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SOME PROPERTIES OF THE PROPORTIONAL ODDS MODEL

Abstract. Marshall and Olkin (1997) introduced a new family of distributions by adding a tilt parameter. The same family was obtained by Kirmani and Gupta (2001) as the proportional odds model, which had been proposed by Clayton (1974). In this paper, stochastic ordering of distributions from this class and preservation of classes of life distributions by adding a parameter are obtained. The proportional odds family is also considered as a family of weighted distributions.

1. Introduction and summary. Marshall and Olkin (1997) discussed a method of introducing a parameter into a family of distributions to enhance flexibility. Let $\overline{F}$ be a survival function. They defined the family of survival functions

$$(1)\qquad \mathcal{G} = \left\{ \overline{G}_\alpha \colon \overline{G}_\alpha(t) = \frac{\alpha \overline{F}(t)}{1-\overline{\alpha}\overline{F}(t)} = \frac{\alpha\overline{F}(t)}{F(t)+\alpha\overline{F}(t)};\ t\in\mathbb{R},\ \alpha>0,\ \overline{\alpha}=1-\alpha \right\}.$$

We will call (1) the *Marshall–Olkin family*. When the tilt parameter α is 1, we have $\overline{G}_\alpha = \overline{F}$. If F has a density f and a hazard rate r_F, then

$$g_\alpha(t) = \frac{\alpha f(t)}{[1-\overline{\alpha}\overline{F}(t)]^2}$$

is the density function of the distribution G_α and

$$r_\alpha(t) = \frac{1}{1-\overline{\alpha}\overline{F}(t)}\, r_F(t), \qquad -\infty < t < \infty,$$

is the hazard rate of G_α.

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Two particular cases were discussed by Marshall and Olkin (1997). The case that F is an exponential distribution with parameter λ yields a new two-parameter family of distributions which can sometimes be a competitor to the Weibull and gamma families. In particular, they showed that the failure rate function $r_\alpha(t) = \lambda/(1-\overline{\alpha}\exp\{-\lambda t\})$ is increasing for $1 < \alpha < \infty$ and decreasing for $0 < \alpha < 1$. They also calculated the expectation, modal value and Laplace transform. When F is the Weibull distribution, then G_α is a new three-parameter distribution function and may be considered as a competitor to the three-parameter Weibull distribution. Marshall and Olkin (1997) also discussed a stability property of G_α.

It turns out that the family of distributions defined by Marshall and Olkin (1997) is the same as in the *proportional odds (PO) model*, introduced by Clayton (1974) and considered by Bennett (1983) and Kirmani and Gupta (2001). The odds on surviving beyond time t are given by the *odds function*:

$$\theta_F(t) = \frac{\overline{F}(t)}{F(t)}.$$

To introduce the proportional odds model we take two distribution functions F and G. We will say that the survival time random variables which have distributions F and G respectively satisfy the PO model with proportionality constant α if

$$\theta_G(t) = \alpha\theta_F(t).$$

This is equivalent to (1).

In this paper the results of Marshall and Olkin (1997) and Kirmani and Gupta (2001) are generalized and extended. In Section 2 we set up notations and terminology, and review Parzen's (1979) approach to classification of probability distributions. In Section 3 we obtain stochastic ordering of distributions from the family $\mathcal{G}$. Proofs for the dispersive and convex ordering are based on the concept of density-quantile function. That section contains a discussion of preservation of classes of life distributions. In order to get our results, we will also use properties of weighted distributions.

2. Preliminaries. Let X and Y be random variables with distribution functions F and G respectively and let f, g be their respective density functions, if they exist. Denote by $\overline{F} = 1-F$ the tail (survival function) of F, by $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in (0,1)$, the quantile function and by $F^{-1}(0)$ and $F^{-1}(1)$ the lower and upper bounds of the support S_F of F, which is an interval, and analogously for G. The function $r_F(x) = f(x)/\overline{F}(x)$ is called the *hazard rate function* of F and $\breve{r}_F(x) = f(x)/F(x)$ is called the *reversed hazard rate function* of F if f exists.

We denote by $G^{-1}F$ the composition of G^{-1} and F (similarly for other functions). *Increasing* is used in place of *nondecreasing*, and *decreasing* in place of *nonincreasing.*

The distribution $F_w(x) = \int_{-\infty}^{x} w(u)\,dF(u)/E[w(X)]$ is called the *weighted distribution* associated with F, where $w\colon \mathbb{R} \to \mathbb{R}^+$ is a *weight function* for which $0 < E[w(X)] < \infty$.

2.1. Density-quantile function. The function $\frac{d}{du}F^{-1}(u)=1/fF^{-1}(u)$ is called the *quantile-density* of the distribution F, and $fF^{-1}(u)$ is the *density-quantile* function. The distribution F is uniquely determined by the function $fF^{-1}(u)$ up to a location parameter. Parzen (1979) and Schuster (1984) used a density-quantile approach to the tail classification of probability laws into one of three types of tail behavior: short, medium, or long. Parzen classifies the right-tail behavior of a probability function with density f according to the value of the right-tail exponent p, defined by $fF^{-1}(u) \sim (1-u)^p$, i.e. $fF^{-1}(u)/(1-u)^p$ tends to a finite constant as u tends to 1. If f is differentiable, then $p = \lim_{u\to 1^-}(1-u)(-f'F^{-1}(u)/[fF^{-1}(u)]^2)$. This implies that

$$fF^{-1}(u) = L(u)(1-u)^p, \tag{2}$$

where $L(u)$ is a *slowly varying function* (svf) from the left at $u = 1$.

The parameter ranges $p < 1$, $p = 1$, and $p > 1$ correspond to short tails (or limited type), medium tails (or exponential type), and long tails (or Cauchy type) respectively.

2.2. Stochastic orders. We say that F *is smaller than* G *in the likelihood ratio order* ($F \le_{\mathrm{lr}} G$) if $g(x)/f(x)$ is increasing. We say that F *is smaller than* G *in the hazard rate order* ($F \le_{\mathrm{hr}} G$) if $\overline{G}(x)/\overline{F}(x)$ is increasing, or equivalently $r_F(x) \ge r_G(x)$ for every x whenever F and G are absolutely continuous. We say that F *is smaller than* G *in the reversed hazard rate order* ($F \le_{\mathrm{rh}} G$) if $G(x)/F(x)$ is increasing, or equivalently $\check{r}_F(x) \le \check{r}_G(x)$ for every x whenever F and G are absolutely continuous. We say that F is *stochastically smaller than* G ($F \le_{\mathrm{st}} G$) if $F(x) \ge G(x)$ for every x, or equivalently $\overline{F}(x) \le \overline{G}(x)$ for every x. We say that F *is smaller than* G *in the dispersive order* ($F \le_{\mathrm{disp}} G$) if $G^{-1}F(x) - x$ is increasing.

Let now X and Y be positive random variables. We say that F *is smaller than* G *in the convex order* ($F \le_{\mathrm{c}} G$) if $G^{-1}F$ is convex on S_F. We say that F *is smaller than* G *in the star order* ($F \le_{*} G$) if $G^{-1}F$ is star-shaped on S_F, i.e. $G^{-1}F(x)/x$ is increasing for $x > 0$. We say that F *is smaller than* G *in the superadditive order* ($F \le_{\mathrm{su}} G$) if $G^{-1}F$ is superadditive.

For more details (equivalent definitions) and properties of stochastic orders see Shaked and Shanthikumar (2007) or Müller and Stoyan (2000).

It is well known (see Shaked and Shanthikumar, 2007) that

$$
\begin{array}{ccc}
F \leq_{\mathrm{lr}} G & \Longrightarrow & F \leq_{\mathrm{hr}} G \\
\Big\Downarrow & & \Big\Downarrow \\
F \leq_{\mathrm{rh}} G & \Longrightarrow & F \leq_{\mathrm{st}} G
\end{array} \tag{3}
$$

$$F \leq_{\mathrm{c}} G \Rightarrow F \leq_{*} G \Rightarrow F \leq_{\mathrm{su}} G.$$

The following lemma, which will be needed later, is a direct consequence of the above definitions.

Lemma 1. *Let F and G be absolutely continuous distributions with densities f and g respectively and $F(0) = G(0) = 0$. Then:*

(i) $F \leq_{\mathrm{c}} G \Leftrightarrow fF^{-1}(u)/gG^{-1}(u)$ *is increasing for* $u \in (0,1)$.
(ii) $F \leq_{\mathrm{disp}} G \Leftrightarrow fF^{-1}(u)/gG^{-1}(u) \geq 1$, $u \in (0,1)$.

2.3. Classes of life distributions. A distribution F is said to be IFR [DFR] if $\log \overline{F}$ is concave [convex] on S_F. A distribution F with $F(0) = 0$ and S_F is said to be IFRA [DFRA] if $-\log \overline{F}(x)/x$ is increasing [decreasing] on S_F, or equivalently, $\overline{F}^{\alpha}(x) \leq [\geq]\ \overline{F}(\alpha x)$ for every $\alpha \in (0,1)$ and $x \in S_F$. A distribution F with $S_F = [a,b]$, $-\infty \leq a < b < \infty$, is said to be IRFR [DRFR] if $\log F$ is convex [concave] on S_F. A distribution F with $S_F = [0,\infty)$ is said to be NBU [NWU] if $\overline{F}(x+y) \leq [\geq]\ \overline{F}(x)\overline{F}(y)$ for all $x, y, x+y \in S_F$. For more details and properties see Barlow and Proschan (1975) or Lai and Xie (2006).

3. Results

3.1. The density-quantile function of the Marshall–Olkin family. It is easy to compute the quantile function and density-quantile function of the distribution G_α:

$$G_\alpha^{-1}(t) = F^{-1}\left(\frac{t\alpha}{1-t+t\alpha}\right), \qquad t \in (0,1),$$

$$g_\alpha G_\alpha^{-1}(t) = \frac{1}{\alpha}\,(1-t+t\alpha)^2 fF^{-1}\left(\frac{t\alpha}{1-t+t\alpha}\right), \qquad t \in (0,1). \tag{4}$$

It is interesting to see a connection between the distributions F and G_α in Parzen's (1979) tail classification.

Lemma 2. *The distributions G_α, $\alpha > 0$, and F are of the same type in Parzen's classification.*

Proof. From (2) and (4) we get

$$g_\alpha G_\alpha^{-1}(t) = \frac{1}{\alpha}\,(1-t+t\alpha)^{2-p} L\left(\frac{t\alpha}{1-t+t\alpha}\right)(1-t)^p, \qquad t \in (0,1).$$

It is easy to check that the functions $(1-t+t\alpha)^{2-p}$ and $L(t\alpha/(1-t+t\alpha))$ are s.v.f. at $t=1$. Since the product of slowly varying functions is slowly varying (see Seneta (1976)), we get $g_\alpha G_\alpha^{-1}(t) = L_G(t)(1-t)^p$, where L_G is a s.v.f. The value of the right-tail exponent is the same for F and G_α, and the lemma follows. ∎

We will use quantile-density functions to study the dispersive and convex ordering in the Marshall–Olkin family of distributions.

Consider now the generalized Pareto distribution F with the density-quantile function of the form $fF^{-1}(t) = c(1-t)^p$. For example, when $p = 1 - 1/c$, then F is the beta distribution with the density function $f(x) = c(1-x)^{c-1}$; when $p=1$, then F is the exponential distribution with mean c; when $p = 1+1/c$, then F is the classical Pareto distribution with parameter $1/c$.

Theorem 1. *Let $fF^{-1}(t) = c(1-t)^p$.*

(i) *If $0<\alpha<\beta<\infty$ and $p>2$, or $0<\beta<\alpha<\infty$ and $p<2$, then $G_\alpha \le_{\mathrm{c}} G_\beta$.*
(ii) *If $0<\alpha<\beta<\infty$ and $p \ge 1$, then $G_\alpha \le_{\mathrm{disp}} G_\beta$.*

Proof. According to (4), compute

$$\varphi(t) = \frac{g_\alpha G_\alpha^{-1}(t)}{g_\beta G_\beta^{-1}(t)} = \frac{\beta[1+t(\beta-1)]^{p-2}}{\alpha[1+t(\alpha-1)]^{p-2}}, \qquad t \in (0,1). \tag{5}$$

Now, consider

$$\frac{d}{dt}\varphi(t) = \frac{\beta}{\alpha}\,\frac{(p-2)(\beta-\alpha)}{[1+(\alpha-1)t]^2}\left[\frac{1+(\beta-1)t}{1+(\alpha-1)t}\right]^{p-3}, \qquad t\in(0,1).$$

The sign of this expression depends only on the sign of $(p-2)(\beta-\alpha)$. We find that $\frac{d}{dt}\varphi(t)$ is positive if either $0<\alpha<\beta<\infty$ and $p>2$, or $0<\beta<\alpha<\infty$ and $p<2$, and then $\varphi(t)$ is increasing. Otherwise the function $\varphi(t)$ is decreasing. Combined with Lemma 1(i), this gives (i).

Next, it is easy to check that

$$\lim_{t\to 0}\varphi(t) = \frac{\beta}{\alpha} \quad \text{and} \quad \lim_{t\to 1}\varphi(t) = \left(\frac{\beta}{\alpha}\right)^{p-1}.$$

Thus, $\varphi(t) \ge 1$ if $0<\alpha<\beta<\infty$ and $p\ge 1$. Applying Lemma 1(ii) completes the proof of (ii). ∎

Remark 1. Let $0<\alpha<\beta<\infty$. The above theorem says that $G_\alpha \le_{\mathrm{disp}} G_\beta$ for medium and long tailed distributions in Parzen's classification. When F is a short tailed distribution, then G_α and G_β are not ordered with respect to the dispersive order, but $G_\beta \le_{\mathrm{c}} G_\alpha$.

REMARK 2. Putting $\beta = 1$ in Theorem 1, we obtain the dispersive and convex order relations between the distributions F and G_α.

For another type of density-quantile function of the form $fF^{-1}(t) = c\big[\ln\big(\frac{1}{1-t}\big)\big]^b(1-t)^p$ we have the following theorem:

THEOREM 2. *Let* $fF^{-1}(t) = c\big[\ln\big(\frac{1}{1-t}\big)\big]^b(1-t)^p$.

(i) *If* $0 < \alpha < \beta < \infty$, $p \geq 2$, *and* $b > 0$, *then* $G_\alpha \leq_{\mathrm{c}} G_\beta$.
(ii) *If* $0 < \alpha < \beta < \infty$, $p \geq 2$, *and* $b \in (0,1)$, *then* $G_\alpha \leq_{\mathrm{disp}} G_\beta$.

Proof. According to (4), compute

$$\varphi_1(t) = \frac{g_\alpha G_\alpha^{-1}(t)}{g_\beta G_\beta^{-1}(t)} = \varphi(t) \cdot \left[\frac{\ln\big(1+\alpha\frac{t}{1-t}\big)}{\ln\big(1+\beta\frac{t}{1-t}\big)}\right]^b, \qquad t \in (0,1),$$

where φ is of the form (5). Now it is sufficient to show that

$$\psi(x) = \frac{\ln(1+\alpha x)}{\ln(1+\beta x)}, \qquad x \in (0,\infty),$$

is increasing. The function $\frac{d}{dx}\psi(x)$ is positive when

$$(1+\beta x)^{(1+\beta x)/\beta} \geq (1+\alpha x)^{(1+\alpha x)/\alpha}. \tag{6}$$

To prove (6), it suffices to show that

$$h(y) = (1+y)^{(1+y)/y}, \qquad y \in (0,\infty),$$

is increasing. But $\frac{d}{dy}h(y)$ is positive when $\ln(1+y) \leq y$, which completes the proof of (i).

The proof of Theorem 2(ii) is similar to that of Theorem 1(ii). ■

The following example shows that for $p < 2$ and $b > 0$ we do not get a convex ordering of the Marshall–Olkin family.

EXAMPLE 1. Consider the Raileigh distribution with density function $f(t) = 2t\exp\{-t^2\}$. Then $fF^{-1}(t) = 2[\ln\big(\frac{1}{1-t}\big)]^{1/2}(1-t)$ and hence

$$g_\alpha G_\alpha^{-1}(t) = \frac{2(1-t+\alpha t)}{\alpha}\left[\ln\left(\frac{1-t+\alpha t}{1-t}\right)\right]^{1/2}(1-t).$$

Let now $\alpha = 2$ and $\beta = 3$. Then

$$\varphi(t) = \frac{g_2 G_2^{-1}(t)}{g_3 G_3^{-1}(t)} = \frac{3(1+t)}{2(1+2t)}\left[\ln\left(\frac{1+t}{1-t}\right)\ln\left(\frac{1-t}{1+2t}\right)\right]^{1/2}.$$

We can compute that $\varphi(0.1) = 1.148$, $\varphi(0.9) = 0.956$, and $\varphi(0.98) = 0.962$, so $\varphi(t)$ is neither increasing nor decreasing.

REMARK 3. Theorems 1 and 2 remain true if we replace the convex order by the star or superadditive order.

Consider now two distribution functions F_1 and F_2. Let

$$\overline{G}_{i,\alpha}(t) = \alpha\overline{F}_i(t)/[1-\alpha\overline{F_i}(t)], \quad i=1,2.$$

Kirmani and Gupta (2001) showed that $G_{1,\alpha}^{-1}G_{2,\alpha}(t) = F_1^{-1}F_2(t)$, so it does not depend on the parameter α. Therefore the dispersive, convex, star, superadditive, and stochastic orders, which are defined by the function $F_1^{-1}F_2(t)$, are not affected by adding the parameter. This can also be proved using the quantile-density function. To do this, it is sufficient to compute

$$\frac{g_{1,\alpha}G_{1,\alpha}^{-1}(t)}{g_{2,\alpha}G_{2,\alpha}^{-1}(t)} = \frac{f_1F_1^{-1}\left(\frac{t\alpha}{1-t+t\alpha}\right)}{f_2F_2^{-1}\left(\frac{t\alpha}{1-t+t\alpha}\right)}$$

and next use Lemma 1.

3.2. Other stochastic orderings. The following theorem concerns the likelihood ratio order in the Marshall–Olkin family.

THEOREM 3. *If F is an absolutely continuous distribution and $0<\alpha<\beta<\infty$, then $G_\alpha \le_{\mathrm{lr}} G_\beta$.*

Proof. It is easy to compute

$$\frac{g_\beta(t)}{g_\alpha(t)} = \frac{\beta}{\alpha}\,\frac{[1-\overline{\alpha}\overline{F}(t)]^2}{[1-\overline{\beta}\overline{F}(t)]^2}.$$

Then

$$\frac{d}{dt}\left[\frac{g_\beta(t)}{g_\alpha(t)}\right] = \frac{2\beta(\beta-\alpha)f(t)[1-\overline{\alpha}\overline{F}(t)]}{\alpha[1-\overline{\beta}\overline{F}(t)]^3}.$$

The sign of this derivative only depends on the sign of $\beta-\alpha$. Thus, since $0<\alpha<\beta<\infty$, the function g_β/g_α is increasing, which completes the proof by the definition of the likelihood ratio order. ■

It is obvious that $G_1 = F$. The following results, which were partially proved by Kirmani and Gupta (2001), are immediate consequences of (3) and Theorem 3.

COROLLARY 1.

(i) *If $0<\alpha<1$, then $G_\alpha \le_{\mathrm{lr}} F$.*
(ii) *If $\alpha \ge 1$, then $F \le_{\mathrm{lr}} G_\alpha$.*

COROLLARY 2.

(i) *If $0<\alpha<\beta<\infty$, then $G_\alpha \le_{\mathrm{hr}} G_\beta$ and $G_\alpha \le_{\mathrm{rh}} G_\beta$.*
(ii) *If $0<\alpha<1$, then $G_\alpha \le_{\mathrm{hr}} F$ and $G_\alpha \le_{\mathrm{rh}} F$.*
(iii) *If $\alpha \ge 1$, then $F \le_{\mathrm{hr}} G_\alpha$ and $F \le_{\mathrm{rh}} G_\alpha$.*

The distribution G_α may be considered as a weighted distribution with weight function $w(u) = 1/[1-\overline{\alpha}\overline{F}(u)]^2$, which is monotone. Many authors, e.g. Rao (1985), Patil and Rao (1985), Jain and Nanda (1999), Bartoszewicz and Skolimowska (2006), studied weighted distributions. Thus all theorems which were formulated for weighted distributions can be expressed in terms of distributions of type G_α. For example, we have

LEMMA 3 (Bartoszewicz and Skolimowska (2006)). $F \le_{\mathrm{lr}} G \Leftrightarrow F_w \le_{\mathrm{lr}} G_w$.

This immediately implies

COROLLARY 3. *If F_i, $i = 1,2$, are absolutely continuous distributions and $\overline{G}_{i,\alpha}(t) = \alpha\overline{F}_i(t)/[1-\alpha\overline{F_i}(t)]$, $i = 1,2$, then $F_1 \le_{\mathrm{lr}} F_2 \Leftrightarrow G_{1,\alpha} \le_{\mathrm{lr}} G_{2,\alpha}$.*

Consider the cumulative hazard function $\Lambda_F = -\log \overline{F}$, and analogously for G_α. Deshpande and Sengupta (1994) studied a partial ordering characterized as follows: G_α is said to be *ageing faster than* F ($G_\alpha \prec_c F$) if and only if $\Lambda_{G_\alpha}\Lambda_F^{-1}$ is convex on $[0,\infty)$. It is obvious that $\Lambda_F^{-1}(t) = F^{-1}(1-\exp\{-t\})$. We can compute

$$\Lambda_{G_\alpha}\Lambda_F^{-1}(t) = -\log \frac{\alpha e^{-t}}{1-\overline{\alpha}e^{-t}}. \tag{7}$$

The above function is independent of F. It is trivial that (7) is equal to $-\log(\overline{G}_\alpha^E)$, where $\overline{G}_\alpha^E$ denotes the survival function of the Marshall–Olkin distribution in the case when F is an exponential distribution. A simple computation gives that (7) is convex when $\alpha > 1$. Thus, we immediately get

LEMMA 4. *If $\alpha > [<]\ 1$, then $G_\alpha \prec_c [\succ_c]\ F$.*

3.3. Preservation of classes of life distributions. Marshall and Olkin (1997) proved that if F is an exponential distribution, then the distribution G_α is NBU. This can be extended to general classes of distributions which are IFR, IFRA, IRFR or NBU. Kirmani and Gupta (2001) proved that if $1 < \alpha < \infty$ and F is IFR, IFRA, or NBU then G_α is IFR, IFRA, or NBU respectively, and analogously for $0 < \alpha < 1$ and the dual classes, DFR, DFRA and NWU. First, this result can be extended to the case of IRFR (or DRFR) distributions.

THEOREM 4. *Let F be absolutely continuous with $F(0) = 0$. If F is IRFR [DRFR] and $1 < \alpha < \infty$ $[0 < \alpha < 1]$, then G_α is IRFR [DRFR].*

Proof. Let F be IRFR [DRFR]. Then $\breve{r}_F$ is an increasing [decreasing] function. The inverse hazard function of G_α is $\breve{r}_\alpha(t) = \breve{r}_F(t)\alpha/[1-\overline{\alpha}\overline{F}(t)]$. It is easy to see that the function $\alpha/[1-\overline{\alpha}\overline{F}(t)]$ is increasing when $\alpha > 1$ and decreasing when $\alpha \in (0,1)$, and thus the theorem follows. ∎

The above theorem and the results of Kirmani and Gupta (2001) do not cover all cases. The following examples show that using those results, in case $0 < \alpha < 1$ and F is IFR we can say nothing about G_α.

EXAMPLE 2. Consider the Weibull distribution with survival function $\overline{F}(t) = \exp\{-(\lambda t)^\beta\}$, $t \ge 0$, $\beta > 1$, which is IFR. Theorem 6 of Kirmani and

Gupta (2001) shows that if $1 < \alpha < \infty$, then G_α is IFR. Now, let $0 < \alpha < 1$. Then the hazard rate of G_α, $r_\alpha(t) = \lambda^\beta \beta t^{\beta-1}/(1-\overline{\alpha}\exp\{-\lambda t\}^\beta)$, is initially increasing and eventually increasing, but there may be one interval where it is decreasing (see Marshall and Olkin (1997)), thus G_α is neither IFR nor DFR.

EXAMPLE 3. Consider a distribution with survival function $\overline{F}(t) = \exp\{-e^t\}$, which is IFR. Let $0 < \alpha < 1$. Then we have $\overline{G}_\alpha(t) = \alpha\exp\{-e^t\}/(1-\overline{\alpha}\exp\{-e^t\})$. After some calculation we find that the hazard rate function of G_α is increasing for all t, and hence G_α is IFR.

In Example 2 one can also apply theorems on weighted distributions. The following theorem on classes of life distributions is a reformulation of Theorem 2 of Bartoszewicz and Skolimowska (2006):

THEOREM 5.

(i) *If* $1 < \alpha < \infty$ $[0 < \alpha < 1]$ *and* $g_\alpha/\overline{F}$ *is decreasing* [*increasing*], *then* G_α *is DFR* [*IFR*].
(ii) *If* $1 < \alpha < \infty$ $[0 < \alpha < 1]$ *and* g_α/F *is decreasing* [*increasing*], *then* G_α *is DRFR* [*IRFR*].
(iii) *If* $g_\alpha/\overline{F}$, $\alpha > 0$ *is decreasing, then* G_α *is DRFR.*
(iv) *If* g_α/F, $\alpha > 0$ *is increasing, then* G_α *is IFR.*

In order to get an interesting corollary, we need the following lemma:

LEMMA 5 (Bartoszewicz (1985, 1997), Bagai, Kochar (1986)).

(i) *If* $F \le_{\mathrm{hr}} G$ *and* F *or* G *is DFR, then* $F \le_{\mathrm{disp}} G$.
(ii) *If* $F \le_{\mathrm{hr}} G$ *and* F *or* G *is IRFR, then* $G \le_{\mathrm{disp}} F$.

Lemma 5, Corollary 2, Theorem 4, and Theorem 6 of Kirmani and Gupta (2001) imply

COROLLARY 4.

(i) *If* $0 < \alpha < \beta < 1$ *and* F *is DFR, then* $G_\alpha \le_{\mathrm{disp}} G_\beta$.
(ii) *If* $1 < \alpha < \beta < \infty$ *and* F *is IRFR, then* $G_\beta \le_{\mathrm{disp}} G_\alpha$.

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