}
$$

where (cf. (3.15))

$$
\beta_1 = \frac{\psi - c_1^2\zeta}{2\eta}, \qquad \beta_2 = \frac{(\psi - c_1^2\zeta)(\zeta + \sqrt{\zeta^2 - 4\eta}) - 2\eta(\varphi - c_1^2)}{2\eta\sqrt{\zeta^2 - 4\eta}},
$$
$$
B_i = \overline{B}_i\xi^2, \qquad i = 1, 2.
$$

Moreover,

$$
\begin{aligned}
e^{-\lambda_1 t} - e^{ic_1|\xi|t-\beta_1 t} &= O(|\xi|^{-1})te^{ic_1|\xi|t-\beta_1 t}, \\
e^{-\lambda_2 t} - e^{-ic_1|\xi|t-\beta_1 t} &= O(|\xi|^{-1})te^{-ic_1|\xi|t-\beta_1 t}, \\
e^{-\lambda_3 t} - e^{-\bar{B}_2\xi^2 t-\beta_2 t} &= O(\xi^{-2})te^{-\bar{B}_2\xi^2 t-\beta_2 t}, \\
e^{-\lambda_4 t} - e^{-\bar{B}_1\xi^2 t+\beta_2 t} &= O(\xi^{-2})te^{-\bar{B}_1\xi^2 t+\beta_2 t}.
\end{aligned}
$$

Thus

$$\hat{G}_{11}(t,\xi) = O(1)e^{-ct} + O(|\xi|^{-1})e^{-ct}.$$

For the other elements of the matrix $\hat{G}$ we obtain (c and B are universal positive constants):

1. For $|\xi| < r_1$:

$$
\begin{aligned}
\hat{G}_{11}(t,\xi) &= O(\xi)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{12}(t,\xi) &= O(\xi)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{13}(t,\xi) &= O(\xi)e^{-c\xi^2 t} + O(\xi^2)te^{-c\xi^2 t}, \\
\hat{G}_{14}(t,\xi) &= O(\xi)e^{-c\xi^2 t} + O(\xi^2)te^{-c\xi^2 t}, \\
\hat{G}_{21}(t,\xi) &= O(1)e^{-c\xi^2 t}, \\
\hat{G}_{22}(t,\xi) &= O(|\xi|)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{23}(t,\xi) &= O(\xi)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{24}(t,\xi) &= O(\xi)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{31}(t,\xi) &= O(1)e^{-c\xi^2 t}, \\
\hat{G}_{32}(t,\xi) &= O(\xi)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{33}(t,\xi) &= O(|\xi|)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{34}(t,\xi) &= O(|\xi|)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{41}(t,\xi) &= O(1)e^{-c\xi^2 t}, \\
\hat{G}_{42}(t,\xi) &= O(\xi)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{43}(t,\xi) &= O(|\xi|)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}, \\
\hat{G}_{44}(t,\xi) &= O(|\xi|)e^{-c\xi^2 t} + O(1)e^{-c\xi^2 t}.
\end{aligned}
$$

2. For $|\xi| > r_2$:

$$
\begin{aligned}
\hat{G}_{11}(t,\xi) &= O(1)e^{-ct} + O(|\xi|^{-1})e^{-ct}, \\
\hat{G}_{12}(t,\xi) &= O(1)e^{-ct}, \\
\hat{G}_{13}(t,\xi) &= O(\xi^{-1})te^{-B\xi^2 t\pm\beta t} + O(1)e^{-B\xi^2 t\pm\beta t},
\end{aligned}
$$

$$
\begin{aligned}
\hat{G}_{14}(t,\xi) &= O(\xi^{-1})te^{-B\xi^2t\pm\beta t} + O(1)e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{21}(t,\xi) &= O(\xi^{-1})e^{-ct} + O(1)e^{-ct},\\
\hat{G}_{22}(t,\xi) &= O(|\xi|^{-1})te^{-ct} + O(1)e^{-ct},\\
\hat{G}_{23}(t,\xi) &= O(\xi^{-2})te^{-ct} + O(|\xi|^{-1})e^{-ct} + O(\xi^{-1})e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{24}(t,\xi) &= O(\xi^{-2})te^{-ct} + O(|\xi|^{-1})e^{-ct} + O(\xi^{-1})e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{31}(t,\xi) &= O(\xi^{-3})te^{-ct} + O(\xi^{-2})e^{-ct} + O(\xi^{-2})e^{-B\xi^2t\pm\beta t}\\
&\quad + O(\xi^{-1})e^{-ct},\\
\hat{G}_{32}(t,\xi) &= O(\xi^{-2})te^{-ct} + O(|\xi|^{-1})e^{-ct} + O(\xi^{-1})e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{33}(t,\xi) &= O(\xi^{-2})te^{-ct} + O(|\xi^{-1})e^{-ct} + O(\xi^{-1})e^{-B\xi^2t\pm\beta t}\\
&\quad + O(1)e^{-B\xi^2t\pm\beta t} + O(|\xi|^{-1})e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{34}(t,\xi) &= O(\xi^{-2})te^{-ct} + O(\xi^{-1})e^{-ct} + O(\xi^{-1})e^{-B\xi^2t\pm\beta t}\\
&\quad + O(1)e^{-B\xi^2t\pm\beta t} + O(|\xi|^{-1})e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{41}(t,\xi) &= O(\xi^{-3})te^{-ct} + O(\xi^{-2})e^{-ct} + O(\xi^{-2})e^{-B\xi^2t\pm\beta t}\\
&\quad + O(\xi^{-1})e^{-B\xi^2t\pm\beta t} + O(\xi^{-2})e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{42}(t,\xi) &= O(\xi^{-2})te^{-ct} + O(|\xi|^{-1})e^{-ct} + O(\xi^{-1})e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{43}(t,\xi) &= O(\xi^{-2})te^{-ct} + O(|\xi|^{-1})e^{-ct} + O(\xi^{-1})e^{-B\xi^2t\pm\beta t}\\
&\quad + O(1)e^{-B\xi^2t\pm\beta t} + O(|\xi|^{-1})e^{-B\xi^2t\pm\beta t},\\
\hat{G}_{44}(t,\xi) &= O(\xi^{-2})te^{-ct} + O(|\xi|^{-1})e^{-ct} + O(\xi^{-1})e^{-B\xi^2t\pm\beta t}\\
&\quad + O(1)e^{-B\xi^2t\pm\beta t} + O(|\xi|^{-1})e^{-B\xi^2t\pm\beta t}.
\end{aligned}
$$

Summarizing the above considerations we have:

COROLLARY 4.1. *There exist positive constants* r_1 *and* r_2 *such that:*

1. *If* $|\xi| < r_1$, *then*

$$\hat{G}_{ij}(t,\xi) = O(\xi)e^{-c\xi^2t} + O(1)e^{-c\xi^2t}, \quad i,j = 1,2,3,4, \tag{4.7}$$

2. *If* $r_1 < |\xi| < r_2$, *then*

$$\exists\, C > 0 \quad |\hat{G}_{ij}(t,\xi)| < C, \quad \forall t > 0,\ i,j = 1,2,3,4. \tag{4.8}$$

3. *If* $|\xi| > r_2$, *then*

$$\hat{G}_{ij}(t,\xi) = O(\xi)e^{-ct} + O(1)e^{-B\xi^2t\pm\beta t}, \quad i,j = 1,2,3,4. \tag{4.9}$$

Proof. Properties 1 and 3 are simple consequences of the above considerations. We prove the second property. Let ξ_0 be the value of ξ for which the characteristic equation (3.1) has multiple roots. We can show that the limit of $\hat{G}_{ij}(t,\xi)$ is bounded at the point ξ_0. Let $\lambda_i(\xi_0) = \lambda_j(\xi_0)$

for some $i, j = 1, 2, 3, 4$ (cf. Remark 3.2). In particular, if we assume that $\lambda_1(\xi_0) = \lambda_2(\xi_0)$, then there exists $C > 0$ for which $|\hat{G}_{ij}(t,\xi)| < C$ for $i, j = 1, 2, 3, 4$ and $t > 0$, $|\xi| > 0$. This follows by tedious calculations. For example, consider $\hat{G}_{11}(t,\xi)$. We write it (cf. (4.6)) in the following form:

$$\begin{aligned}\hat{G}_{11}(t,\xi) &= a_{11}e^{-\lambda_1 t} + b_{11}e^{-\lambda_2 t} + c_{11}e^{-\lambda_3 t} + d_{11}e^{-\lambda_4 t}\\ &= -\frac{\lambda_2\lambda_3\lambda_4(\lambda_1 - B_1)(\lambda_1 - B_2)}{B_1B_2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\, e^{-\lambda_1 t}\\ &\quad - \frac{\lambda_1\lambda_3\lambda_4(\lambda_2 - B_1)(\lambda_2 - B_2)}{B_1B_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\, e^{-\lambda_2 t}\\ &\quad - \frac{\lambda_1\lambda_2\lambda_4(\lambda_3 - B_1)(\lambda_3 - B_2)}{B_1B_2(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}\, e^{-\lambda_3 t}\\ &\quad - \frac{\lambda_1\lambda_2\lambda_3(\lambda_4 - B_1)(\lambda_4 - B_2)}{B_1B_2(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}\, e^{-\lambda_4 t}.\end{aligned}$$

We write the first two terms in the form

$$\begin{aligned}&a_{11}e^{-\lambda_1 t} + b_{11}e^{-\lambda_2 t}\\ &\quad= \frac{\lambda_2\lambda_3\lambda_4(\lambda_1 - B_1)(\lambda_1 - B_2)}{B_1B_2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\, e^{-\lambda_1 t}(1 - e^{(\lambda_1-\lambda_2)t})\\ &\quad\quad - \frac{\lambda_3\lambda_4[\lambda_2(\lambda_1 - B_1)(\lambda_1 - B_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)]}{B_1B_2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\, e^{-\lambda_2 t}.\\ &\quad\quad + \frac{\lambda_3\lambda_4[\lambda_1(\lambda_2 - B_1)(\lambda_2 - B_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)]}{B_1B_2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\, e^{-\lambda_2 t}.\end{aligned}$$

After some calculations we have

$$\begin{aligned}&a_{11}e^{-\lambda_1 t} + b_{11}e^{-\lambda_2 t}\\ &\quad= \frac{\lambda_2\lambda_3\lambda_4(\lambda_1 - B_1)(\lambda_1 - B_2)}{B_1B_2(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\, e^{-\lambda_1 t}\sum_{n=1}^{\infty}\frac{t^n}{n!}\,(\lambda_1-\lambda_2)^{n-1}\\ &\quad\quad - \frac{\lambda_3\lambda_4[(\lambda_3\lambda_4 - \lambda_1\lambda_2)(\lambda_1\lambda_2 - B_1B_2)]}{B_1B_2(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\, e^{-\lambda_2 t}\\ &\quad\quad - \frac{\lambda_3\lambda_4[(\lambda_1+\lambda_2-\lambda_3-\lambda_4)(B_1+B_2-B_1B_2(\lambda_1+\lambda_2))]}{B_1B_2(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\, e^{-\lambda_2 t}.\end{aligned}$$

Then

$$\begin{aligned}\lim_{\xi\to\xi_0}\hat{G}_{11}(t,\xi) &= \frac{\lambda_1\lambda_3\lambda_4(\lambda_1 - B_1)(\lambda_1 - B_2)}{B_1B_2(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\, te^{-\lambda_1 t}\\ &\quad - \frac{\lambda_3\lambda_4[(\lambda_3\lambda_4 - \lambda_1^2)(\lambda_1^2 - B_1B_2)]}{B_1B_2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}\, e^{-\lambda_1 t}\end{aligned}$$

$$- \frac{\lambda_3\lambda_4[(2\lambda_1-\lambda_3-\lambda_4)(B_1+B_2-2B_1B_2\lambda_1)]}{B_1B_2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}\,e^{-\lambda_1 t}$$
$$- \frac{\lambda_1^2\lambda_4(\lambda_3-B_1)(\lambda_3-B_2)}{B_1B_2(\lambda_3-\lambda_1)^2(\lambda_3-\lambda_4)}\,e^{-\lambda_3 t} - \frac{\lambda_1^2\lambda_3(\lambda_4-B_1)(\lambda_4-B_2)}{B_1B_2(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)}\,e^{-\lambda_4 t}.$$

The above limit is bounded, because $\lambda_1(\xi_0) \neq \lambda_3(\xi_0) \neq \lambda_4(\xi_0)$ (cf. Remark 3.2). If the characteristic equation (3.1) has two double roots, i.e. $\lambda_1(\xi_0) = \lambda_2(\xi_0)$ and $\lambda_3(\xi_0) = \lambda_4(\xi_0)$, then we proceed as above, but we analyse two pairs of terms of $\hat{G}_{11}(t,\xi)$. ∎

5. L^∞-L^1 and L^2-L^2 time decay estimates for the initial-value problem (1.1)–(1.2)**.** Basing on the considerations of the previous section we prove the L^∞-L^1 and L^2-L^2 time decay estimates. Writing

$$U_i(t,x) = \sum_{j=1}^{4} G_{ij}(t,x) *_x U_{0j}(x), \qquad i = 1,\ldots,4, \tag{5.1}$$

we have

$$\|U(t,\cdot)\|_{L^\infty} \le \sum_{i=1}^{4}\sum_{j=1}^{4} \|G_{ij}(t,\cdot) *_{\cdot} U_{0j}(\cdot)\|_{L^\infty}.$$

By the properties of the Fourier transformation (cf. [19], [22]), we obtain

$$\begin{aligned}\|G_{ij}(t,\cdot) *_{\cdot} U_{0j}(\cdot)\|_{L^\infty} &= \|F^{-1}[\hat{G}_{ij}(t,\xi)\hat{U}_{0j}(\xi)](\cdot)\|_{L^\infty} \\ &\le C\|\hat{G}_{ij}(t,\cdot)\hat{U}_{0j}(\cdot)\|_{L^1} \le C\|\hat{G}_{ij}(t,\cdot)\|_{L^1}\|U_{0j}\|_{L^1}.\end{aligned}$$

Now, we prove the following lemma.

LEMMA 5.1. *Let*

$$f(t,\xi,p,q) = \begin{cases} t^p\xi^q e^{-c\xi^2 t} & \text{for } |\xi| < r_1, \\ 0 & \text{for } |\xi| > r_1, \end{cases}$$

for $t \in \mathbb{R}_+$, $\xi \in \mathbb{R}$, and $p,q \ge 0$, $c, r_1 > 0$. Then

$$\|f(t,\cdot,p,q)\|_{L^1} \le C(1+t)^{p-(q+1)/2}, \tag{5.2}$$

$$\|f(t,\cdot,p,q)\|_{L^\infty} \le \begin{cases} C(1+t)^{p-q/2}, & q \ge 1, \\ t^p, & q = 0. \end{cases} \tag{5.3}$$

Sketch of proof. For $t > 1$ we have

$$\|f(t,\cdot,p,q)\|_{L^1} = \int\limits_{|\xi|<r_1} |t^p\xi^q e^{-c\xi^2 t}|\,d\xi$$

$$\overset{[\zeta=\xi\sqrt{t}]}{=} t^{p-1/2}\int\limits_{\sqrt{t}r_1}^{\sqrt{t}r_1} |t^{-q/2}\zeta^q e^{-c\zeta^2}|\,d\zeta \le t^{p-(q+1)/2}\int\limits_{\mathbb{R}} |\zeta^q e^{-c\zeta^2}|\,d\zeta \le Ct^{p-(q+1)/2}.$$

For $t \le 1$ we have

$$\|f(t,\cdot,p,q)\|_{L^1} = \int\limits_{|\xi|<r_1} |t^p \xi^q e^{-c\xi^2 t}|\, d\xi \le C \int\limits_{|\xi|<r_1} |\xi^q|\, d\xi \le C$$

where C is a universal positive constant. From the above we get (5.2).

The proof of (5.3) is a consequence of the analysis of the extreme points of f. ∎

LEMMA 5.2. *Let $f(\xi)$ be a bounded function which vanishes inside the ring $r_1 < |\xi| < r_2$ and let*

$$\operatorname{Re}\lambda_i(\xi) > C_3 > 0 \qquad \forall \xi \ne 0,\ i = 1,2,3,4.$$

Then

$$\|f(\xi)e^{-\lambda_i(\xi)t}\|_{L^1} \le Ce^{-ct}, \qquad \forall t > 0,\ i = 1,2,3,4,$$
$$\|f(\xi)e^{-\lambda_i(\xi)t}\|_{L^\infty} \le Ce^{-ct}, \qquad \forall t > 0,$$

where the norms are taken with respect to ξ.

The proof is simple and we omit it.

LEMMA 5.3. *Let*

$$f(t,\xi,p,q) = \begin{cases} t^p\xi^q, & |\xi| > r_2, \\ 0, & |\xi| < r_2, \end{cases}$$

for $t \in \mathbb{R}_+,\ \xi \in \mathbb{R},\ p,q \in \mathbb{Z},\ q < -1$. Then for $c, \bar{B}, \beta > 0$ we have

$$\|f(t,\xi,p,q)e^{-ct}\|_{L^1}, \|f(t,\xi,p,q)e^{-\bar{B}\xi^2 t \pm \beta t}\|_{L^1} \le Ce^{-ct} \tag{5.4}$$

and

$$\|f(t,\xi,p,q)e^{-ct}\|_{L^\infty}, \|f(t,\xi,p,q)e^{-\bar{B}\xi^2 t \pm \beta t}\|_{L^\infty} \le Ce^{-ct}. \tag{5.5}$$

Proof. We have

$$\|f(t,\xi,p,q)e^{-ct}\|_{L^1} = \int\limits_{|\xi|>r_2} |t^p\xi^q e^{-ct}|\, d\xi = t^p e^{-ct} \int\limits_{|\xi|>r_2} |\xi^q|\, d\xi \le Ce^{-ct}$$

$$\begin{aligned}\|f(t,\xi,p,q)&e^{-\bar{B}\xi^2 t \pm \beta t}\|_{L^1} \\ &= \int\limits_{|\xi|>r_2} |t^p\xi^q e^{-\bar{B}\xi^2 t \pm \beta t}|\, d\xi = t^p \int\limits_{|\xi|>r_2} |\xi^q e^{-\bar{B}\xi^2 t \pm \beta t}|\, d\xi \le Ce^{-ct}.\end{aligned}$$

We skip the proof of (5.5). ∎

We consider the term $G_{ij}(t,x) *_x U_{0j}(x)$ in three intervals:

$$\begin{aligned}\|\hat{G}_{ij}(t,\cdot)\hat{U}_{0j}(\cdot)\|_{L^1} &= \Big(\int\limits_{|\xi|<r_1} + \int\limits_{r_1<|\xi|<r_2} + \int\limits_{|\xi|>r_2}\Big)|\hat{G}_{ij}(t,\xi)\hat{U}_{0j}(\xi)|\, d\xi \\ &= I_{ij}^1 + I_{ij}^2 + I_{ij}^3.\end{aligned}$$

We estimate the above integrals using (3.19) and Theorem 3.1. First,

$$I_{ij}^1 \leq C\|U_0\|_{L^1} \int\limits_{|\xi|<r_1} |e^{-c\xi^2 t}|\, d\xi \leq C(1+t)^{-1/2}\|U_0\|_{L^1}.$$

We estimate the second integral using (4.8) and Lemma 3.1:

$$I_{ij}^2 \leq Ce^{-ct}\|U_0\|_{L^1}.$$

The last integral satisfies (cf. (4.9) and Lemma 5.1)

$$\begin{aligned} I_{ij}^3 &= \int\limits_{|\xi|>r_2} |\xi e^{-ct}\hat{U}_{0j}(\xi)|\, d\xi + \int\limits_{|\xi|>r_2} |e^{-\bar{B}\xi^2 t \pm \beta t}\hat{U}_{0j}(\xi)|\, d\xi \\ &= e^{-ct} \int\limits_{|\xi|>r_2} |\xi^{-s+1}|\,|\xi^s \hat{U}_{0j}|\, d\xi + \int\limits_{|\xi|>r_2} |\xi^{-s}|\,|\xi^s \hat{U}_{0j} e^{-\bar{B}\xi^2 t \pm \beta t}|\, d\xi \\ &\leq Ce^{-ct}\|U_0\|_{L^{s,1}}, \quad s>2. \end{aligned}$$

We summarize the above considerations in the following formula:

$$\sum_{i,j=1}^{4} \|G_{ij}(t,\cdot) *_x U_{0j}(\cdot)\|_{L^\infty} \leq C(1+t)^{-/2}\|U_0\|_{L^{s,1}} \quad \forall t>0,\ s>2.$$

In view of (5.1) we have

$$\|U(t,\cdot)\|_{L^2} \leq C\sum_{i=1}^{4}\sum_{j=1}^{4} \|G_{ij}(t,\cdot) *_x U_{0j}(\cdot)\|_{L^2}.$$

Then using the properties of the Fourier transformation (cf. [1], [20]), we obtain

$$\begin{aligned} \|G_{ij}(t,\cdot) *. U_{0j}(\cdot)\|_{L^2} &= \|F^{-1}[\hat{G}_{ij}(t,\xi)\hat{U}_{0j}(\xi)](\cdot)\|_{L^2} \\ &= \|\hat{G}_{ij}(t,\cdot)\hat{U}_{0j}(\cdot)\|_{L^2} \leq \|\hat{G}_{ij}(t,\cdot)\|_{L^\infty}\|U_{0j}\|_{L^2}. \end{aligned} \tag{5.6}$$

Now, we consider $G_{ij}(t,x) *_x U_{0j}(x)$, using the second row of (5.16). We have

$$\begin{aligned} \|\hat{G}_{ij}(t,\cdot)\hat{U}_{0j}(\cdot)\|_{L^2}^2 &= \Big(\int\limits_{|\xi|<r_1} + \int\limits_{r_1<|\xi|<r_2} + \int\limits_{|\xi|>r_2}\Big)|\hat{G}_{ij}(t,\xi)\hat{U}_{0j}|^2\, d\xi \\ &= I_{ij}^1 + I_{ij}^2 + I_{ij}^3. \end{aligned}$$

Next we estimate the above integrals (cf. Theorem 3.1 and Lemmas 3.1, 5.1):

$$I_{ij}^1 \leq C\|U_0\|_{L^2}^2 \operatorname*{ess\,sup}_{|\xi|<r_1} |e^{-c\xi^2 t}|^2 \leq C\|U_0\|_{L^2}^2, \tag{5.7}$$

$$I_{ij}^2 \leq Ce^{-ct}\|U_0\|_{L^2}^2, \tag{5.8}$$

$$\begin{aligned} I_{ij}^3 &= \int\limits_{|\xi|>r_2} |\xi e^{-ct}\hat{U}_{0j}(\xi)|^2\, d\xi + \int\limits_{|\xi|>r_2} |e^{-\bar{B}\xi^2 t \pm \beta t}\hat{U}_{0j}(\xi)|^2\, d\xi \\ &\leq Ce^{-ct}\|U_0\|_{L^{1,2}}^2. \end{aligned} \tag{5.9}$$

From (5.7)–(5.9) we have

$$\sum_{i,j=1}^{4} \|G_{ij}(t,\cdot) *_x U_{0j}(\cdot)\|_{L^2} \le C\|U_0\|_{L^{1,2}}, \qquad \forall t > 0.$$

6. L^p-L^q time decay estimates for the Cauchy problem (1.1)–(1.2) **and the proof of the main theorem.** We summarize the considerations of Section 5 as follows:

$$\|(\bar U,\bar\theta)(t,\cdot)\|_{L^\infty} \le C(1+t)^{-1/2}\|(\bar U_0,\bar\theta_0)\|_{L^{s,1}}, \qquad \forall t>0,\ s>2 \tag{6.1}$$

and

$$\|(\bar U,\bar\theta)(t,\cdot)\|_{L^2} \le C\|(\bar U_0,\bar\theta_0)\|_{L^{1,2}}, \quad \forall t>0. \tag{6.2}$$

The proof of the main theorem will be preceded by the following lemma.

LEMMA 6.1. *Let T be a linear operator satisfying the following conditions:*

$$T : W^{m,1} \to L^\infty \ \textit{is bounded with norm } M_0, \tag{6.3}$$

$$T : L^{k,2} \to L^2 \quad \textit{is bounded with norm } M_1, \tag{6.4}$$

where $k,m \in \mathbb{N}$. Let $1<p<2<q<\infty$, $1/p+1/q=1$, $\Theta = 2/q$, $N\in\mathbb{N}$ and $N > (1-\Theta)m + \Theta k \ge 0$. Then

$$T : W^{N,p} \to L^q \ \textit{is bounded with norm } M$$

where

$$M \le c(k,m,p) M_0^{1-\Theta} M_1^{\Theta}.$$

Sketch of proof. Lemma 6.1 is a consequence of the imbedding theorems for Besov spaces B^s_{pq} and Bessel spaces $L^{s,p}$ (cf. [1], [22], [24]). We have

$$B^m_{11} \hookrightarrow W^{m,1},$$
$$B^k_{22} \hookrightarrow W^{k,2}.$$

Therefore, by (6.1)–(6.2)

$$T : B^m_{11} \to L^\infty \text{ is bounded with norm } M_0,$$
$$T : B^k_{22} \to L^2 \ \text{ is bounded with norm } M_1.$$

Then by the interpolation theorem (cf. [2]) we have:

$$[L^\infty, L^2]_\Theta = L^q, \qquad q = 2/\Theta,\ 0<\Theta<1, \tag{6.5}$$

$$[B^m_{11}, B^k_{22}]_\Theta = B^{(1-\Theta)m+\Theta k}_{pp}, \tag{6.6}$$

therefore

$$W^{N,p} \equiv L^{N,p} \equiv L^{(1-\Theta)m+\Theta k+\varepsilon,p} \hookrightarrow B^{(1-\Theta)m+\Theta k}_{pp}.$$

On the other hand, by the Riesz–Thorin theorem (cf. Theorem 2.4) the operator

$$T : [B^m_{11}; B^k_{22}]_\Theta \to [L^\infty, L^2]_\Theta$$

is bounded by M, and $M \le M_0^{1-\Theta} M_1^\Theta$. Now basing on (6.5) and (6.6) we conclude the proof. ∎

Proof of the Main Theorem 1.1. Let $T^\star$ be the linear operator defined by

$$T^\star(t,\cdot)f = G(t,\cdot) \star f \tag{6.7}$$

where $G(t,\cdot) = \mathcal{F}^{-1}\hat{G}(t,\cdot)$ is given by formula (4.5), and $f \in L^{s,1}(\mathbb{R}) \cap L^{1,2}(\mathbb{R})$, $s > 2$, are some functions corresponding to the initial value U_0. By (6.1), (6.2) and (6.7) the operator $T^\star$ is linear and bounded and has the following properties:

$$\begin{aligned} &T^\star : L^{s,1}(\mathbb{R}) \to L^\infty, \quad &&\|T^\star\| \le C(1+t)^{-1/2},\ s > 2,\\ &T^\star : L^{1,2}(\mathbb{R}) \to L^2, \quad &&\|T^\star\| \le C. \end{aligned}$$

In view of Lemma 6.1 the linear operator $T^\star$ maps $W^{N,p}(\mathbb{R})$ to the space $L^q(\mathbb{R})$, i.e.

$$T^\star : W^{N,p}(\mathbb{R}) \to L^q(\mathbb{R})$$

and is bounded by M, where $1 < p < q < \infty$, $1/p + 1/q = 1$, $\Theta = 2/q$, $N \in \mathbb{N}$ and the condition $N > (1-\Theta)s + \Theta \ge 0$ in this case has the form

$$N > (2/p - 1)(s-1) + 1 \ge 0.$$

The constant M has the form

$$M = C(s,p)(1+t)^{1/2-1/p}.$$

This ends the proof of the Main Theorem 1.1. ∎

7. Summary. In this paper we proved L^p-L^q time decay estimates for the solution of three coupled partial differential equations of the second order describing the process of thermodiffusion in a solid body (in one-dimensional space). The same approach can be applied to other hyperbolic-parabolic or hyperbolic systems of equations used in continuum mechanics. In the subsequent paper (cf. [25]) we shall apply Theorem 1.1 in the proof of global (in time) existence of solution of the Cauchy problem for the nonlinear hyperbolic-parabolic system of partial differential equations describing the thermodiffusion in a solid body associated with the linear system (1.1)–(1.2).

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Arkadiusz Szymaniec
Institute of Mathematics and Cryptology
Faculty of Cybernetics
Military University of Technology
S. Kaliskiego 2
00-908 Warszawa, Poland
E-mail: aszymaniec@wat.edu.pl

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