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Beata Sikora (Gliwice)

ON CONSTRAINED CONTROLLABILITY OF DYNAMICAL SYSTEMS WITH MULTIPLE DELAYS IN CONTROL

Abstract. Linear, continuous dynamical systems with multiple delays in control are studied. Their relative and absolute controllability with constrained control is discussed. Definitions of various types of constrained relative and absolute controllability for linear systems with delays in control are introduced. Criteria of relative and absolute controllability with constrained control are established. Constraints on control values are considered. Mutual implications between constrained relative controllability of systems with and without delays are studied as well as implications between constrained relative and absolute controllability of systems with delay in control. The results are illustrated by examples.

1. Introduction. Investigating the controllability of dynamical systems is one of the main elements in their analysis. Owing to the abundance of mathematical models of dynamical systems with delays, the controllability problem for such systems is especially important. Delay dynamical systems occur in many fields of science, industry, medicine, biology and economy.

In this article we analyse dynamical systems with delays in control. We discuss linear dynamical systems with multiple, time-dependent delays in control. The monograph [3] provides necessary and sufficient conditions of relative and absolute controllability for linear, continuous dynamical systems with multiple delays in control, but these criteria only concern unconstrained controls. Since, in practice, controls are always constrained, we investigate the relative and absolute controllability with delays and constraints in control. We illustrate our analysis by examples.

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2. Mathematical model. We consider linear, continuous, finite-dimensional dynamical systems with time-dependent, multiple delays in control described by the ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + \sum_{i=0}^{M} B_i(t)u(v_i(t)), \qquad t \geq t_0, \tag{1}$$

where

- $x(t) \in \mathbb{R}^M$ is the instantaneous state n-vector,
- $u \in L^2_{\mathrm{loc}}([0,\infty), \mathbb{R}^m)$ is the control,
- $A(t)$ is an $M \times M$ matrix with elements $a_{kj} \in L^1_{\mathrm{loc}}([0,\infty), \mathbb{R})$, $k, j = 1, \ldots, M$,
- $B_i(t)$, $i = 0, 1, \ldots, M$, are $M \times m$ matrices with elements $b_{ikj} \in L^2_{\mathrm{loc}}([0,\infty), \mathbb{R})$, $k = 1, \ldots, M$, $j = 1, \ldots, m$,
- $v_i : [t_0, \infty) \to \mathbb{R}$, $i = 0, 1, \ldots, M$, are absolutely continuous, strictly increasing functions, satisfying

  $$v_M(t) < v_{M-1} < \cdots < v_k(t) < \cdots < v_1(t) < v_0(t) = t, \qquad t \in [t_0, \infty),$$

  where $v_i(t) = t - h_i(t)$ and $h_i(t) \geq 0$, $i = 0, 1, \ldots, M$, are time-dependent delays in control.

Let $S \subset \mathbb{R}^M$ and $U \subset \mathbb{R}^m$ be any non-empty sets. Let $L^2([t_0,t], \mathbb{R}^m)$ denote the Hilbert space of square integrable functions defined in the time interval $[t_0, t]$ with values in $\mathbb{R}^m$. The set $L^2([t_0,t_1], U)$ of square integrable functions in $[t_0, t_1]$ with values in U is the *set of admissible controls* for the dynamical system (1). For a given initial condition $z(t_0) = \{x_0, u_{t_0}\} \in \mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$, where $x_0 = x(t_0) \in \mathbb{R}^M$ and u_{t_0} is a given initial value in $[v_M(t_0), t_0]$, and an admissible control $u \in L^2([t_0,t], U)$, for every $t \geq t_0$ there exists a unique, absolutely continuous solution $x(t, z(t_0), u)$ of the differential equation (1). This solution has the form (cf. [3])

$$x(t, z(t_0), u) = F(t, t_0)x(t_0) + \int_{t_0}^{t} F(t,\tau) \sum_{i=0}^{M} B_i(\tau)u(v_i(\tau))\, d\tau, \tag{2}$$

where $F(t,\tau)$ is the $n \times n$ transition matrix of the linear system

$$\dot{x}(t) = A(t)x(t).$$

The initial condition $z(t_0)$ is called an *initial complete state* of the dynamical system (1). In the case of a dynamical system with delays, only a complete state $z(t) = (x(t), u_t(s))$, where $u_t(s) = u(s)$ for $s \in [v_M(t), t)$, fully describes the behaviour of the dynamical system at time t.

3. Relative controllability. In this paper, constraints put directly on control values will be considered. Constraints of this type frequently occur in practical problems connected with, among others, optimal control of industrial processes or mathematical modelling of economic processes.

3.1. *Basic definitions.* In this section we define various types of relative controllability with constrained values of control for the dynamical system (1) in the time interval $[t_0, t_1]$.

Definition 1. The dynamical system (1) is said to be *relatively U-controllable in the time interval* $[t_0, t_1]$ *from the complete state* $z(t_0) \in \mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$ *into the set* $S \subset \mathbb{R}^M$ if for every vector $\widetilde{x} \in S$, there exists an admissible control $\widetilde{u} \in L^2([t_0, t_1], U)$ such that the corresponding trajectory $x(t, z(t_0), \widetilde{u})$ of (1) satisfies $x(t_1, z(t_0), \widetilde{u}) = \widetilde{x}$.

Definition 2. The dynamical system (1) is said to be (*globally*) *relatively U-controllable in the time interval* $[t_0, t_1]$ *into the set* S if it is relatively U-controllable in $[t_0, t_1]$ into S for every initial complete state $z(t_0) \in \mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$.

Definition 3. The dynamical system (1) is said to be (*globally*) *relatively U-controllable from* t_0 *into the set* S if for every initial complete state $z(t_0) \in \mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$, there exists $t_1 \in [0, \infty)$ such that (1) is relatively U-controllable in $[t_0, t_1]$ into S.

If $S = \mathbb{R}^M$, then we talk about (*global*) *relative U-controllability in* $[t_0, t_1]$. When $S = \{0\}$, we talk about *relative null U-controllability in* $[t_0, t_1]$ *from the complete state* $z(t_0)$, and (*global*) *relative null U-controllability in* $[t_0, t_1]$.

Assume that S is a linear variety in $\mathbb{R}^M$ of the form

$$S = \{x \in \mathbb{R}^M : \mathbf{L}x = c\}, \tag{3}$$

where $\mathbf{L}$ is a known $p \times M$ matrix of rank p and $c \in \mathbb{R}^p$ is a given vector. If $\mathbf{L} = \mathbf{I}_M$ (the $M \times M$ unit matrix) and $c = 0$, we get $S = \{0\}$.

There is also a related notion of attainable set. The *attainable set* from the initial complete state $z(t_0)$ at time $t \geq t_0$ for the dynamical system (1) is defined, just as for systems without constraints and delays [3], by

$$K_U([t_0, t], z(t_0)) = \Big\{x \in \mathbb{R}^M : x = F(t, t_0)x(t_0) + \int_{t_0}^{t} F(t, \tau) \sum_{i-1}^{M} B_i(\tau)u(v_i(\tau))\, d\tau,\ u \in L^2([t_0, t], U)\Big\}.$$

3.2. *Controllability results.* In order to formulate criteria for various types of controllability with constrained controls for the dynamical system (1), with the assumption that the final set is of the form (3), let us introduce a scalar function $J : \mathbb{R}^M \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$, connected with the attainable set $K_U([t_0, t], z(t_0))$ of the system (1) and defined by

$$J(z(t_0),t,a) = v^{\mathrm{T}}\mathbf{L}F(t,t_0)x(t_0) + \int_{t_0}^{t} \sup\Big\{a^{\mathrm{T}}\mathbf{L}F(t,\tau)\sum_{i=0}^{M} B_i(\tau)u(v_i(\tau)) : u \in L^2([t_0,t],U)\Big\}\,d\tau - a^{\mathrm{T}}c,$$

where $a \in \mathbb{R}^p$ is any vector; it is called the *supporting function* of the attainable set. An application of supporting functions for dynamical systems without delays can be found in [9].

Using the absolute continuity of the v_i and applying their inverses $r_i : [v_i(t_0), v_i(t_1)] \to [t_0,t_1]$, $i = 0,1,\ldots,M$, we can write the solution of (1) in the following form:

$$x(t,z(t_0),u) = F(t,t_0)x(t_0) + \sum_{i=0}^{M}\int_{v_i(t_0)}^{v_i(t_1)} F(t,r_i(\tau))B_i(r_i(\tau))\dot{r}_i(\tau)u(\tau)\,d\tau.$$

Let us fix a final time $t_1 > 0$. Without loss of generality, for simplicity of notation, we may assume that $t_0 = v_k(t_1)$ for some $k \geq M$. If such a k does not exist, then we introduce an additional delay h_k with control matrix $B_k(t) = 0$. Then the solution (2) of the dynamical system (1) has, at time t_1, the form (see [3])

$$\begin{aligned} x(t_1,z(t_0),u) &= F(t_1,t_0)x(t_0) + \sum_{i=0}^{k}\int_{v_i(t_0)}^{t_0} F(t_1,r_i(\tau))B_i(r_i(\tau))\dot{r}_i(\tau)u_{t_0}(\tau)\,d\tau \\ &\quad + \sum_{i=k+1}^{M}\int_{v_i(t_0)}^{v_i(t_1)} F(t_1,r_i(\tau))B_i(r_i(\tau))\dot{r}_i(\tau)u_{t_0}(\tau)\,d\tau \\ &\quad + \sum_{i=0}^{k}\int_{t_0}^{v_i(t_1)} F(t_1,r_i(\tau))B_i(r_i(\tau))\dot{r}_i(\tau)u(\tau)\,d\tau. \end{aligned}$$

The first three terms on the right hand side depend only on $z(t_0)$, but not on u. To simplify notation we set (see [3])

$$\begin{aligned} (4)\quad q(z(t_0)) &= x(t_0) + \sum_{i=0}^{k}\int_{v_i(t_0)}^{t_0} F(t_0,r_i(\tau))B_i(r_i(\tau))\dot{r}_i(\tau)u_{t_0}(\tau)\,d\tau \\ &\quad + \sum_{i=k+1}^{M}\int_{v_i(t_0)}^{v_i(t_1)} F(t_0,r_i(\tau))B_i(r_i(\tau))\dot{r}_i(\tau)u_{t_0}(\tau)\,d\tau \in \mathbb{R}^M, \end{aligned}$$

and for $t \in [v_{i+1}(t_1), v_i(t_1))$, $i = 0,1,\ldots,k-1$,

$$(5)\qquad B_{t_1}(t) = \sum_{j=0}^{i} F(t_0,r_j(t))B_j(r_j(t))\dot{r}_j(t).$$

LEMMA 1 ([3]). *Let*

$$\dot{y}(t) = A(t)y(t) + B_{t_1}(t)u(t), \qquad t \in [t_0, t_1], \tag{6}$$

be a linear, time-dependent dynamical system without delays in control. Then

$$x(t, z(t_0), u) = y(t, q(z(t_0)), u), \qquad t \in [t_0, t_1].$$

By Lemma 1, the relative controllability in $[t_0, t_1]$ of the dynamical system (1) and the controllability in $[t_0, t_1]$ of the dynamical system (6) without delays in control are equivalent.

The function $J(z(t_0), t, a)$, for $t = t_1$, has the form

$$\begin{aligned} J(z(t_0), t_1, a) &= a^{\mathrm{T}}\mathbf{L}F(t_1, t_0)q(z(t_0)) \\ &\quad + \int_{t_0}^{t_1} \sup\{a^{\mathrm{T}}\mathbf{L}F(t_1, \tau)B_{t_1}(\tau)u(\tau) : u \in L^2([t_0, t_1], U)\}\, d\tau - a^{\mathrm{T}}c. \end{aligned} \tag{7}$$

Here $B_{t_1}(t)$ is a matrix of square integrable functions in $[t_0, t_1]$, so the integral in the above formula is properly defined.

Now we can formulate a criterion of relative controllability for (1).

THEOREM 1. *Let U be a compact set and $E \subset \mathbb{R}^p$ be any set containing* 0 *as an interior point. Then the dynamical system* (1) *with delays in control is relatively U-controllable from the complete state $z(t_0) \in \mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$ into the set S of the form* (3) *if and only if for some $t_1 \in [t_0, \infty)$,*

$$\min\{J(z(t_0), t_1, a) : a \in E\} = 0$$

or, equivalently,

$$J(z(t_0), t_1, a) \geq 0 \qquad \text{for every } a \in E,$$

where $J(z(t_0), t_1, a)$ is defined by (7).

Proof. By Lemma 1 the attainable set $K_U([t_0, t_1], z(t_0))$ for (1) is

$$\begin{aligned} K_U([t_0, t_1], z(t_0)) = \Big\{ x \in \mathbb{R}^M : x &= F(t_1, t_0)q(z(t_0)) \\ &+ \int_{t_0}^{t_1} F(t_1, \tau)B_{t_1}(\tau)u(\tau)\, d\tau,\ u \in L^2([t_0, t_1], U) \Big\}. \end{aligned}$$

This set is convex and compact. Indeed, to prove its compactness, we will show that every sequence of points $x_1(t_1), x_2(t_1), \ldots$ in $K_U([t_0, t_1], z(t_0))$ has a subsequence convergent to some $\overline{x}(t_1) \in K_U([t_0, t_1], z(t_0))$. Since the set $L^2([t_0, t_1], U)$ of admissible controls is weakly compact in $L^2([t_0, t_1], \mathbb{R}^m)$ (see [4, Lemma 1A, p. 169]), there exists a subsequence of controls $u_{k_i} \in$

$L^2([t_0,t_1],U)$ weakly convergent to some control $\overline{u}$ such that

$$\lim_{k_i\to\infty}\int_{t_0}^{t_1} F(t_1,\tau)B_{t_1}(\tau)u_{k_i}(\tau)\,d\tau = \int_{t_0}^{t_1} F(t_1,\tau)B_{t_1}(\tau)\overline{u}(\tau)\,d\tau.$$

Let $\overline{x}(t)$ be the solution corresponding to $\overline{u}(t)$. Then in $[t_0,t_1]$ we have

$$\overline{x}(t) = F(t,t_0)q(z(t_0)) + \int_{t_0}^{t} F(t,\tau)B_{t_1}(\tau)\overline{u}(\tau)\,d\tau = \lim_{k_i\to\infty} x_{k_i}(t).$$

Therefore

$$\lim_{k_i\to\infty} x_{k_i}(t_1) = \overline{x}(t_1) \in K_U([t_0,t_1],z(t_0)).$$

The convexity of $K_U([t_0,t_1],z(t_0))$ is proved in [5], [6].

It follows that the set $\widetilde{K}_U([t_0,t_1],z(t_0))$ of the form

$$\widetilde{K}_U([t_0,t_1],z(t_0)) = \{y\in\mathbb{R}^p : y = \mathbf{L}x,\ x\in K_U([t_0,t_1],z(t_0))\}$$

is also convex and compact. An initial complete state x_0 can be steered to the set S in time $t_1>0$ if and only if the vector c and the set $\widetilde{K}_U([t_0,t_1],z(t_0))$ cannot be strictly separated by a hyperplane, that is, if

$$a^{\mathrm{T}}c \le \sup\{a^{\mathrm{T}}\widetilde{x} : \widetilde{x}\in\widetilde{K}_U([t_0,t_1],z(t_0))\}$$

for all vectors $a\in\mathbb{R}^p$. This follows from a theorem about separating convex sets [2].

The above inequality can be equivalently written as follows:

$$a^{\mathrm{T}}\mathbf{L}F(t_1,t_0)q(z(t_0)) + \sup\Big\{\int_{t_0}^{t_1} a^{\mathrm{T}}\mathbf{L}F(t_1,\tau)B_{t_1}(\tau)u(\tau)\,d\tau : u\in L^2([t_0,t_1],U)\Big\} - a^{\mathrm{T}}c \ge 0.$$

Interchanging integration and taking supremum we conclude that $c\in\widetilde{K}_U([t_0,t_1],z(t_0))$ if and only if $J(z(t_0),t_1,a)\ge 0$ for all $a\in\mathbb{R}^p$.

Moreover, we can show that

$$kJ(z(t_0),t_1,a) = J(z(t_0),t_1,ka) \quad \text{for every } k\ge 0,$$

therefore, restricting to vectors $a\in E$, we obtain the assertion of the theorem.

Corollary 1. *Let $U\subset\mathbb{R}^m$ be a compact set and $E\subset\mathbb{R}^M$ be any set containing* 0 *as an interior point. Then the dynamical system* (1) *is relatively null U-controllable from $z(t_0)\in\mathbb{R}^M\times L^2([v_M(t_0),t_0],U)$ if and only if for some $t_1\in[t_0,\infty)$,*

$$\min\{J(z(t_0),t_1,a) : a\in E\} = 0$$

or, equivalently,

$$J(z(t_0),t_1,a)\ge 0 \quad \textit{for every } a\in E.$$

Proof. This follows directly from Theorem 1 for $S = \{0\}$, i.e. for $\mathbf{L} = \mathbf{I_M}$ and $c = 0$. Then E is a subset of $\mathbb{R}^M$.

COROLLARY 2. *The dynamical system* (1) *is relatively U-controllable in* $[t_0, t_1]$ *from* $z(t_0) \in \mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$ *into the set* S *of the form* (3) *if and only if the dynamical system without delays given by the equation*

$$\dot{x}(t) = A(t)x(t) + B_i(t)u(t), \qquad t \in [t_0, t_1], \tag{8}$$

is U-controllable in $[t_0, t_1]$ *from the initial condition* $q(z(t_0))$ *into* S.

Proof. Direct from Lemma 1 and Theorem 1.

COROLLARY 3. *If* (1) *is relatively U-controllable in* $[t_0, t_1]$ *from* $z(t_0) \in \mathbb{R}^M \in L^2([v_M(t_0), t_0], U)$ *into* S *of the form* (3), *then the dynamical system without delays described by the equation*

$$\dot{x}(t) = A(t)x(t) + \widetilde{B}(t)w(t), \qquad t \in [t_0, t_1], \tag{9}$$

where

$$\widetilde{B}(t) = [B_0(t) \vdots B_1(t) \vdots \ldots \vdots B_M(t)]$$

is U-controllable in $[t_0, t_1]$ *from the initial condition* $q(z(t_0))$ *into* S.

Proof. Let $w(t) = [w_0(t), w_1(t), \ldots\ldots, w_M(t)]^T$. Then the solution of (9) has the form

$$\begin{aligned} J(z(t_0), t_1, a) &= a^{\mathrm{T}} \mathbf{L} F(t_0, t_1) q(z(t_0)) \\ &\quad + \int_{t_0}^{t_1} \sup\{a^{\mathrm{T}} \mathbf{L} F(t_1, \tau) \widetilde{B}(\tau) w(\tau) : u \in L^2([t_0, t_1], U)\}\, d\tau - a^{\mathrm{T}} c \\ &= a^{\mathrm{T}} \mathbf{L} F(t_0, t_1) q(z(t_0)) \\ &\quad + \int_{t_0}^{t_1} \sup \Big\{ a^{\mathrm{T}} \mathbf{L} F(t_1, \tau) \sum_{i=0}^{M} B_i(\tau) w_i(\tau) : u \in L^2([t_0, t_1], U) \Big\}\, d\tau - a^{\mathrm{T}} c. \end{aligned}$$

Taking in particular $w_i(t) = u(v_i(t))$, by assumption and Theorem 1 we obtain the assertion.

COROLLARY 4. *Let* $t_1 < h_M$. *If* (1) *is relatively U-controllable in* $[t_0, t_1]$ *from* $z(t_0) = (x_0, 0)$ *into* S *of the form* (3), *then the dynamical system with delays in control of the form*

$$\dot{x}(t) = A(t)x(t) + \sum_{i-0}^{M-1} B_i(t)u(v_i(t)) + C(t)u(v_M(t)), \qquad t \in [t_0, t_1],$$

where $C(t)$ *is any* $n \times m$ *matrix with elements* $c_{kj} \in L^2_{\mathrm{loc}}([t_0, \infty), \mathbb{R})$, $k = 1, \ldots, n$, $j = 1, \ldots, m$, *is relatively U-controllable in* $[t_0, t_1]$ *from* $z(t_0) = (x_0, 0)$ *into* S.

Proof. The assertion follows directly from Theorem 1 and the form of the matrix $B_{t_1}(t)$. For (1), $B_{t_1}(t)$ is defined by (4) for $t \in [v_{i+1}(t_1), v_i(t_1))$, $i = 0, 1, \ldots, k-1$ and $k = 1, \ldots, M$. If $t_1 < h_M$, the matrix $B_M(\cdot)$ does not occur in the above formula. Therefore, by Theorem 1, with $u_{t_0} \equiv 0$, controllability of (1) does not depend on the form of the matrix at the highest delay.

3.3. *Examples.* The examples below illustrate the mutual dependence between U-controllability of a dynamical system without delays and relative U-controllability of a dynamical system with delays in control with the same matrices $A(t)$ and $B_0(t)$.

EXAMPLE 1. Consider the dynamical system (1) of the form

$$\dot{x}(t) = x(t) + u(t), \qquad t \in [0, 2], \tag{10}$$

and the sets $U = [0, 1]$ and $E = [-1, 1]$.

First, we will test the null U-controllability of this dynamical system from the initial state $x_0 = -1$. Since the dynamical system (10) is stationary,

$$J(x_0, 2, a) = x_0 e^2 a + \int_0^2 \sup\{u(\tau) e^{2-\tau} a : u \in L^2([0, 2], U)\}\, d\tau, \qquad a \in E,$$

and we have

$$J(x_0, 2, a) = \begin{cases} x_0 e^2 a & \text{for } a \in [-1, 0], \\ x_0 e^2 a + a(e^2 - 1) & \text{for } a \in (0, 1]. \end{cases}$$

Therefore, (10) is null U-controllable in $[0, 2]$ if and only if

$$\min\{J(x_0, 2, a) : a \in [-1, 1]\} = 0,$$

that is, from initial states $x_0 \in \mathbb{R}$ satisfying the inequality

$$-1 + e^{-2} < x_0 \leq 0.$$

So, (10) is not null U-controllable from $x_0 = -1$.

We now introduce three delays in (10): $h_1 = 1$, $h_2 = 2$, $h_3 = 3$ and assume $u_{t_0}(s) = 1/2$ for $s \in [-3, 0]$, $t_0 = 0$. We get the differential equation

$$\dot{x}(t) = x(t) + u(t) + u(t-1) + u(t-2) + u(t-3), \qquad t \in [0, 2]. \tag{11}$$

Thus, for the dynamical system (11) with delays in control, in $[0, 2]$ we have $v_0(t) = t$, $v_1(t) = t - 1$, $v_2(t) = t - 2$, $v_3(t) = t - 3$, $M = 3$ and $k = 2$. Since $v_2(2) = 0$, it follows that

$$J(z(t_0), 2, a) = q(z(t_0)) e^2 a + \int_0^2 \sup\{u(\tau) B_2(\tau) e^{2-\tau} a : u \in L^2([0, 2], U)\}\, d\tau, \qquad a \in E,$$

where, according to formula (5),

$$B_{t_1}(t) = \begin{cases} e^{-t} + e^{-t-2} & \text{for } t \in [0,1), \\ e^{-t} & \text{for } t \in [1,2). \end{cases}$$

and $q(z(t_0))$ is calculated from (4). After substitution we get

$$\begin{aligned} J(z(t_0), 2, a) = {} & \big(x_0 + 1 - \tfrac{1}{2}e^{-1} - \tfrac{1}{2}e^{-2}\big)e^2 a \\ & + \int_0^1 \sup\{u(\tau)(e^{-\tau} + e^{-\tau-2})e^{2-\tau}a : u \in L^2([0,2], U)\}\, d\tau \\ & + \int_1^2 \sup\{u(\tau)e^{-\tau}e^{2-\tau}a : u \in L^2([0,2], U)\}\, d\tau, \qquad a \in E. \end{aligned}$$

Therefore, for the system (11) we have

$$\begin{aligned} & J(z(t_0), 2, a) \\ & = \begin{cases} \big(x_0 + 1 - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-2}\big)e^2 a & \text{for } t \in [-1, 0], \\ \big(x_0 + 1 - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-2}\big)e^2 a + \big(\frac{1}{2} - e^{-2} + \frac{1}{2}e^2\big)a & \text{for } t \in (0, 1]. \end{cases} \end{aligned}$$

Hence (11) is null relatively U-controllable in $[0,2]$ from the initial states $x_0 \in \mathbb{R}$ satisfying

$$-\tfrac{3}{2} + \tfrac{1}{2}e^{-1} + e^{-4} \le x_0 \le -1 + \tfrac{1}{2}e^{-1} + \tfrac{1}{2}e^{-2}.$$

In particular, after introducing delays to the system we obtain null relative U-controllability from $x_0 = -1$.

Example 2. Consider the system (10) in $[1,2]$ with $U = [0,1]$ and $E = [-1,1]$. It is easy to calculate that it is null U-controllable from x_0 satisfying

$$-1 + e^{-1} \le x_0 \le 0.$$

It is also clear that the system is not null U-controllable from $x_0 = -1$.

We introduce in (10) three time varying delays in control, getting the system

$$\dot{x}(t) = x(t) + u(t) + u(v_1(t)) + u(v_2(t)) + u(v_3(t)), \qquad t \in [1,2], \tag{12}$$

where $v_1(t) = t$, $v_2(t) = t$, $v_3(t) = t$, that is, $M = 3$ and $k = 2$ (since $v_2(2) = 1$).

So, $F(t_1, t_0) = e$ and

$$B_2(t) = \begin{cases} e^{1-t} + \frac{4}{3}e^{1-4/3t} & \text{for } t \in \big[1, \frac{3}{2}\big), \\ e^{1-t} & \text{for } t \in \big[\frac{3}{2}, 2\big). \end{cases}$$

Taking $u_{t_0} = 0$, we get

$$\begin{aligned}J(z(t_0),2,a)&\\&= x_0ea + \int_1^{3/2} \sup\big\{u(\tau)\big(e^{1-\tau}+\tfrac{4}{3}e^{1-4/3\tau}\big)e^{2-\tau}a : u \in L^2([1,2],U)\big\}\,d\tau\\&\quad + \int_{3/2}^{2} \sup\{u(\tau)e^{1-\tau}e^{2-\tau}a : u \in L^2([1,2],U)\}\,d\tau, \qquad a \in E.\end{aligned}$$

Using Theorem 1 we find that the system (12) is null relatively U-controllable in $[1,2]$ from x_0 satisfying

$$-\tfrac{1}{2} - \tfrac{7}{2}e^{-1/3} + \tfrac{1}{2}e^{-2} + \tfrac{7}{4}e^{-3/2} \le x_0 \le 0.$$

So, as in Example 1, after introducing delays we obtain null relative U-controllability from $x_0 = -1$. Moreover, in this case the dynamical system (12) with delays is also null relatively U-controllable from every initial state from which the system (10) without delays is null U-controllable (because $[-1+e^{-1},0] \subset \big[-\tfrac{1}{2} - \tfrac{7}{2}e^{-1/3} + \tfrac{1}{2}e^{-2} + \tfrac{7}{4}e^{-3/2}, 0\big]$).

Example 3. We keep considering the system (10) of the form

$$\dot{x}(t) = x(t) + u(t), \qquad t \in [0,\infty),$$

with $U = [0,1]$ and $E = [-1,1]$. In a time interval $[0,t_1]$ we have

$$J(x_0,t_1,a) = x_0e^{t_1}a + \int_0^{t_1} \sup\{u(\tau)e^{t_1-\tau}a : u \in L^2([0,t_1],U)\}\,d\tau, \qquad a \in E.$$

We get

$$J(x_0,t_1,a) = \begin{cases} x_0e^{t_1}a & \text{for } a \in [-1,0],\\ x_0e^{t_1}a + a(e^{t_1}-1) & \text{for } a \in (0,1],\end{cases}$$

so (10) is null U-controllable if and only if

$$\min\{J(x_0,t_1,a) : a \in [-1,1]\} = 0,$$

that is, from x_0 satisfying

$$-1 < x_0 \le 0.$$

Therefore, the system is not null U-controllable from $x_0 = -1$ in any time interval $[0,t_1]$, for $t_1 \in [0,\infty)$.

Let us now introduce two delays: $h_1 = 1$ with $B_1 = e^t$ and $h_2 = 2$ with $B_2 = 1$. Moreover, assume that $u_{t_0} \equiv 0$. We get the differential equation

$$\dot{x}(t) = x(t) + u(t) + e^t u(t-1) + u(t-2), \qquad t \in [0,\infty). \tag{13}$$

For the system (13) with delays in control, in $[0,t_1]$ we have

$$J(z(t_0),t_1,a) = x_0 e^{t_1} a + \int_0^{t_1} \sup\{u(\tau)B_{t_1}(\tau)e^{t_1-\tau}a : u \in L^2([0,t_1],U)\}\,d\tau, \qquad a \in E.$$

To find k such that $v_k(t_1) = 0$ we introduce a fictitious delay $h_3 = t_1$ with $B_3 = 0$. We calculate:

$$B_{t_1}(t) = \begin{cases} e^{-t} + 1 + e^{-t-2} & \text{for } t \in [0, t_1 - 2), \\ e^{-t} + 1 & \text{for } t \in [t_1 - 2, t_1 - 1), \\ e^{-t} & \text{for } t \in [t_1 - 1, t_1). \end{cases}$$

Then

$$J(z(t_0),t_1,a) = x_0 e^{t_1} a + \int_0^{t_1-1} \sup\{u(\tau)(e^{-\tau}+1)e^{t_1-\tau}a : u \in L^2([0,t_1],U)\}\,d\tau + \int_{t_1-1}^{t_1} \sup\{u(\tau)e^{-\tau}e^{t_1-\tau}a : u \in L^2([0,t],U)\}\,d\tau, \qquad a \in E.$$

For the system (13) we get

$$J(z(t_0),t_1,a) = \begin{cases} x_0 e^{t_1} a & \text{for } a \in [-1,0], \\ x_0 e^{t_1} a + \left(\frac{3}{2}e^{t_1} - e - \frac{1}{2}e^{-t_1}\right)a & \text{for } a \in (0,1]. \end{cases}$$

Therefore, the system (13) is null relatively U-controllable from $x_0 \in \mathbb{R}$ satisfying

$$-\tfrac{3}{2} < x_0 \le 0.$$

In this way we also obtain the null relative U-controllability from $x_0 = -1$, without diminishing the set of initial states from which one can reach zero as a final state.

Examples 1 and 2 show that after introducing delays in control in a system without delays which is not U-controllable from a given initial state, we can obtain its relative U-controllability from that state. Moreover, Example 2 shows that for properly selected delays the set of initial states may increase.

We can also look for a connection between U-controllability in $[t_0, t_1]$ of the dynamical system without delays of the form

$$\dot{x}(t) = A(t)x(t) + B_0(t)u(t), \tag{14}$$

and relative U-controllability of (1) with the same matrices $A(t)$ and $B_0(t)$. Examples 1 and 3 show that there is no general implication here.

4. Absolute controllability. The relative U-controllability of the dynamical system (1) makes sense in any time interval $[t_0, t_1]$. Assume now that $[t_0, t_1]$ is long, i.e.

$$t_0 < v_M(t_1).$$

Then we can consider absolute controllability, where for given initial conditions the final segment of the system's trajectory should be a given function. This is called *functional controllability.*

4.1. *Definitions.* Basing on the definition of absolute controllability for dynamical systems with delays in control [3] we introduce the notion of absolute U-controllability.

Definition 4. The dynamical system (1) is said to be *absolutely U-controllable in the time interval* $[t_0, t_1]$ *from the complete state* $z(t_0) \in \mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$ *into the set* S if for any function $\widetilde{w} \in L^2([v_M(t_1), t_1], U)$, there exists a control $\widetilde{u} \in L_2([t_0, v_M(t_1)], U)$ such that the corresponding trajectory $x(t, z(t_0), \widetilde{w}, \widetilde{u})$ of (1) satisfies

$$x(t_1, z(t_0), \widetilde{w}, \widetilde{u}) \in S.$$

Definition 5. The dynamical system (1) is said to be (*globally*) *absolutely U-controllable in* $[t_0, t_1]$ *into* S if it is absolutely U-controllable $[t_0, t_1]$ into S for every $z(t_0) \in \mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$.

4.2. *Controllability criterion.* As in the relative controllability case the lemma below allows us to replace studying the absolute controllability of the dynamical system (1) with delays in control by studying the controllability of a certain dynamical system without delays.

Lemma 2 ([3]). *The dynamical system* (1) *is* (*globally*) *absolutely controllable in* $[t_0, t_1]$ *if and only if the dynamical system without delays in control of the form*

$$\dot{x}(t) = A(t)x(t) + \widehat{B}(t)u(t), \tag{15}$$

where

$$\widehat{B}(t) = \sum_{i=0}^{M} F(t, r_i(t))B_i(r_i(t))\dot{r}_i(t), \tag{16}$$

is controllable in $[t_0, v_M(t_1)]$.

As in the relative U-controllability case, we formulate a necessary and sufficient condition for absolute U-controllability of (1).

Theorem 2. *Let U be a compact set and $E \subset \mathbb{R}^p$ be any set containing* 0 *as an interior point. Then the dynamical system* (1) *with delays in control is absolutely U-controllable from the complete state* $z(t_0) \in$

$\mathbb{R}^M \times L^2([v_M(t_0), t_0], U)$ *into the set* S *of the form* (3) *if and only if for some* $t_1 \in [t_0, \infty)$,

$$\min\{J(z(t_0), z(t_1), v_M(t_1), a) : a \in E\} = 0$$

or, *equivalently*,

$$J(z(t_0), z(t_1), v_M(t_1), a) \geq 0 \quad \text{for every } a \in E,$$

where

$$\begin{aligned}(17)\quad & J(z(t_0), z(t_1), v_M(t_1), a)\\ &= v^T \mathbf{L} F(t_1, t_0)\Big[x(t_0) + \sum_{i=0}^{M} \int_{v_i(t_0)}^{t_0} F(t_0, r_i(\tau)) B_i(r_i(\tau)) \dot{r}_i(\tau) u_{t_0}(\tau)\, d\tau\\ &\quad + \sum_{i=0}^{M} \int_{v_M(t_1)}^{v_i(t_1)} F(t_0, r_i(\tau)) B_i(r_i(\tau)) \dot{r}_i(\tau) u_{t_1}(\tau)\, d\tau\Big]\\ &\quad + \int_{t_0}^{v_M(t_1)} \sup\{a^{\mathrm{T}} \mathbf{L} F(t_1, \tau) \widehat{B}(\tau) u(\tau) : u \in L^2([t_0, v_M(t_1)], U)\}\, d\tau - a^{\mathrm{T}} c.\end{aligned}$$

Proof. The proof proceeds analogously to that of Theorem 1, taking into consideration that $K_U([t_0, t_1], z(t_0))$ is the attainable set of the system (15) with constrained control $u(t) \in U$.

4.3. *Example.* The example below shows that relative U-controllability in $[0, t_1]$ does not imply absolute U-controllability in the same interval. By Definitions 1 and 4, the converse is true: absolute U-controllability in $[0, t_1]$ implies relative U-controllability in that interval.

EXAMPLE 4. Consider a dynamical system with two delays in control of the form

$$(18)\qquad \dot{x}(t) = x(t) + u(t) + e^t u(t-1) + u(t-2), \qquad t \in [0, \infty),$$

and the sets $U = [0, 1], E = [-1, 1]$. In Example 3 it has been shown that this system is relatively null U-controllable in $[0, t_1]$ from all initial states $x_0 \in \mathbb{R}$ satisfying

$$-\tfrac{3}{2} < x_0 \leq 0,$$

with $u_{t_0} \equiv 0$. Assume that $t_1 > 2$. It is easy to verify that the dynamical system (18) is relative null U-controllable in $[0, t_1]$. We will study its absolute null U-controllability in $[0, t_1]$.

The function $J(z(t_0), z(t_1), v_M(t_1), a)$ has the following form:

$$
\begin{aligned}
(19)\quad & J(z(t_0), z(t_1), v_M(t_1), a) \\
&= ve^{t_1}\Big[x_0 + \sum_{i=0}^{M} \int\limits_{v_i(0)}^{0} e^{-r_i(\tau)} B_i(r_i(\tau))\dot{r}_i(\tau) u_{t_0}(\tau)\, d\tau \\
&\quad + \sum_{i=0}^{M} \int\limits_{v_M(t_1)}^{v_i(t_1)} e^{-r_i(\tau)} B_i(r_i(\tau))\dot{r}_i(\tau) u_{t_1}(\tau)\, d\tau\Big] \\
&\quad + \int\limits_{0}^{v_M(t_1)} \sup\{u(\tau)e^{t_1-\tau}\widehat{B}(\tau)a : u \in L^2([0, v_M(t_1)], U)\}\, d\tau,
\end{aligned}
$$

where $v_M(t_1) = v_2(t_1) = t_1 - 2$ and

$$\widehat{B}(t) = 1 + e^t + e^{-2}.$$

Taking $u_{t_0}(t) = 0$ and $u_{t_1}(t) = 1/2$, we get

$$
\begin{aligned}
& J(z(t_0), z(t_1), t_1 - 2, a) \\
&= ae^{t_1}\Big[x_0 + \frac{1}{2}\sum_{i=0}^{2} \int\limits_{v_2(t_1)}^{v_i(t_1)} e^{-r_i(\tau)} B_i(r_i(\tau))\dot{r}_i(\tau)\, d\tau\Big] \\
&\quad + \int\limits_{0}^{t_1-2} \sup\{u(\tau)e^{t_1-\tau}(1 + e^{\tau} + e^{-2})a : u \in L^2([0, v_2(t_1)], U)\}\, d\tau.
\end{aligned}
$$

Finally, for $a \in E$, the above equality takes the form

$$
\begin{aligned}
& J(z(t_0), z(t_1), t_1 - 2, a) \\
&= \begin{cases}
ae^{t_1}\big(x_0 + \frac{1}{2}e^{t_1} - \frac{1}{2} + \frac{1}{2}e^{-2}\big) & \text{for } a \in [-1, 0], \\
ae^{t_1}\big(x_0 + \frac{1}{2}e^{t_1} - \frac{1}{2} + \frac{1}{2}e^{-2}\big) & \\
\qquad + a(t_1e^{t_1} - e^{t_1} + e^{t_1-2} - e^{-2} - 1) & \text{for } a \in (0, 1].
\end{cases}
\end{aligned}
$$

By Theorem 2, the dynamical system (18) is not absolutely null U-controllable in $[0, t_1]$ for $t_1 > 2$, because

$$\min\{J(z(t_0), z(t_1), v_M(t_1), a) : a \in E\}$$

does not exist for $t_1 \to \infty$.

5. Concluding remarks. The results obtained in this article extend those in [1], [8] and [9] to systems with delays in control. Controllability results for dynamical systems with delays in state and with constrained controls can be found in [10].

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Silesian University of Technology
Kaszubska 23
44-100 Gliwice, Poland
E-mail: bsikora@zeus.polsl.gliwice.pl

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