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## AXIOMATIZATION OF VALUES OF COOPERATIVE GAMES USING A FAIRNESS PROPERTY

*Abstract.* We propose new systems of axioms which characterize four types of values of cooperative games: the Banzhaf value, the Deegan–Packel value, the least square prenucleolus and the least square nucleolus. The common element used in these axiomatizations is a fairness property. It requires that if to a cooperative game we add another game in which two given players are symmetric, then their payoffs change by the same amount. In our analysis we will use an idea applied by R. van den Brink (2001) to obtain an axiomatic characterization of the Shapley value.

**Introduction.** The article is devoted to axiomatizations of four types of values of cooperative games: the Banzhaf value, the Deegan–Packel value, the least square prenucleolus and the least square nucleolus. In contrast to the existing well known axiomatic theorems concerning these values, we use a fairness property as a common feature of all systems of axioms. This property requires that if to a cooperative game we add another game in which two given players are symmetric, then their payoffs change by the same amount.

Fairness was used by R. van den Brink (2001) to axiomatize the Shapley value. Our article is in a sense a continuation of his research. We will show that by the use of fairness one can obtain a relatively simple collection of conditions describing the four values mentioned above. In the case of linear values we can omit additivity. We employ the dummy player, amalgamation and fairness axioms to characterize the Banzhaf value. In the case of the Deegan–Packel value the system of axioms includes the quasi-efficiency, zero-player and fairness properties. Moreover, we will prove that fairness is a

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2000 *Mathematics Subject Classification*: Primary 91A12.

*Key words and phrases*: cooperative game, value, Banzhaf value, Deegan–Packel value, least square prenucleolus, least square nucleolus.

good tool to characterize the least square nucleolus, which is not additive and probably for this reason has not been axiomatized yet. This has turned out to be a simple consequence of the axiomatization of the least square prenucleolus, which consists of the efficiency, additivity and fairness axioms and a new inter-game balance axiom. All our new results are presented in Theorems 6–9 at the beginning of Section 2.

**1. Definitions and fundamental facts.** In this section we introduce some main notions, assumptions and relevant facts of cooperative game theory.

**1.1. Cooperative games and axioms.** Let  $n$  be a natural number, fixed throughout the paper. An  $n$ -person transferable utility cooperative game (briefly: *TU-game*) is defined by the set of players  $N = \{1, \dots, n\}$  (a *grand coalition*) and by a function  $v : 2^N \rightarrow \mathbb{R}$  with  $v(\emptyset) = 0$ , called the *characteristic function* of the game. Therefore, when  $N$  is fixed, a TU-game  $(N, v)$  can be identified with  $v$ . A game  $v$  is called *additive* if  $v(S \cup K) = v(S) + v(K)$  for any disjoint sets  $S, K \subseteq N$ , called *coalitions*. The cardinality of a coalition  $S$  will be denoted by  $|S|$ . The *unanimity game*  $u_T, T \subseteq N$ , is such that  $u_T(S) = 1$  if  $T \subseteq S$  and  $u_T(S) = 0$  otherwise, for any  $S \subseteq N$ .

A *value* of the game  $v$  is defined to be a function  $\varphi(v) = (\varphi_1(v), \dots, \varphi_n(v))$  which assigns to the game  $v$  a vector from  $\mathbb{R}^n$ . An  $n$ -dimensional vector  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  is called a *preimputation* if

$$\sum_{i=1}^n x_i = v(N).$$

If, additionally,  $x_i \geq v(\{i\})$  for all  $i \in N$ , then  $x$  is called an *imputation*. The sets of all preimputations and imputations of the game  $v$  will be denoted by **P** and **PI**, respectively. Let  $G_N$  be the set of all games  $v$ .

Now we formulate several fundamental properties used most often in various axiomatic theorems. They are basic for our considerations. The first group of Axioms 1–7 is standard.

Let  $\varphi$  be a value on  $G_N$  and  $v \in G_N$  be any  $n$ -person game.

**AXIOM 1 (Efficiency).** A value  $\varphi$  is *efficient* if for any game  $v$ ,

$$\sum_{i=1}^n \varphi_i(v) = v(N).$$

This means that the vector  $\varphi(v)$  is a preimputation.

**AXIOM 2 (Super-efficiency).** A value  $\varphi$  is *super-efficient* if it is efficient and the vector  $\varphi(v)$  is an imputation for any game  $v$  for which the set **PI** is nonempty.

AXIOM 3 (Equal treatment). If for two different players  $i, j \in N$  and all  $S \subseteq N \setminus \{i, j\}$  we have  $v(S \cup \{i\}) = v(S \cup \{j\})$  (in this case players  $i$  and  $j$  are called *symmetric*) then  $\varphi_i(v) = \varphi_j(v)$ .

Let  $v \in G_N$  and  $\sigma$  be a permutation of the set  $N$ . We define a new game  $\sigma v$  by  $\sigma v(\sigma(K)) = v(K)$  for any  $K \subseteq N$ , where  $\sigma(K) = \{\sigma(i) : i \in K\}$ .

AXIOM 4 (Symmetry). A value  $\varphi$  is *symmetric* if  $\varphi_{\sigma(i)}(\sigma v) = \varphi_i(v)$  for any  $i \in N$ , any game  $v$ , and any permutation  $\sigma$  of the set  $N$ .

Note that symmetry implies equal treatment (cf. L. S. Shapley (1953)).

AXIOM 5 (Additivity). A value  $\varphi$  is *additive* if  $\varphi(v+w) = \varphi(v) + \varphi(w)$  for any two  $n$ -person cooperative games  $v, w$ , where  $(v+w)(S) = v(S) + w(S)$  for any  $S \subseteq N$ .

AXIOM 6 (Dummy player property). If  $i \in N$  is a *dummy player* of the game  $v$ , i.e. if  $v(S \cup \{i\}) = v(S) + v(\{i\})$  for any  $S \subseteq N \setminus \{i\}$ , then  $\varphi_i(v) = v(\{i\})$ .

AXIOM 7 (Inessential game property). If  $v$  is an additive game then  $\varphi_i(v) = v(\{i\})$  for any  $i \in N$ .

The next axiom describes an interesting property which is a common feature of all the values of cooperative games analyzed in the paper. It was introduced by R. van den Brink (2001) to get a new axiomatization of the Shapley value.

AXIOM 8 (Fairness). A value  $\varphi$  has the *fairness property* if for any two symmetric players  $i$  and  $j$  in a game  $w$  we have  $\varphi_i(v+w) - \varphi_i(v) = \varphi_j(v+w) - \varphi_j(v)$  for any game  $v$ , i.e. the payoffs of both players change by the same amount.

E. Lehrer (1988) introduced the notion of amalgamation of two players and the corresponding new axiom in the description of the Banzhaf value.

DEFINITION 1 (E. Lehrer (1988)). An *amalgamation* of two different players  $i, j$  of an  $n$ -person game  $v$  is a transformation from the game  $v$  into the  $(n-1)$ -person game  $v_{(ij)}$  with the set of players  $(N \setminus \{i, j\}) \cup \{p\}$ , where  $p$  denotes a player represented by the coalition  $\{i, j\}$ . The characteristic function of this game is defined by

$$v_{(ij)}(K) = \begin{cases} v(K) & \text{if } p \notin K, \\ v((K \setminus \{p\}) \cup \{i, j\}) & \text{if } p \in K, \end{cases}$$

for any set  $K \subseteq (N \setminus \{i, j\}) \cup \{p\}$ .

To formulate the next axiom we need to introduce the set of all games  $v$  with grand coalition  $S \subseteq N$ , that is,

$$\tilde{G}_N = \bigcup_{S \subseteq N} G_S.$$

AXIOM 9 (Amalgamation property). For any two different players  $i, j \in N$  and any  $v \in \tilde{G}_N$ ,  $\varphi_p(v_{(ij)}) = \varphi_i(v) + \varphi_j(v)$ .

The next two axioms were used in the axiomatic description of the Deegan–Packel value (J. Deegan and E. W. Packel (1979)).

AXIOM 10 (Quasi-efficiency). A value  $\varphi$  is *quasi-efficient* if for any game  $v$ ,

$$\sum_{i=1}^n \varphi_i(v) = \sum_{S \subseteq N} v(S).$$

AXIOM 11 (Zero-player property). If  $i \in N$  is a zero-player of a game  $v$  then  $\varphi_i(v) = 0$ . (A player  $i \in N$  is called a *zero-player* if for any  $K \subseteq N$ ,  $v(K) = 0$  whenever  $i \in K$ .)

L. M. Ruiz *et al.* (1996) proposed a new value, the least square prenucleolus, where the next axiom was used.

For a game  $v$  and  $i \in N$ , we define

$$g_i(v) = \sum_{\substack{K \subseteq N \\ i \in K}} v(K).$$

AXIOM 12 (Average marginal contribution monotonicity (AMC)). For any  $i, j \in N$  and any game  $v$ , if  $g_i(v) \geq g_j(v)$  then  $\varphi_i(v) \geq \varphi_j(v)$ .

The last axiom is new. It is applied in the axiomatization of the least square prenucleolus and the least square nucleolus (values defined by L. M. Ruiz *et al.* (1996)), proposed in the present paper.

AXIOM 13 (Inter-game balance). Let  $v, w \in G_N$ . For any  $i, j \in N$  if  $\varphi_i(v + w) = \varphi_j(v + w)$ , then  $\varphi_i(v) - \varphi_j(w) = \varphi_j(v) - \varphi_i(w)$ .

This property means that if the values of two players in the sum of two games are equal then the differences between the values of these players in the respective games are opposite (or, in other words, the sums of the values for these players in the two games are equal). It is easy to check that additivity implies inter-game balance. Let  $\varphi$  be an additive solution on  $G_N$ . Then for any  $v, w \in G_N$  and  $i, j \in N$  if  $\varphi_i(v + w) = \varphi_j(v + w)$  then  $\varphi_i(v) + \varphi_i(w) = \varphi_i(v + w) = \varphi_j(v + w) = \varphi_j(v) + \varphi_j(w)$ , which is equivalent to Axiom 13. Therefore the inter-game balance axiom seems weaker than additivity. We show later on that it is satisfied by the least square nucleolus, which is not additive.

**1.2. Basic values and their properties.** We first recall formulae for some most important types of values of cooperative games, and next quote some relevant theorems.

DEFINITION 2 (L. S. Shapley (1953)). The *Shapley value* of player  $i \in N$  in a game  $v \in G_N$  is defined as

$$\text{Sh}_i(v) = \sum_{S \subseteq N} \frac{|S|!(n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S)).$$

DEFINITION 3 (J. F. Banzhaf III (1965)). The *Banzhaf value* of player  $i \in N$  in a game  $v \in G_N$  is defined as

$$\text{B}_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N} (v(S \cup \{i\}) - v(S)).$$

DEFINITION 4 (J. Deegan and E. W. Packel (1979)). The *Deegan–Packel value* of player  $i \in N$  in a game  $v \in G_N$  is defined by

$$\text{DP}_i(v) = \sum_{\substack{K \subseteq N \\ i \in K}} \frac{v(K)}{|K|}.$$

Denote by  $S_1, \dots, S_{2^n}$  all the subsets of a set  $N$ . For a vector  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and a coalition  $S$  we define

$$x(S) = \sum_{i \in S} x_i.$$

A vector  $r(v, x) \in \mathbb{R}^{2^n}$  with coordinates of the form

$$r_i(v, x) = v(S_i) - x(S_i) \quad \text{for } i = 1, \dots, 2^n$$

is called the *excess vector*.

DEFINITION 5 (L. M. Ruiz *et al.* (1996)). The *least square prenucleolus* (briefly: *LS-prenucleolus*) of a game  $v \in G_N$  is a preimputation  $x$  of this game which satisfies

$$\sum_{i=1}^{2^n} (r_i(v, x) - \overline{r(v, x)})^2 = \min_{y \in \mathbf{P}} \sum_{i=1}^{2^n} (r_i(v, y) - \overline{r(v, y)})^2,$$

where  $\overline{r(v, x)}$  denotes the arithmetic mean of all coordinates of  $r(v, x)$ .

DEFINITION 6 (L. M. Ruiz *et al.* (1996)). The *least square nucleolus* (briefly: *LS-nucleolus*) of a game  $v \in G_N$  is an imputation  $x$  of this game which satisfies

$$\sum_{i=1}^{2^n} (r_i(v, x) - \overline{r(v, x)})^2 = \min_{y \in \mathbf{PI}} \sum_{i=1}^{2^n} (r_i(v, y) - \overline{r(v, y)})^2.$$

Now we will quote several theorems characterizing basic values defined above. During the last fifty years many axiomatizations based on various sets of properties have been formulated and proved. We have chosen only those results which will be useful in our research.

The well known axiomatization of the Shapley value (L. S. Shapley (1953)) states that  $\text{Sh}(v)$  is the unique value satisfying the efficiency, dummy player, equal treatment and additivity axioms. R. van den Brink (2001) proved that equal treatment and additivity can be replaced by the fairness axiom.

**THEOREM 1.** *A value  $\varphi$  on  $G_N$  satisfies the efficiency, dummy player and fairness axioms if and only if  $\varphi(v) = \text{Sh}(v)$  for all  $v \in G_N$ .*

**REMARK.** The first axiomatization of the Shapley value without the additivity axiom was given by H. P. Young (1985). Moreover, R. van den Brink (2001) noted some connections between fairness and other axioms. We state them together with a third simple property in the following theorem.

**THEOREM 2.** *Let  $\varphi$  be a value on  $G_N$ .*

- (a) *If  $\varphi$  satisfies the equal treatment and additivity axioms, then it also satisfies the fairness axiom.*
- (b) *If  $\varphi$  satisfies the dummy player and fairness axioms, then it also satisfies the equal treatment axiom.*
- (c) *If  $\varphi$  satisfies the zero-player and fairness axioms, then it also satisfies the equal treatment axiom.*

In general, the Banzhaf value  $B(v)$  is not efficient. On account of this difficulty, the first axiomatization of this value was given relatively late. E. Lehrer based his important result on the amalgamation axiom.

**THEOREM 3** (E. Lehrer (1988)). *The Banzhaf value  $\varphi(v) = B(v)$  for all  $v \in G_N$  is the unique value on  $\tilde{G}_N$  which satisfies the dummy player, equal treatment, amalgamation and additivity axioms.*

**THEOREM 4** (J. Deegan and E. W. Packel (1979)). *The value  $\varphi(v) = \text{DP}(v)$  for all  $v \in G_N$  is the unique value on  $G_N$  which satisfies the symmetry, quasi-efficiency, zero-player and additivity axioms.*

L. M. Ruiz *et al.* (1996) proved that for any  $v \in G_N$  and any player  $i \in N$  the LS-prenucleolus  $L(v)$  can be expressed by

$$(1) \quad L_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}} \left( ng_i(v) - \sum_{d \in N} g_d(v) \right).$$

It is worth noting that the LS-prenucleolus is the additive normalization of the Banzhaf value, i.e., for  $i \in N$ ,

$$L_i(v) = B_i(v) + \left( v(N) - \sum_{d=1}^n B_d(v) \right) / n.$$

In the article by L. M. Ruiz *et al.* (1996) the following was proved.

**THEOREM 5.** *The LS-prenucleolus is the unique value on  $G_N$  which satisfies the efficiency, average marginal contribution monotonicity, inessential game and additivity axioms.*

The fact that the LS-nucleolus is not additive is the main reason for the lack of its effective axiomatization. In this article we will try to provide some consistent set of axioms to characterize this value.

The LS-nucleolus can be calculated using the following algorithm.

**ALGORITHM 1** (L. M. Ruiz *et al.* (1996)). Construct a sequence of pairs  $(\mathbf{x}_i, T_i)$ ,  $i = 1, 2, \dots$ , where  $\mathbf{x}_i = (x_{i1}, \dots, x_{in})$  is a preimputation and  $T_i$  is a subset of  $N$  inductively defined by

- (a)  $\mathbf{x}_1 := L(v)$  and  $T_1 := \{j \in N : x_{1j} < v(\{j\})\}$ ,
- (b)  $\mathbf{x}_{i+1}$  is the solution of the problem

$$\min \sum_{S \subseteq N} (v(S) - y(S))^2$$

such that  $\sum_{j \in N} y_j = v(N)$  and  $y_j = v(\{j\}) \forall j \in M_i = \bigcup_{k=1}^i T_k$ .

The sequence terminates when  $T_i = \emptyset$ . This procedure ends after at most  $n - 1$  steps and the final vector  $\mathbf{x}_i$  is the LS-nucleolus. One can easily check that the solution  $\mathbf{x}_{i+1} = (y_1, \dots, y_n)$  of the problem in (b) for  $j \notin M_i$  is of the form

$$\mathbf{x}_{i+1,j} = \frac{v(N) - \mathbf{x}_i(M_i)}{n - |M_i|} + \frac{1}{(n - |M_i|)2^{n-2}} \left( (n - |M_i|)g_i(v) - \sum_{k \notin M_i} g_k(v) \right)$$

for every  $i = 2, 3, \dots$

**2. Main results.** As mentioned earlier, the fairness axiom is a useful property describing a feature of “just” cooperative games and used to construct a specific axiomatization of the Shapley value (cf. Theorem 1). It turns out, however, that this axiom is also very helpful in an axiomatic description of other values: the Banzhaf value, the Deegan–Packel value, the LS-prenucleolus and the LS-nucleolus. As those values are completely different from one another, this can be rather surprising. Below we formulate Theorems 6–9, describing the four values axiomatically with the use of the fairness axiom.

The first theorem shows that in the classical axiomatic characterization of the Banzhaf value (cf. Theorem 3) the equal treatment and additivity properties can be replaced by the fairness axiom.

**THEOREM 6.** *A value  $\varphi$  on  $\tilde{G}_N$  coincides with the Banzhaf value if and only if it satisfies the amalgamation, dummy player and fairness axioms.*

In our second theorem a new axiomatization of the Deegan–Packel value consists of two axioms used in the classical approach (cf. Theorem 3) and the fairness property instead of symmetry and additivity.

**THEOREM 7.** *The Deegan–Packel value is the unique value on  $G_N$  which satisfies the quasi-efficiency, zero-player and fairness axioms.*

The third theorem gives an axiomatization of the LS-prenucleolus. It is based on the fairness axiom and on the inter-game balance property (cf. Axiom 13).

**THEOREM 8.** *A value  $\varphi$  on  $G_N$  satisfies the efficiency, inter-game balance, average marginal contribution monotonicity, fairness and inessential game axioms if and only if  $\varphi$  is the LS-prenucleolus.*

The last theorem gives an axiomatic description of the LS-nucleolus; it is a simple consequence of Theorem 8. This is the first effective axiomatization of this value.

**THEOREM 9.** *A value  $\varphi$  on  $G_N$  satisfies the super-efficiency, inter-game balance, average marginal contribution monotonicity, fairness and inessential game axioms if and only if  $\varphi$  is the LS-nucleolus.*

The proofs of Theorems 6–9 are given in Subsections 2.1–2.4.

**2.1. Proof of Theorem 6.** We begin with the following lemma.

**LEMMA 1.** *Assume that a value  $\varphi$  on  $\tilde{G}_N$  satisfies the amalgamation, dummy player and fairness axioms. Then for any nonempty coalition  $T \subseteq N$ , unanimity game  $u_T$  and any constant  $c$ ,*

$$(2) \quad \varphi_i(c \cdot u_T) = \begin{cases} c/2^{|T|-1} & \text{if } i \in T, \\ 0 & \text{if } i \notin T. \end{cases}$$

*Proof.* The proof is similar to the corresponding proofs for other axiomatizations (A. S. Nowak (1997), for example). It will be carried out by induction on  $t$ ,  $1 \leq t \leq n$ . First notice that  $\varphi_i(c \cdot u_T)$  of the form (2) coincides with the Banzhaf value  $B_i(c_i \cdot u_T)$ . This follows easily from Definition 3.

Formula (2) holds for  $|T| = 1$ , because then each player in the game  $u_T$  is dummy. Suppose that it is true for some  $n$  and for any coalition  $T$  with  $|T| = t < n$ . Let  $\omega_{\mathcal{T}}$  be an  $n$ -person unanimity game with  $|\mathcal{T}| = t + 1$ . Amalgamate some two different players  $i, j \in \mathcal{T}$ . Then according to

Definition 1,  $\omega_{\mathcal{T}(ij)} = \omega_{\mathcal{T}^*}$  is the  $n$ -person unanimity game of the coalition  $\mathcal{T}^* = (\mathcal{T} \setminus \{i, j\}) \cup \{p\}$  with  $|\mathcal{T}^*| = t$ .

By the induction hypothesis  $\varphi_p(c \cdot \omega_{\mathcal{T}^*}) = c/2^{|\mathcal{T}^*|-1} = c/2^{t-1}$  and according to the amalgamation property  $\varphi_i(c \cdot \omega_{\mathcal{T}}) + \varphi_j(c \cdot \omega_{\mathcal{T}}) = c/2^{t-1}$ .

The fairness and dummy player axioms imply equal treatment (by Theorem 2(b)). Hence, since players  $i$  and  $j$  are symmetric in the game  $c \cdot \omega_{\mathcal{T}}$ , we have  $\varphi_i(c \cdot \omega_{\mathcal{T}}) = \varphi_j(c \cdot \omega_{\mathcal{T}})$ , and consequently,  $\varphi_i(c \cdot \omega_{\mathcal{T}}) = c/2^t = c/2^{|\mathcal{T}|-1}$  for  $i \in \mathcal{T}$ .

On the other hand, the dummy player axiom implies that  $\varphi_i(c \cdot \omega_{\mathcal{T}}) = 0$  if  $i \notin \mathcal{T}$ . Therefore, by the induction principle, formula (2) holds. Thus  $\varphi(c \cdot u_{\mathcal{T}}) = B(c \cdot u_{\mathcal{T}})$  for any unanimity game  $u_{\mathcal{T}}$  and any real  $c$ , ending the proof of the lemma. ■

In the proof of the theorem, we will use the game graph method proposed by R. van den Brink (2001) for the Shapley value. For completeness we repeat some arguments from that paper.

( $\Rightarrow$ ) The Banzhaf value satisfies the amalgamation and dummy players axioms (cf. Theorem 3). Because it is additive and has the equal treatment property, according to Theorem 2(a), it also satisfies fairness.

( $\Leftarrow$ ) Fix a value  $\varphi$  on  $\tilde{G}_N$  and assume that it has all the properties mentioned in the theorem.

We will prove that  $\varphi = B$ . It is clear that  $\varphi = B$  for all one-player games (in this case  $v = c \cdot u_{\{i\}}$  for some real number  $c$ ). Suppose that  $\varphi = B$  for all  $m$ -person games where  $m \leq n - 1$ . Let then  $v$  be an  $n$ -person game. It can be uniquely represented as a linear combination of unanimity games (cf. L. S. Shapley (1953)):

$$(3) \quad v = \sum_{S \subseteq N} \eta_S u_S,$$

where  $\eta_S$  are constants. Let  $\delta(v)$  be the number of nonzero coefficients  $\eta_S$  in (3). We will use induction on  $\delta(v)$ .

If  $\delta(v) = 0$ , then  $v$  is the *null game*, i.e.  $v(S) = 0$  for all  $S \subseteq N$ . The dummy player property implies that  $\varphi_i(v) = 0 = B_i(v)$  for any  $i \in N$ . If  $\delta(v) = 1$  then  $v = \eta_S u_S$  for some  $S \subseteq N$  with  $\eta_S \neq 0$ . Then we have  $\varphi_i(v) = B_i(\eta_S u_S)$  for any  $i \in N$ .

Now assume that  $\varphi(z)$  is uniquely determined and coincides with  $B(z)$  for all games  $z \in G_N$  with  $\delta(z) \leq k$  ( $k \geq 1$ ), and let  $v$  be any  $n$ -person game with  $\delta(v) = k + 1$ . Then  $\delta(v) \geq 2$  and  $n \geq 2$ . Put  $\Delta(v) = \{S \subseteq N : \eta_S \neq 0\}$ . We introduce the following auxiliary definition.

DEFINITION 7. Define the graph  $(N, \mathfrak{R}_v)$  as follows:  $\{i, j\} \in \mathfrak{R}_v$  if and only if  $i, j \in N$ ,  $i \neq j$  and there exists a set  $S \in \Delta(v)$  with  $\{i, j\} \subseteq S$  or  $\{i, j\} \cap S = \emptyset$ . A coalition  $K \subseteq N$  is said to be *connected* in  $\mathfrak{R}_v$  if

either  $\text{card}(K) = 1$  or for any  $i, j \in K$ ,  $i \neq j$ , there is a sequence of players  $p_1, \dots, p_q$  such that  $p_1 = i$ ,  $p_q = j$  and  $\{p_r, p_{r+1}\} \in \mathfrak{R}_v$  for all  $r \in \{1, \dots, q-1\}$ . A connected coalition  $K$  is a *component* in  $\mathfrak{R}_v$  if  $\{i, j\} \notin \mathfrak{R}_v$  whenever  $i \in K$  and  $j \in N \setminus K$ .

The rest of the proof will be divided into two cases.

CASE 1:  $N$  is a component in  $\mathfrak{R}_v$ , i.e.  $(N, \mathfrak{R}_v)$  is a connected graph. Let  $\zeta(v) = \{i \in N : \eta_S \neq 0 \text{ for some } S \ni i\}$ . Fix any  $j \in \zeta(v)$  and let  $S_0 = \{j\}$ . For  $b \in N$  we define recursively the sets

$$(4) \quad S_b = \left\{ i \in N \setminus \bigcup_{\mu=0}^{b-1} S_\mu : \text{there exists an } h \in S_{b-1} \text{ such that } \{i, h\} \in \mathfrak{R}_v \right\}.$$

Clearly,  $S_r \cap S_t = \emptyset$  for  $r \neq t$ . Suppose that  $N \setminus \bigcup_{\mu=0}^{b-1} S_\mu \neq \emptyset$  and  $S_b = \emptyset$ . Then for any  $i \in N \setminus \bigcup_{\mu=0}^{b-1} S_\mu$  and  $h \in \bigcup_{\mu=0}^{b-1} S_\mu$  we have  $\{i, h\} \notin \mathfrak{R}_v$ , which is impossible, because  $N$  is a component in  $\mathfrak{R}_v$ . Thus  $N \setminus \bigcup_{\mu=0}^{b-1} S_\mu = \emptyset$ .

Therefore the finiteness of  $N$  implies that there exists an  $m \in N$  such that  $S_0, S_1, \dots, S_m$  is a partition of  $N$  consisting of nonempty sets.

Let  $\varphi_j(v) = a$  for some  $a \in \mathbb{R}$  and let  $a_j = 0$ . Therefore we can put

$$\varphi_i(v) = a + a_i, \quad i \in N,$$

for some  $a_i$ . Clearly, the set  $\{a_i : i \in S_0\}$  is uniquely determined, since  $S_0 = \{j\}$  and  $a_j = 0$ . Suppose now that all  $a_h$  for  $h \in S_{t-1}$ , for some  $t \geq 1$ , have been uniquely determined.

For any fixed  $i \in S_t$ , there exist  $h \in S_{t-1}$  and  $S \in \Delta(v)$  such that  $\{i, h\} \subseteq S$  or  $\{i, h\} \cap S = \emptyset$ . Then players  $i, h$  are symmetric in the game  $\eta_S u_S$  and from the fairness property we have  $\varphi_i(v) - \varphi_i(v - \eta_S u_S) = \varphi_h(v) - \varphi_h(v - \eta_S u_S)$ . Hence,

$$\begin{aligned} a + a_i &= \varphi_i(v) = \varphi_h(v) - \varphi_h(v - \eta_S u_S) + \varphi_i(v - \eta_S u_S) \\ &= a + a_h - \varphi_h(v - \eta_S u_S) + \varphi_i(v - \eta_S u_S) \end{aligned}$$

and therefore  $a_i = a_h - \varphi_h(v - \eta_S u_S) + \varphi_i(v - \eta_S u_S)$ . However, by our main induction hypothesis (i.e.  $\delta(v - \eta_S u_S) = k$ ) we have  $\varphi_h(v - \eta_S u_S) = B_h(v - \eta_S u_S)$  and  $\varphi_i(v - \eta_S u_S) = B_i(v - \eta_S u_S)$ . It follows, therefore, that  $a_i$  is uniquely determined.

Amalgamate now players  $i$  and  $h$ . Then the amalgamation axiom and induction hypothesis imply

$$2a + a_i + a_h = \varphi_i(v) + \varphi_h(v) = \varphi_p(v_{(ih)}) = B_p(v_{(ih)}).$$

Consequently, also  $a = (B_p(v_{(ih)}) - a_i - a_h)/2$  is uniquely determined. Summarizing, all values  $\varphi_i(v)$ ,  $i = 1, \dots, n$ , are uniquely determined. Thus  $\varphi_i(v) = B_i(v)$  for any  $i = 1, \dots, n$ , because  $B(v)$  satisfies the three axioms considered in Theorem 6 (cf. Theorems 3 and 2(a)).

CASE 2:  $N$  is not a component in  $\mathfrak{R}_v$ . Then  $(N, \mathfrak{R}_v)$  is not a connected graph. This implies that  $\mathfrak{R}_v$  has at least two components. Moreover, any distinct components are disjoint. We will prove that  $\mathfrak{R}_v$  has exactly two components. Suppose  $B_1, B_2$  and  $B_3$  are three distinct components in  $\mathfrak{R}_v$ . Because  $\delta(v) \geq 2$  we have  $\Delta(v) \neq \emptyset$ . Without loss of generality we can assume that there exists  $S \in \Delta(v)$  such that  $S \subseteq B_3$ . If  $i \in B_1$  and  $j \in B_2$  then  $\{i, j\} \cap S = \emptyset$  since different components are disjoint. But then  $\{i, j\} \in \mathfrak{R}_v$ , yielding a contradiction with  $B_1, B_2$  being distinct components in  $\mathfrak{R}_v$ .

Let  $S$  and  $T$  be two components in  $\mathfrak{R}_v$ . Then  $S \cap T = \emptyset$ . If  $S \cup T \neq N$ , then for  $i \in S, j \in T$  and  $h \in N \setminus (S \cup T)$  we have  $\{\{i, h\}, \{j, h\}\} \subseteq \mathfrak{R}_v$ , which is a contradiction with  $T$  and  $S$  being components in  $\mathfrak{R}_v$ . Because  $\delta(v) \geq 2$ ,  $v$  is the sum of two multiplied unanimity games, i.e.  $v = \eta_S u_S + \eta_T u_T$ , where  $\eta_S \neq 0$  and  $\eta_T \neq 0$ .

We will analyze two subcases according to the cardinality of the set  $N$ .

First let  $n \geq 3$ . Without loss of generality we can assume that  $|S| \geq 2$ . Take  $i \in S$  and  $j \in T$  and define the game  $\xi \in G_N$  as

$$\xi = v + \eta_S u_{(S \setminus \{i\}) \cup \{j\}} = \eta_S u_S + \eta_S u_{(S \setminus \{i\}) \cup \{j\}} + \eta_T u_T.$$

Now we determine  $\varphi(\xi)$ . Let  $\varphi_j(\xi) = a$ . By fairness <sup>(1)</sup> we have  $\varphi_i(\xi) - \varphi_i(\eta_T u_T) = \varphi_j(\xi) - \varphi_j(\eta_T u_T)$ . We know by (2) that  $\varphi_i(\eta_T u_T) = 0$  and  $\varphi_j(\eta_T u_T) = \eta_T / (2^{|T|-1})$ . This way,  $\varphi_i(\xi) = \varphi_j(\xi) - \varphi_j(\eta_T u_T) + \varphi_i(\eta_T u_T) = a - \eta_T / (2^{|T|-1})$ . The dummy player axiom implies that

$$\varphi_h(\eta_S u_S + \eta_S u_{(S \setminus \{i\}) \cup \{j\}}) = 0 \quad \text{for } h \in T \setminus \{j\}.$$

Fairness <sup>(2)</sup> and the fact that  $\delta(\eta_S u_{(S \setminus \{i\}) \cup \{j\}}) = 1$  imply that for  $h \in T \setminus \{j\}$ ,

$$\begin{aligned} \varphi_j(\eta_S u_S + \eta_S u_{(S \setminus \{i\}) \cup \{j\}}) &= \varphi_h(\eta_S u_S + \eta_S u_{(S \setminus \{i\}) \cup \{j\}}) - \varphi_h(\eta_S u_{(S \setminus \{i\}) \cup \{j\}}) \\ &\quad + \varphi_j(\eta_S u_{(S \setminus \{i\}) \cup \{j\}}) = \eta_S / 2^{|S|-1}. \end{aligned}$$

Thus, for any  $h \in T \setminus \{j\}$  by fairness <sup>(3)</sup> we have

$$\begin{aligned} \varphi_h(\xi) &= \varphi_j(\xi) - \varphi_j(\eta_S u_S + \eta_S u_{(S \setminus \{i\}) \cup \{j\}}) + \varphi_h(\eta_S u_S + \eta_S u_{(S \setminus \{i\}) \cup \{j\}}) \\ &= a - \eta_S / 2^{|S|-1}. \end{aligned}$$

For any  $h \in S \setminus \{i\}$  by fairness <sup>(4)</sup> we have  $\varphi_h(\xi) - \varphi_h(\eta_S u_{(S \setminus \{i\}) \cup \{j\}}) = \varphi_i(\xi) - \varphi_i(\eta_S u_{(S \setminus \{i\}) \cup \{j\}})$ . We know that  $\varphi_i(\eta_S u_{(S \setminus \{i\}) \cup \{j\}}) = 0$  and  $\varphi_h(\eta_S u_{(S \setminus \{i\}) \cup \{j\}}) = \eta_S / 2^{|S|-1}$  for any  $h \in S \setminus \{i\}$ . Thus

<sup>(1)</sup> Players  $i$  and  $j$  are symmetric in the game  $\eta_S u_S + \eta_S u_{(S \setminus \{i\}) \cup \{j\}}$ .

<sup>(2)</sup> Players  $h$  and  $j$  are symmetric in the game  $\eta_S u_S$ .

<sup>(3)</sup> Players  $h$  and  $j$  are symmetric in the game  $\eta_T u_T$ .

<sup>(4)</sup> Players  $i$  and  $h$  are symmetric in the game  $v$ .

$$\begin{aligned}\varphi_h(\xi) &= \varphi_i(\xi) - \varphi_i(\eta_S u_{(S \setminus \{i\}) \cup \{j\}}) + \varphi_h(\eta_S u_{(S \setminus \{i\}) \cup \{j\}}) \\ &= a - \eta_T/2^{|T|-1} + \eta_S/2^{|S|-1}\end{aligned}$$

for any  $h \in S \setminus \{i\}$ .

We can now recapitulate our considerations as follows:

$$(5) \quad \varphi_h(\xi) = \begin{cases} a & \text{if } h = j, \\ a - \eta_T/2^{|T|-1} & \text{if } h = i, \\ a - \eta_S/2^{|S|-1} & \text{if } h \in T \setminus \{j\}, \\ a - \eta_T/2^{|T|-1} + \eta_S/2^{|S|-1} & \text{if } h \in S \setminus \{i\}. \end{cases}$$

Let  $i \neq \theta \in S$ . Amalgamate players  $h$  and  $\theta$ . Then  $\theta \in S \setminus \{i\}$  and by the amalgamation axiom as well as the induction hypothesis we have

$$\begin{aligned}B_p(\xi_{(h\theta)}) &= \varphi_p(\xi_{(h\theta)}) = \varphi_h(\xi) + \varphi_\theta(\xi) \\ &= \begin{cases} 2a - \eta_T/2^{|T|-1} + \eta_S/2^{|S|-1} & \text{if } h = j, \\ 2a - \eta_T/2^{|T|-2} + \eta_S/2^{|S|-1} & \text{if } h = i, \\ 2a - \eta_T/2^{|T|-1} & \text{if } h \in T \setminus \{j\}, \\ 2a - \eta_T/2^{|T|-2} + \eta_S/2^{|S|-2} & \text{if } h \in S \setminus \{i\}. \end{cases}\end{aligned}$$

Calculating  $a$  in each of these cases we obtain

$$(6) \quad \varphi_h(\xi) = \begin{cases} B_p(\xi_p)/2 + \eta_T/2^{|T|} - \eta_S/2^{|S|} & \text{if } h = j, \\ B_p(\xi_p)/2 + \eta_T/2^{|T|-1} - \eta_S/2^{|S|} & \text{if } h = i, \\ B_p(\xi_p)/2 + \eta_T/2^{|T|} & \text{if } h \in T \setminus \{j\}, \\ B_p(\xi_p)/2 + \eta_T/2^{|T|-1} - \eta_S/2^{|S|-1} & \text{if } h \in S \setminus \{i\}. \end{cases}$$

Now we must determine the value  $\varphi_i(v)$ ,  $i \in N$ . Let  $\varphi_j(v) = b$ . Fairness and the dummy player property imply equal treatment (by Theorem 2(b)) and therefore  $\varphi_h(v) = b$  for all  $h \in T$ . For any  $h \in S \setminus \{i\}$  by fairness (players  $h$  and  $j$  are symmetric in the game  $\eta_S u_{(S \setminus \{i\}) \cup \{j\}}$ ) we have  $\varphi_h(v) - \varphi_h(\xi) = \varphi_j(v) - \varphi_j(\xi)$ , and by (6)

$$\varphi_h(v) = \varphi_j(v) - \varphi_j(\xi) + \varphi_h(\xi) = b + (\eta_T/2^{|T|-1} - \eta_S/2^{|S|-1})/2.$$

From Theorem 2(b) we can also conclude that  $\varphi_h(v) = \varphi_d(v)$  for all  $d \in S$ , since any two players in  $S$  are symmetric in the game  $v$ . This way

$$(7) \quad \varphi_i(v) = \begin{cases} b + \eta_T/2^{|T|} - \eta_S/2^{|S|} & \text{if } i \in S, \\ b & \text{if } i \in T. \end{cases}$$

Now, amalgamate two players  $i \in S$  and  $j \in T$ . Then according to the amalgamation axiom and induction hypothesis we obtain  $B_p(v_{(ij)}) = \varphi_i(v) + \varphi_j(v) = 2b + \eta_T/2^{|T|} - \eta_S/2^{|S|}$ , whence  $b = B_p(v_{(ij)})/2 - \eta_T/2^{|T|+1} + \eta_S/2^{|S|+1}$ . Therefore, it follows by (7) that  $\varphi_i(v)$  is uniquely determined for any  $i \in N$ , when  $n \geq 3$ .

Suppose now that  $n = 2$ . Let  $N = \{1, 2\}$  and repeat the considerations of the previous subcase with obvious modifications ( $|S| = |T| = 1$ ).

Thus  $\varphi(v)$  is uniquely determined and because it satisfies all axioms mentioned in the theorem, we also have  $\varphi(v) = B(v)$ , as in Case 1. ■

**2.2. Proof of Theorem 7.** In the proof we again apply the graph method using a new system of basic games in  $\mathbb{R}^{2^n}$  instead of unanimity games. The idea is similar to the standard R. van den Brink method.

( $\Rightarrow$ ) The Deegan–Packel value (DP) has the quasi-efficiency and zero-player properties (cf. Theorem 4). Because it also satisfies equal treatment (an easily verifiable fact) and additivity, it satisfies fairness (by Theorem 2(a)).

( $\Leftarrow$ ) Fix a value  $\varphi$  on  $G_N$  and assume that it satisfies all the three properties appearing in Theorem 7. For  $S \subseteq N$ , let  $w_S$  be a basic game, i.e.

$$(8) \quad w_s(K) = \begin{cases} 1 & \text{if } K = S \\ 0 & \text{if } K \neq S \end{cases} \quad \text{for any } K \subseteq N.$$

Then according to quasi-efficiency, zero-player and equal treatment (a consequence of Theorem 2(b)), we have

$$(9) \quad \varphi_i(c \cdot w_s) = \begin{cases} c/|S| & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \quad \text{for any constant } c.$$

We will prove that  $\varphi = \text{DP}$ . It is clear that  $\varphi = \text{DP}$  for all one-player games. Suppose that  $\varphi = \text{DP}$  for all  $m$ -person games where  $m \leq n - 1$ , and let  $v$  be any  $n$ -person game. It is known (cf. J. Deegan and E. W. Packel (1979)) that  $v$  can be uniquely represented as a linear combination

$$(10) \quad v = \sum_{S \subseteq N} c_S w_S,$$

where  $c_S$  are constants. Let  $d(v)$  be the number of nonzero coefficients  $c_S$  in (10). We will use induction on  $d(v)$ .

If  $d(v) = 0$ , then by the zero-player axiom,  $\varphi_i(v) = 0 = \text{DP}_i(v)$  for any  $i \in N$ . If  $d(v) = 1$  then  $v = c_S \cdot w_S$  for some  $S \subseteq N$ . By earlier conclusions we have  $\varphi_i(v) = c_S/|S| = \text{DP}_i(c_S \cdot w_S)$  for any  $i \in S$  and  $\varphi_i(v) = 0 = \text{DP}_i(c_S \cdot w_S)$  for any  $i \in N \setminus S$ .

The game graph in Subsection 2.1 was constructed for the structure of unanimity games. It is easy to obtain a similar construction for the representation consisting of the basic games  $\{w_S\}$ . Assume that  $\varphi(z)$  coincides with  $\text{DP}(z)$  for all games  $z \in G_N$  with  $d(z) \leq k$  ( $k \geq 1$ ) and let  $d(v) = k + 1$ . Then  $d(v) \geq 2$  and  $n \geq 2$ . Let  $D(v) = \{S \subseteq N : c_S \neq 0\}$ . Define the graph  $(N, \Gamma_v)$  as in Definition 7 with  $D(v)$  instead of  $\Delta(v)$ . We consider two cases.

CASE 1:  $N$  is a component in  $\Gamma_v$ . Let  $\vartheta(v) = \{i \in N : c_S \neq 0 \text{ for some } S \ni i\}$ . Fix any  $j \in \vartheta(v)$  and let  $S_0 = \{j\}$ . For  $b \in N$ , we define recursively

the sets  $S_b$  as in (4), where  $\mathfrak{R}_v$  is replaced by  $\Gamma_v$ . As in Subsection 2.1 we can prove that there is  $m \in N$  such that  $S_0, S_1, \dots, S_m$  is a partition of  $N$  consisting of nonempty sets. We can also repeat the further considerations for  $i \in S_t$  and some  $h \in S_{t-1}$  to get  $a_i = a_h - \text{DP}_h(v - c_S \cdot w_S) + \text{DP}_i(v - c_S \cdot w_S)$ , with some constant  $a_h$  determined inductively. Hence, all  $a_i$  with  $i \in S_t$  are uniquely determined, implying the same for all  $a_i, i \in N$ .

Quasi-efficiency implies that

$$\sum_{i=1}^n \varphi_i(v) = n \cdot a + \sum_{i=1}^n a_i = \sum_{S \subseteq N} v(S).$$

Therefore the constant  $a$  is also uniquely determined. By the equalities  $\varphi_i(v) = a + a_i$  for  $i \in N$ , we conclude that all values  $\varphi_i(v), i = 1, \dots, n$ , are uniquely determined. Thus  $\varphi_i(v) = \text{DP}_i(v)$  for any  $i = 1, \dots, n$ , because DP satisfies the three axioms considered in Theorem 7 (see part ( $\Rightarrow$ )).

CASE 2:  $N$  is not a component in  $\Gamma_v$ . In exactly the same way as in Case 2 in the previous subsection, we conclude that there are coalitions  $S$  and  $T$  with  $S \cap T = \emptyset$  and  $S \cup T = N$  such that  $v = c_S w_S + c_T w_T$  with  $c_S \neq 0$  and  $c_T \neq 0$ . We consider two subcases.

If  $n \geq 3$ , then without loss of generality we can assume that  $|S| \geq 2$ . Take an  $i \in S$  and  $j \in T$  and define the game  $\chi \in G_N$  as

$$\chi = v + c_S w_{(S \setminus \{i\}) \cup \{j\}} = c_S w_S + c_S w_{(S \setminus \{i\}) \cup \{j\}} + c_T w_T.$$

Now, we can repeat the reasoning of the analogous subcase leading to formula (5), replacing  $\xi, u, (2)$  and the dummy player axiom by  $\chi, w, (10)$  and the zero-player axiom, respectively. Then the result, instead of (5), will be

$$(11) \quad \varphi_h(\chi) = \begin{cases} a & \text{if } h = j, \\ a - c_T/|T| & \text{if } h = i, \\ a - c_S/|S| & \text{if } h \in T \setminus \{j\}, \\ a - c_T/|T| + c_S/|S| & \text{if } h \in S \setminus \{i\}. \end{cases}$$

By quasi-efficiency we have

$$2c_S + c_T = \sum_{S \subseteq N} \chi(S) = \sum_{i=1}^n \varphi_i(v) = n \cdot a - \frac{|S|}{|T|} c_T + \frac{|S| - |T|}{|S|} c_S.$$

Hence the constant  $a$ , and thereby value  $\varphi_h(\chi)$  for  $h \in N$ , is uniquely determined.

Now we must determine  $\varphi_i(v), i \in N$ . Let  $\varphi_j(v) = b$ . Fairness and the zero-player property imply equal treatment (by Theorem 2(c)) and therefore  $\varphi_h(v) = b$  for all  $h \in T$ . For any  $h \in S \setminus \{i\}$  by fairness we have  $\varphi_h(v) - \varphi_h(\chi) = \varphi_j(v) - \varphi_j(\chi)$  since then  $h$  and  $j$  are symmetric players in the game  $c_S w_{(S \setminus \{i\}) \cup \{j\}}$ . Therefore, by (11) we have  $\varphi_h(v) = \varphi_j(v) - \varphi_j(\chi) + \varphi_h(\chi) =$

$b - (c_T/|T| - c_S/|S|)$  for  $h \in S \setminus \{i\}$ . According to Theorem 2(c) we can also conclude that  $\varphi_i(v) = \varphi_d(v)$  for all  $d \in S$ , since  $i$  and any  $d$  in  $S$  are symmetric. Thus

$$\varphi_i(v) = \begin{cases} b - c_T/|T| + c_S/|S| & \text{if } i \in S, \\ b & \text{if } i \in T. \end{cases}$$

But quasi-efficiency implies

$$c_S + c_T = \sum_{K \subseteq N} v(K) = \sum_{i=1}^n \varphi_i(v) = n \cdot b - \frac{|S|}{|T|} c_T + c_S.$$

Hence the constant  $b$ , and thereby the value  $\varphi_i(v)$  for  $i \in N$ , are uniquely determined for  $n \geq 3$ .

Now, let  $n = 2$ . Therefore  $N = \{1, 2\}$ . Then  $v = c_{\{1\}}w_{\{1\}} + c_{\{2\}}w_{\{2\}}$ . Let  $\varphi_2(v) = a$ . The zero-player property implies that  $\varphi_2((c_{\{1\}} - c_{\{2\}})w_{\{1\}}) = 0$ . By quasi-efficiency we have  $\varphi_1((c_{\{1\}} - c_{\{2\}})w_{\{1\}}) = c_{\{1\}} - c_{\{2\}}$ . Fairness implies that

$$\varphi_1(v) - \varphi_1((c_{\{1\}} - c_{\{2\}})w_{\{1\}}) = \varphi_2(v) - \varphi_2((c_{\{1\}} - c_{\{2\}})w_{\{1\}})$$

and so  $\varphi_1(v) = a + c_{\{1\}} - c_{\{2\}}$ . From the quasi-efficiency axiom we obtain  $c_{\{1\}} + c_{\{2\}} = \varphi_1(v) + \varphi_2(v) = 2a + c_{\{1\}} - c_{\{2\}}$ , and hence  $a = c_{\{2\}}$ .

Thus  $\varphi(v)$  is uniquely determined and because it satisfies all axioms mentioned in Theorem 7, by part ( $\Rightarrow$ ), we finally have  $\varphi(v) = \text{DP}(v)$ . ■

**2.3. Proof of Theorem 8.** ( $\Rightarrow$ ) It is known that the least square prenucleolus (L) satisfies the efficiency, average marginal contribution monotonicity (AMC), and inessential game and additivity axioms (cf. Theorem 5). It is easy to verify that additivity implies inter-game balance. On the other hand, the LS-prenucleolus is a symmetric value; therefore it also has the equal treatment property. Thus, by Theorem 2(a) it satisfies fairness.

( $\Leftarrow$ ) Fix a value  $\varphi$  on  $G_N$  and assume that it satisfies all the conditions mentioned in the theorem. For  $S \subseteq N$ , let  $w_S \in G_N$  be a basic game defined in (8).

By efficiency and AMC, there exist constants  $\alpha_S$ ,  $S \subseteq N$ , such that for real  $c_S$ ,

$$(12) \quad \varphi_i(c_S \cdot w_S) = \begin{cases} c_S \cdot \alpha_S/|S| & \text{if } i \in S, \\ -c_S \cdot \alpha_S/(n - |S|) & \text{if } i \notin S \end{cases}$$

with  $\alpha_N = 1$ .

Let  $S_1$  and  $S_2$  be two disjoint nonempty subsets of  $N$ . Then, by AMC, we have  $\varphi_i(w_{S_1} + w_{S_2}) = \varphi_j(w_{S_1} + w_{S_2})$  for  $i \in S_1, j \in S_2$ . Thus, by inter-game balance it follows that  $\varphi_i(w_{S_1}) - \varphi_j(w_{S_2}) = \varphi_j(w_{S_1}) - \varphi_i(w_{S_2})$ , whence, by (12),

$$\frac{\alpha_{S_1}}{|S_1|} - \frac{\alpha_{S_2}}{|S_2|} = -\frac{\alpha_{S_1}}{n - |S_1|} + \frac{\alpha_{S_2}}{n - |S_2|},$$

and consequently,

$$(13) \quad \alpha_{S_1}/(|S_1|(n - |S_1|)) = \alpha_{S_2}/(|S_2|(n - |S_2|))$$

if  $S_1 \neq \emptyset$ ,  $S_2 \neq \emptyset$  and  $S_1 \cap S_2 = \emptyset$ .

We will now prove that (13) holds for all nonempty subsets of  $N$ . If  $S_1 \cap S_2 \neq \emptyset$  and  $S_1 \cup S_2 \neq N$  then (13) is true for  $S_1$  and  $N \setminus (S_1 \cup S_2)$  as well as for  $S_2$  and  $N \setminus (S_1 \cup S_2)$ , and therefore for  $S_1$  and  $S_2$ . If  $S_1 \cap S_2 \neq \emptyset$  and  $S_1 \cup S_2 = N$  then (13) holds for  $S_1$  and  $N \setminus S_1$  as well as for  $S_2$  and  $N \setminus S_2$ , and thus for  $N \setminus S_1$  and  $N \setminus S_2$  and finally for  $S_1$  and  $S_2$ . Then

$$(14) \quad \alpha := \alpha_S/(|S|(n - |S|))$$

for any nonempty  $S \subsetneq N$  is a constant. Thus  $\alpha$  is well defined when  $n = |N| > 1$ . For  $|N| = 1$ , let  $\alpha$  be arbitrary.

Let

$$(15) \quad \lambda_i(v) = \frac{v(N)}{n} + \alpha \left( n \sum_{\substack{K \subseteq N \\ i \in K}} v(K) - \sum_{p \in N} \sum_{\substack{K \subseteq N \\ p \in K}} v(K) \right)$$

for any  $i \in N$ .

We will prove that the value  $\varphi$  is uniquely determined as  $\varphi_i(v) = \lambda_i(v)$  for any  $i \in N$ .

We apply all the assumptions and notations concerning the game graph method introduced in Subsection 2.2.

It is clear (by the efficiency axiom and (15)) that  $\varphi = \lambda$  for all one-player games. Suppose that  $\varphi = \lambda$  for all  $m$ -person games where  $m \leq n - 1$ , and let  $v$  be an  $n$ -person game. We will use induction on  $d(v)$ .

If  $d(v) = 0$ , then the efficiency and AMC properties imply that  $\varphi_i(v) = 0 = \lambda_i(v)$  for any  $i \in N$ . If  $d(v) = 1$  then  $v = c_S w_S$  for some  $S \subseteq N$ . By (12), (14) and (15) we easily get  $\varphi_i(v) = \lambda_i(c_S \cdot w_S)$  for all  $i \in N$ .

Assume therefore that  $\varphi(z)$  coincides with  $\lambda(z)$  for all games  $z \in G_N$  with  $d(z) \leq k$  (for some  $k \geq 1$ ) and let  $d(v) = k + 1$ . Then  $d(v) \geq 2$  and  $n \geq 2$ . We consider two cases.

CASE 1:  $N$  is a component in  $\Gamma_v$ . Fix any  $j \in \vartheta(v)$  and let  $S_0 = \{j\}$ , where  $\vartheta(v)$  and  $S_b$  are defined as in Subsection 2.2. We know that there exists an  $m \in N$  such that  $S_0, S_1, \dots, S_m$  is a partition of  $N$  consisting of nonempty sets. We can repeat those considerations for  $i \in S_t$  and some  $h \in S_{t-1}$  to get  $a_i = a_h - \lambda_h(v - c_S \cdot w_S) + \lambda_i(v - c_S \cdot w_S)$ , with some constant  $a_h$  determined inductively. It follows that all  $a_i$  with  $i \in S_t$  are uniquely determined, implying the same for all  $a_i$ ,  $i \in N$ . Just as in Subsection 2.2, we can prove that  $a$  and, consequently, the values  $\varphi_i(v)$ ,  $i = 1, \dots, n$ , are uniquely determined. Hence  $\varphi_i(v) = \lambda_i(v)$  for all  $i \in N$ , because  $\lambda$  satisfies the first four axioms considered in Theorem 8 (we have not used the inessential game property of  $\varphi$  yet).

CASE 2:  $N$  is not a component in  $\Gamma_v$ . In exactly the same way as in Case 2 in Subsections 2.1 and 2.2 we conclude that  $\Gamma_v$  has exactly two components, denoted by  $S$  and  $T$ , with  $S \cap T = \emptyset$  and  $S \cup T = N$ , and therefore  $v$  is the sum of two basic games:  $v = c_S w_S + c_T w_T$  with  $c_S \neq 0$  and  $c_T \neq 0$ .

Since  $g_i(c_S w_S + c_T w_T) = g_j(c_S w_S + c_T w_T)$  for  $i, j \in S$ , the efficiency and AMC axioms imply that there exists a constant  $\beta_S$  such that

$$(16) \quad \varphi_i(c_S w_S + c_T w_T) = \begin{cases} \beta_S/|S| & \text{if } i \in S, \\ -\beta_S/(n - |S|) & \text{if } i \in T \end{cases}$$

(note that  $|T| = n - |S|$ ).

Define an  $n$ -person game  $z = c_S w_S + c_T w_T + (c_S - c_T)w_T = c_S(w_S + w_T)$ . By the AMC axiom we have  $\varphi_i(z) = \varphi_j(z)$  for any  $i \in S$  and  $j \in T$  and then, according to the inter-game balance property,

$$\varphi_i(c_S w_S + c_T w_T) - \varphi_j(c_S w_S + c_T w_T) = \varphi_j((c_S - c_T)w_T) - \varphi_i((c_S - c_T)w_T).$$

Hence, using (16) and (12), we get

$$\frac{\beta_S}{|S|} + \frac{\beta_S}{n - |S|} = (c_S - c_T)\alpha(n - |T|) + (c_S - c_T)\alpha|T| = (c_S - c_T)\alpha \cdot n$$

and therefore

$$\beta_S = \frac{(c_S - c_T)\alpha \cdot n \cdot |S| \cdot (n - |S|)}{|S| + n - |S|} = (c_S - c_T)\alpha|S|(n - |S|).$$

Now, using (16), (14) and (15) we deduce that  $\varphi_i(v) = \lambda_i(v)$  for any  $i \in N$ .

This way we have proved that  $\varphi$  coincides with  $\lambda$  on  $G_N$ . The inessential game property (applied to  $\lambda$ ) implies that  $\alpha = 1/(n2^{n-2})$ . Thus, we finally obtain  $\lambda = L$ . The proof is complete. ■

**2.4. Proof of Theorem 9.** ( $\Rightarrow$ ) It is well known (cf. L. M. Ruiz *et al.* (1996)) that the LS-nucleolus (LN) satisfies the super-efficiency, average marginal contribution monotonicity and inessential game axioms. We must prove that it also has the inter-game balance and fairness properties.

One can easily check that for any  $v \in G_N$  and  $i, j \in N$ ,  $i \neq j$ , we have (cf. L. M. Ruiz *et al.* (1996))

$$(17) \quad \text{LN}_i(v) - \text{LN}_j(v) = \frac{1}{2^{n-2}} (g_i(v) - g_j(v)).$$

Suppose that  $v, w \in G_N$  and  $\text{LN}_i(v+w) = \text{LN}_j(v+w)$  for some  $i, j \in N$ ,  $i \neq j$ . Then by (17), we have  $g_i(v+w) = g_j(v+w)$ . Thus, by additivity of  $g$ , we obtain  $g_i(v) - g_j(w) = g_j(v) - g_i(w)$  and by (17),  $\text{LN}_i(v) - \text{LN}_j(w) = \text{LN}_j(v) - \text{LN}_i(w)$ . Therefore the LS-nucleolus satisfies the inter-game balance axiom.

Let  $w \in G_N$  and  $i, j \in N$ ,  $i \neq j$ , be two players such that  $w(S \cup \{i\}) = w(S \cup \{j\})$  (i.e. players  $i, j$  are symmetric in the game  $w$ ). Then we have  $g_i(w) = g_j(w)$ . Therefore

$$g_i(v+w) - g_i(v) - g_i(w) = g_j(v+w) - g_j(v) - g_j(w)$$

and  $g_i(v+w) - g_i(v) = g_j(v+w) - g_j(v)$  for any  $v \in G_N$ . By (17) we have  $LN_i(v+w) - LN_i(v) = LN_j(v+w) - LN_j(v)$ . Thus the LS-nucleolus satisfies fairness.

( $\Leftarrow$ ) Suppose that a value  $\varphi$  on  $G_N$  satisfies all axioms given in the theorem. Then it is an imputation satisfying the assumptions of Theorem 8. According to that theorem and the conclusions <sup>(5)</sup> made by L. M. Ruiz *et al.* (1996) we have  $\varphi = LN$ .

**Acknowledgements.** I thank Professor José M. Zarzuelo (University of the Basque Country, Bilbao, Spain) for useful explanations concerning the properties of the LS-nucleolus. I am also grateful to the anonymous referee for patient and careful reading of the paper as well as for detailed comments and suggestions, which were helpful to improve my work.

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*Received on 16.7.2002;*  
*revised version on 23.4.2004*

(1649)

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<sup>(5)</sup> An LS-prenucleolus of a game  $v$  which is an imputation of this game coincides with its LS-nucleolus.