BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 54, No. 1, 2006

COMPLEX ANALYSIS

Regularity and Uniqueness of Solutions to Boundary Blow-up Problems for the Complex Monge–Ampère Operator

by

Björn IVARSSON

Presented by Józef SICIAK

Summary. We prove that plurisubharmonic solutions to certain boundary blow-up problems for the complex Monge–Ampère operator are Lipschitz continuous. We also prove that in certain cases these solutions are unique.

1. Introduction. In [3], Cheng and Yau studied the problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)\right) = f(z)e^{Ku(z)}, & z \in \Omega, \\ \lim\limits_{z\to z_0} u(z) = \infty \quad \text{for all } z_0 \in \partial\Omega, \end{cases}$$

where Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, f is a smooth strictly positive function and $K > 0$ a constant. They showed that there is a unique smooth plurisubharmonic solution to this problem. In this paper we study a similar problem, namely

$$(1)\qquad \begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)\right) = f(z, u(z)), & z \in \Omega, \\ \lim\limits_{z\to z_0} u(z) = \infty \quad \text{for all } z_0 \in \partial\Omega, \end{cases}$$

where the right hand side is a function $f \in C^\infty(\overline{\Omega} \times \mathbb{R})$ which is strictly positive, increasing in the second variable and satisfies the following three conditions:

2000 *Mathematics Subject Classification*: 32W20, (32U10, 35B65, 35J60).
Key words and phrases: complex Monge–Ampère operator, interior regularity, plurisubharmonic function, strongly pseudoconvex domain.

A: There exist functions $h \in C^\infty(\overline{\Omega})$ and $f_1 \in C^\infty(\mathbb{R})$ and two strictly positive constants c_1 and c_2 such that

$$\lim_{t\to\infty} \frac{f(z,t)}{f_1(t)} = h(z)$$

uniformly in Ω and $c_1 f_1(t) \le f(z,t) \le c_2 f_1(t)$ for all $(z,t) \in \Omega \times \mathbb{R}$.

B: The function f_1 is strictly positive and increasing.

C: The function

$$\Psi_n(a) = \int_a^\infty ((n+1)F(y))^{-1/(n+1)}\, dy$$

exists for $a > 0$, where $F'(s) = f_1(s)$ and $F(0) = 0$.

Certain aspects of this problem has been studied by the author and Matero in [7].

The following theorem proven by Caffarelli, Kohn, Nirenberg and Spruck in [2] will be useful.

THEOREM 1.1. *Let Ω be a bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let $f \in C^\infty(\overline{\Omega} \times \mathbb{R})$ be a strictly positive function which is increasing in the second variable. Let $\varphi \in C^\infty(\partial\Omega)$. Then the problem*

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) = f(z,u) & \text{in } \Omega, \\ u = \varphi \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

has a unique strictly plurisubharmonic solution u. Moreover, $u \in C^\infty(\overline{\Omega})$.

This result is used to construct solutions to Problem (1). A sequence of plurisubharmonic functions u_N which solve Problem (2) on certain pseudoconvex domains Ω_N is constructed. We construct upper and lower bounds for these solutions and since $\Omega = \bigcup_N \Omega_N$ we can conclude that the sequence u_N converges to a solution for Problem (1) on Ω. This is done in Section 2. In Section 3 the regularity of the solution is studied in some special cases. There it is assumed that the right hand side f depends only on u, that is, $f(z,u) = f(u)$, and also satisfies an extra condition. The extra assumption is used to get a priori estimates for the first derivatives of solutions, which lets us conclude that solutions to Problem (1) are Lipschitz under these assumptions. Finally, in Section 4 uniqueness of solutions is studied. Here the right hand side can depend on the z-variable but we need to make another extra assumption. This extra assumption together with estimates on the boundary behavior of the solution, which were proved in [7], lets us conclude that solutions to Problem (1) are unique.

We will use the notation

$$u_j = \frac{\partial u}{\partial z_j}, \qquad u_{\overline{k}} = \frac{\partial u}{\partial \overline{z}_k}, \qquad u_{j\overline{k}} = \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}.$$

2. Construction of solutions. In order to prove existence of a solution of the problem

$$\begin{cases} \det\,(u_{j\overline{k}}(z)) = f(z, u(z)), \quad z \in \Omega, \\ \lim\limits_{z \to z_0} u(z) = \infty \quad \text{for all } z_0 \in \partial\Omega, \end{cases}$$

we shall begin by constructing approximate solutions. Let $\varrho\colon \Omega \to \mathbb{R}$ be a strictly negative plurisubharmonic function such that $\varrho \in C^\infty(\overline{\Omega})$ and $\lim_{z\to z_0} \varrho(z) = 0$ for all $z_0 \in \partial\Omega$. Take a strictly increasing convex function $g\colon \mathbb{R}^- \to \mathbb{R}$ such that $\lim_{x\to 0-} g(x) = \infty$. Put $\varphi(z) = g(\varrho(z))$. This is a plurisubharmonic function which satisfies $\lim_{z\to z_0} \varphi(z) = \infty$ for all $z_0 \in \partial\Omega$. Let

$$(\varrho^{j\overline{k}}) = (\varrho_{j\overline{k}})^{-1}$$

and

$$\|d\varrho\|_\varrho^2 = \varrho^{j\overline{k}} \varrho_j \varrho_{\overline{k}}.$$

We see that

$$\frac{\partial \varphi}{\partial \overline{z}_k} = \varrho_{\overline{k}} g'(\varrho)$$

and

$$\varphi_{j\overline{k}} = \varrho_{j\overline{k}} g'(\varrho) + \varrho_j \varrho_{\overline{k}} g''(\varrho).$$

Let $M_{j\overline{k}}$ be the minor

$$\det \begin{pmatrix} \varrho_{1\overline{1}} & \cdots & \varrho_{1\overline{(k-1)}} & \varrho_{1\overline{(k+1)}} & \cdots & \varrho_{1\overline{n}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varrho_{(j-1)\overline{1}} & \cdots & \varrho_{(j-1)\overline{(k-1)}} & \varrho_{(j-1)\overline{(k+1)}} & \cdots & \varrho_{(j-1)\overline{n}} \\ \varrho_{(j+1)\overline{1}} & \cdots & \varrho_{(j+1)\overline{(k-1)}} & \varrho_{(j+1)\overline{(k+1)}} & \cdots & \varrho_{(j+1)\overline{n}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varrho_{n\overline{1}} & \cdots & \varrho_{n\overline{(k-1)}} & \varrho_{n\overline{(k+1)}} & \cdots & \varrho_{n\overline{n}} \end{pmatrix}.$$

We see that

$$\begin{aligned} \det\,(\varphi_{j\overline{k}}) &= \det\,(\varrho_{j\overline{k}} g'(\varrho) + \varrho_j \varrho_{\overline{k}} g''(\varrho)) \\ &= g'(\varrho)^n \det\,(\varrho_{j\overline{k}}) + g''(\varrho) g'(\varrho)^{n-1} \sum_{j,k=1}^n M_{jk} \varrho_j \varrho_{\overline{k}} \\ &= (g'(\varrho)^n + \|d\varrho\|_\varrho^2 g''(\varrho) g'(\varrho)^{n-1}) \det\,(\varrho_{j\overline{k}}). \end{aligned}$$

Since

$$g'(\varrho(z)) = 1/(g^{-1})'(\varphi(z))$$

and

$$g''(\varrho(z)) = -(g^{-1})''(\varphi(z))/(g^{-1})'(\varphi(z))^3,$$

this can be rewritten as

$$\det(\varphi_{j\bar{k}}) = \frac{1}{(g^{-1})'(\varphi(z))^n}\det(\varrho_{j\bar{k}}) - \frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}}\,\|d\varrho\|_\varrho^2\det(\varrho_{j\bar{k}}).$$

We shall show that we can choose g so that

$$\frac{1}{(g^{-1})'(\varphi(z))^n} \leq -\frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}}\,\|d\varrho\|_\varrho^2$$

near the boundary. This will show that the last term is the important term.

Hopf's lemma, sometimes also referred to as Zaremba's principle, implies that a plurisubharmonic function $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ which satisfies $u(z) < u(z_0)$ for all $z \in \Omega$ and a boundary point z_0 also satisfies $(\partial u/\partial\nu)(z_0) < 0$ where ν denotes the inward-pointing normal to $\partial\Omega$. A proof of Hopf's lemma can be found in Taylor's book [9]. Since every boundary point is a global maximum for ϱ and $\partial\Omega$ is compact we see that $\|d\varrho\|_\varrho^2 > \varepsilon$, for some $\varepsilon > 0$, near the boundary.

We are interested in solving

$$\begin{cases} \det(u_{j\bar{k}}(z)) = f(z, u(z)), & z \in \Omega, \\ \lim\limits_{z\to z_0} u(z) = \infty \quad \text{for all } z_0 \in \partial\Omega, \end{cases}$$

where f is strictly positive, increasing in the second variable and satisfies conditions $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. We deduce what g should be by solving

$$-\frac{(g^{-1})''(x)}{(g^{-1})'(x)^{n+2}} = f_1(x).$$

Rewriting this we get

$$\frac{d}{dx}\left(\frac{1}{(n+1)(g^{-1})'(x)^{n+1}}\right) = f_1(x).$$

Integrating we see that

$$\frac{1}{(g^{-1})'(x)^{n+1}} = (n+1)F(x).$$

This implies that

$$g^{-1}(x) = \int ((n+1)F(x))^{-1/(n+1)}\,dx.$$

In particular, we can choose $g^{-1}(x) = -\Psi_n(x)$. Making this choice we get

$$(g^{-1})'(x) = ((n+1)F(x))^{-1/(n+1)}.$$

Let us now turn to the question if

$$\frac{1}{(g^{-1})'(\varphi(z))^n} \leq -\frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}} \, \|d\varrho\|_\varrho^2$$

near the boundary. But this is the same as

$$\varepsilon^{-1} \leq -\frac{(g^{-1})''(x)}{(g^{-1})'(x)^2} = \frac{d}{dx}\left(\frac{1}{(g^{-1})'(x)}\right)$$

as x tends to ∞. Here ε is the infimum of $\|d\varrho\|_\varrho^2$ in some neighborhood of the boundary. Assume that

$$\frac{d}{dx}\left(\frac{1}{(g^{-1})'(x)}\right) = \frac{d}{dx}((n+1)F(x))^{1/(n+1)} < \varepsilon^{-1}$$

for large x. We get

$$((n+1)F(x))^{1/(n+1)} < \varepsilon^{-1}x + C$$

for large x but this contradicts the integrability of $((n+1)F(x))^{-1/(n+1)}$. Hence

$$\frac{d}{dx}\left(\frac{1}{(g^{-1})'(x)}\right) \geq \varepsilon^{-1}$$

and we conclude that

$$\frac{1}{(g^{-1})'(\varphi(z))^n} \leq -\frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}} \, \|d\varrho\|_\varrho^2$$

near the boundary.

Having this at our disposal we can construct plurisubharmonic functions which are approximate solutions to the problem we are interested in. Namely, given f and f_1 use the method above to choose g. Take a plurisubharmonic function ϱ which solves

$$\begin{cases} \det\left(\varrho_{j\bar{k}}(z)\right) = 1, & z \in \Omega, \\ \lim\limits_{z\to z_0} \varrho(z) = 0 & \text{for all } z_0 \in \partial\Omega. \end{cases}$$

By Theorem 1.1 we know that $\varrho \in C^\infty(\overline{\Omega})$. It is also strictly plurisubharmonic on $\overline{\Omega}$. Hence $\|d\varrho\|_\varrho^2 \in C^\infty(\overline{\Omega})$. Put $\varphi = g \circ \varrho$. We see that $\lim_{z\to z_0} \varphi(z) = \infty$ for all $z_0 \in \partial\Omega$ and

$$\det\left(\varphi_{j\bar{k}}(z)\right) = \frac{1}{(g^{-1})'(\varphi)^n} - \frac{(g^{-1})''(\varphi)}{(g^{-1})'(\varphi)^{n+2}} \, \|d\varrho\|_\varrho^2 = \kappa(z) f_1(\varphi)$$

where

$$0 < C \leq \kappa(z) = \|d\varrho\|_\varrho^2 - \frac{(g^{-1})'(\varphi)^2}{(g^{-1})''(\varphi)} \leq C'.$$

The existence of the upper bound C' is clear. The lower bound is a little trickier. At points near the boundary we know, by Hopf's lemma, that $\|d\varrho\|_\varrho^2 > \varepsilon$.

At points where $\|d\varrho\|_\varrho^2 \le \varepsilon$ (these points are not close to the boundary), we see that φ is bounded and hence

$$-\frac{(g^{-1})'(\varphi)^2}{(g^{-1})''(\varphi)} \ge \varepsilon'$$

for some $\varepsilon' > 0$. We see that there is a lower bound C so that $0 < C < \kappa(z)$.

Now notice that $\varrho_K = K\varrho$ satisfies

$$\begin{cases} \det\left(\varrho_{K,j\bar{k}}(z)\right) = K^n, & z \in \Omega, \\ \lim\limits_{z\to z_0} \varrho_K(z) = 0 & \text{for all } z_0 \in \partial\Omega. \end{cases}$$

The function $\varphi_K = g \circ \varrho_K$ satisfies $g^{-1}(\varrho_K) = Kg^{-1}(\varrho)$. Therefore

$$\begin{aligned} \det\left(\varphi_{K,j\bar{k}}(z)\right) &= \det\left(\varrho_{K,j\bar{k}}(z)\right)\left(\|d\varrho_K\|_{\varrho_K}^2 - \frac{(g^{-1})'(\varphi_K)^2}{(g^{-1})''(\varphi_K)}\right) f_1(\varphi_K) \\ &= K^n\left(K\|d\varrho\|_\varrho^2 - \frac{(g^{-1})'(\varphi_K)^2}{(g^{-1})''(\varphi_K)}\right) f_1(\varphi_K) \\ &= K^{n+1}\left(\|d\varrho\|_{\varrho_K}^2 - \frac{(g^{-1})'(\varphi)^2}{(g^{-1})''(\varphi)}\right) f_1(\varphi_K) = K^{n+1}\kappa(z) f_1(\varphi_K). \end{aligned}$$

We see that by choosing K and $\widetilde{K}$ suitably we have $\widetilde{K}^{n+1}\kappa \le c_1$ and $c_2 \le K^{n+1}\kappa$. Let $\Omega_N = \{z \in \Omega;\ \varphi_K(z) < N\}$ and u_N be the solution of

$$\begin{cases} \det\left(u_{N,j\bar{k}}(z)\right) = f(z, u_N(z)), & z \in \Omega_N, \\ \lim\limits_{z\to z_0} u_N(z) = N & \text{for all } z_0 \in \partial\Omega_N. \end{cases}$$

which exists by Theorem 1.1. By Lemma 2.2 in [6] we get $\varphi_K \le u_N \le u_{N+1} \le \varphi_{\widetilde{K}}$ on Ω_N. Define $u(z) = \lim_{N\to\infty} u_N(z)$. We now investigate the regularity of u.

3. A priori estimate of first derivatives of solutions. In this section we assume that $f(z, u) = f(u)$ is a function satisfying **B**, **C** and the technical condition

$$\frac{n-1}{n+1} \le \frac{F(x)f'(x)}{f(x)^2}.$$

We shall estimate the norm of the gradient of u_N on compact subsets of Ω. We do this by studying the functions $v_N = |\nabla u_N|^2 (g^{-1})'(u_N)^2$. Notice that $|\nabla \varrho_K|^2 = |\nabla\varphi|^2(g^{-1})'(\varphi)^2 \le C$ and that $v_N = |\nabla u_N|^2(g^{-1})'(u_N)^2 \le |\nabla\varphi|^2(g^{-1})'(\varphi)^2$ on $\partial\Omega_N$ since $u_N = \varphi$ on $\partial\Omega_N$ and $\varphi \le u_N$ in Ω_N. We claim that $\sup(v_N(z);\ z \in \Omega_N) \le \sup(v_N(z);\ z \in \partial\Omega_N) \le C$. We shall show that v_N does not have any interior maximum in Ω_N to establish the claim. This calculation was inspired by Bo Guan's work on the regularity of the pluricomplex Green function [4], [5]. Readers interested in the regularity of the pluricomplex Green function should also consult Błocki's paper [1].

Assume that a local maximum for v_N is attained at $p \in \Omega_N$. We know that $\nabla v_N(p) = 0$. Choose coordinates near p so that $u_{N,jk}(p) = u_{N,\overline{jk}}(p) = 0$ and $u_{N,j\bar{k}}(p) = 0$ if $j \neq k$. It is known that such coordinates can be found if $\nabla u_N(p) \neq 0$, which is the case at a maximum point of v_N. A proof can be extracted from the calculation on page 130 of [8]. Remember that

$$v_N = \sum_{l=1}^{n} u_{N,l} u_{N,\bar{l}} (g^{-1})'(u_N)^2$$

and hence

$$\begin{aligned} v_{N,j} = \sum_{l=1}^{n} &(u_{N,l} u_{N,j\bar{l}} (g^{-1})'(u_N)^2 \\ &+ u_{N,jl} u_{N,\bar{l}} (g^{-1})'(u_N)^2 + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,l} u_{N,\bar{l}} u_{N,j}). \end{aligned}$$

Evaluating this at p yields

$$\begin{aligned} v_{N,j} &= \sum_{l=1}^{n} (u_{N,l} u_{N,j\bar{l}} (g^{-1})'(u_N)^2 + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,l} u_{N,\bar{l}} u_{N,j}) \\ &= u_{N,j} u_{N,j\bar{j}} (g^{-1})'(u_N)^2 + \sum_{l=1}^{n} 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,l} u_{N,\bar{l}} u_{N,j} \\ &= u_{N,j} (g^{-1})'(u_N)^2 \left(u_{N,j\bar{j}} + 2\frac{(g^{-1})''(u_N)}{(g^{-1})'(u_N)} |\nabla u_N|^2 \right) = 0. \end{aligned}$$

At the relevant local maximum point we have $|\nabla u_N| > 0$ and therefore

$$\prod_{j=1}^{n} \left(u_{N,j\bar{j}} + 2\,\frac{(g^{-1})''(u_N)}{(g^{-1})'(u_N)}\, |\nabla u_N|^2 \right) = 0.$$

Thus we have

$$|\nabla u_N|^2 = -\frac{(g^{-1})'(u_N)}{2(g^{-1})''(u_N)}\, u_{N,j\bar{j}}$$

for some j. We see that

$$|\nabla u_N|^2 \leq -\frac{(g^{-1})'(u_N)}{2(g^{-1})''(u_N)} \sum_{j=1}^{n} u_{N,j\bar{j}}$$

with equality if and only if $u_{N,j\bar{j}} = 0$ for all but one j. Hence

$$|\nabla u_N|^2 < -\frac{(g^{-1})'(u_N)}{2(g^{-1})''(u_N)} \sum_{j=1}^{n} u_{N,j\bar{j}}$$

because otherwise $\det(u_{N,j\bar{k}}) = 0$. Remembering that

$$(g^{-1})'(x) = ((n+1)F(x))^{-1/(n+1)}$$

we get

$$|\nabla u_N|^2 < \frac{(n+1)F(u_N)}{2f(u_N)} \sum_{j=1}^{n} u_{N,j\bar{j}}.$$

So far we have only used the fact that p is a critical point. Now we shall use the fact that it is a local maximum point. We have $\log\det(u_{N,j\bar{k}}) = \log f(u)$. Differentiating we see that

$$\frac{\partial}{\partial z_j}\log\det(u_{N,k\bar{l}}) = \sum_{k,l=1}^{n} \frac{M_{k\bar{l}}}{\det(u_{N,k\bar{l}})}\, u_{N,k\bar{l}j} = \sum_{k,l=1}^{n} u_N^{k\bar{l}} u_{N,k\bar{l}j} = \sum_{l=1}^{n} u_N^{l\bar{l}} u_{N,l\bar{l}j}$$

and hence we get the relation

$$\sum_{l=1}^{n} u_N^{l\bar{l}} u_{N,l\bar{l}j} = \frac{f'(u_N)}{f(u_N)}\, u_{N,j}.$$

We also have

$$\sum_{l=1}^{n} u_N^{l\bar{l}} u_{N,l\bar{l}\bar{j}} = \frac{f'(u_N)}{f(u_N)}\, u_{N,\bar{j}}.$$

If we differentiate v_N twice we get

$$\begin{aligned}
v_{N,j\bar{k}} = {} & (g^{-1})'(u_N)^2 \sum_{l=1}^{n} (u_{N,lj\bar{k}} u_{N,\bar{l}} + u_{N,l\bar{k}} u_{N,j\bar{l}} + u_{N,l} u_{N,j\bar{l}\bar{k}}) \\
& + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,\bar{k}} \sum_{l=1}^{n} u_{N,l} u_{N,j\bar{l}} \\
& + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,j} \sum_{l=1}^{n} u_{N,l\bar{k}} u_{N,\bar{l}} \\
& + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,j\bar{k}} \sum_{l=1}^{n} u_{N,l} u_{N,\bar{l}} \\
& + (2(g^{-1})'(u_N)(g^{-1})'''(u_N) + 2(g^{-1})''(u_N)^2) u_{N,j} u_{N,\bar{k}} \sum_{l=1}^{n} u_{N,l} u_{N,\bar{l}}.
\end{aligned}$$

Here we have used the fact that the Hessian of u_N is diagonal to simplify the expression. Since p is assumed to be a local maximum point we know that

$$\sum_{j,k=1}^{n} u_N^{j\bar{k}} v_{N,j\bar{k}} \le 0.$$

Therefore

$$
\begin{aligned}
&\sum_{j,k=1}^{n} u_N^{j\bar{k}} v_{N,j\bar{k}} = \sum_{j=1}^{n} u_N^{j\bar{j}} v_{N,j\bar{j}} \\
&= (g^{-1})'(u_N)^2 \Big(\sum_{j,l=1}^{n} (u_N^{j\bar{j}} u_{N,lj\bar{j}} u_{N,\bar{l}} + u_N^{j\bar{j}} u_{N,l} u_{N,j\bar{l}\bar{j}}) + \sum_{j=1}^{n} u_{N,j\bar{j}} \Big) \\
&\quad + (4+2n)(g^{-1})'(u_N)(g^{-1})''(u_N) \sum_{j=1}^{n} u_{N,\bar{j}} u_{N,j} \\
&\quad + (2(g^{-1})'(u_N)(g^{-1})'''(u_N) + 2(g^{-1})''(u_N)^2) \sum_{j,l=1}^{n} u_N^{j\bar{j}} u_{N,j} u_{N,\bar{j}} u_{N,l} u_{N,\bar{l}} \\
&= 2(g^{-1})'(u_N)^2 \frac{f'(u_N)}{f(u_N)} |\nabla u_N|^2 + (g^{-1})'(u_N)^2 \sum_{j=1}^{n} u_{N,j\bar{j}} \\
&\quad + (4+2n)(g^{-1})'(u_N)(g^{-1})''(u_N)|\nabla u_N|^2 \\
&\quad + (2(g^{-1})'(u_N)(g^{-1})'''(u_N) + 2(g^{-1})''(u_N)^2)|\nabla u_N|^2 \sum_{j=1}^{n} u_N^{j\bar{j}} u_{N,j} u_{N,\bar{j}} \leq 0
\end{aligned}
$$

at p. We need to analyze $\sum_{j=1}^{n} u_N^{j\bar{j}} u_{N,j} u_{N,\bar{j}}$. At p we have

$$
u_{N,j\bar{j}} = -2 \frac{(g^{-1})''(u_N)}{(g^{-1})'(u_N)} |\nabla u_N|^2
$$

if $u_{N,j} \neq 0$. Therefore

$$
\sum_{j=1}^{n} u_N^{j\bar{j}} u_{N,j} u_{N,\bar{j}} = \sum_{j=1}^{n} \frac{u_{N,j} u_{N,\bar{j}}}{u_{N,j\bar{j}}} = -\frac{(g^{-1})'(u_N)}{2(g^{-1})''(u_N)}.
$$

Using this gives the inequality

$$
\sum_{j=1}^{n} u_{N,j\bar{j}} \leq |\nabla u_N|^2 \bigg(\frac{(g^{-1})'''(u_N)}{(g^{-1})''(u_N)} - \frac{2f'(u_N)}{f(u_N)} - \frac{(3+2n)(g^{-1})''(u_N)}{(g^{-1})'(u_N)} \bigg).
$$

We have

$$
\begin{aligned}
(g^{-1})'(x) &= ((n+1)F(x))^{-1/(n+1)}, \\
(g^{-1})''(x) &= -f(x)((n+1)F(x))^{-1-1/(n+1)}, \\
(g^{-1})'''(x) &= -f'(x)((n+1)F(x))^{-1-1/(n+1)} \\
&\quad + (n+2)f(x)^2((n+1)F(x))^{-2-1/(n+1)}.
\end{aligned}
$$

Hence

$$\frac{(g^{-1})'''(u_N)}{(g^{-1})''(u_N)} - \frac{2f'(u_N)}{f(u_N)} - \frac{(3+2n)(g^{-1})''(u_N)}{(g^{-1})'(u_N)}$$
$$= \frac{(g^{-1})'''(u_N)}{(g^{-1})''(u_N)} - \frac{2f'(u_N)}{f(u_N)} + \frac{(3+2n)f(u_N)}{(n+1)F(u_N)} = -\frac{f'(u_N)}{f(u_N)} + \frac{f(u_N)}{F(u_N)}.$$

Combining the two inequalities

$$|\nabla u_N|^2 < \frac{(n+1)F(u_N)}{2f(u_N)} \sum_{j=1}^n u_{N,j\bar{j}}$$

and

$$\sum_{j=1}^n u_{N,j\bar{j}} \le |\nabla u_N|^2 \left(\frac{f(u_N)}{F(u_N)} - \frac{f'(u_N)}{f(u_N)} \right)$$

yields

$$|\nabla u_N|^2 < \frac{n+1}{2} \frac{F(u_N)}{f(u_N)} \sum_{j=1}^n u_{N,j\bar{j}} \le \frac{n+1}{2} \left(1 - \frac{F(u_N)f'(u_N)}{f(u_N)^2} \right) |\nabla u_N|^2,$$

which gives a contradiction if

$$\frac{n+1}{2} \left(1 - \frac{F(u_N)f'(u_N)}{f(u_N)^2} \right) \le 1.$$

We see that, on the assumption

$$\frac{n-1}{n+1} \le \frac{F(u_N)f'(u_N)}{f(u_N)^2},$$

the function $|\nabla u_N|^2 (g^{-1})'(u_N)^2$ attains its maximum on the boundary and hence we have

$$|\nabla u_N|^2 (g^{-1})'(u_N)^2 \le C$$

on Ω_N. Since any compact set $K \subseteq \Omega$ is contained in Ω_N for sufficiently large N we have proven that

$$\sup(|\nabla u_N(z)|^2 (g^{-1})'(u_N(z));\, z \in K) < C$$

for all N large enough. Hence

$$|\nabla u_N(z)|^2 \le C g'(u_N(z))^2$$

in K and since $u_N(z) \le \varphi(z) \le C$ in K we see that $\|u_N\|_{C^1(K)} \le C$. Since the sequence of u_N's converges uniformly on compacts we can conclude that u is Lipschitz. We state this in a theorem.

THEOREM 3.1. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Suppose that f satisfies* **B**, **C** *and*

$$\frac{n-1}{n+1} \le \frac{f'(x)F(x)}{f(x)^2}.$$

Then the problem

$$\begin{cases} (dd^c u)^n = f(u(z)), & z \in \Omega \\ \lim\limits_{z \to z_0} u(z) = \infty & \textit{for all } z_0 \in \partial\Omega, \end{cases}$$

has a solution u *that is Lipschitz.*

Remark 3.2. Note that $f(u) = e^{Ku}$, $K > 0$, and $f(u) = u^\gamma$ (suitably modified for $u < 1$) where $\gamma \geq (n-1)/2$, satisfies all the conditions in the theorem.

4. Uniqueness. We shall now establish a uniqueness result. Uniqueness for boundary blow-up problems is not as straightforward as for the Dirichlet problem. This is because the comparison principles in [6] and [7] are not formulated with the situation in mind where *both* plurisubharmonic functions tend to ∞ as we approach the boundary.

We need the following definition and theorem from [7].

Definition 4.1. Assume that $\Omega = \{z \in \mathbb{C}^n;\ \varrho(z) < 0\}$ where $\varrho \in C^\infty(\overline{\Omega})$. For $z_0 \in \partial\Omega$ suppose that $|\nabla\varrho(z_0)| = 1$. Let $\Pi(z_0)$ be the product of the eigenvalues of the form

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_j \partial \overline{z}_k}(z_0)\, dz_j \wedge d\overline{z}_k$$

restricted to the vector space $\{w \in \mathbb{C}^n;\ \sum_{j=1}^n \frac{\partial\varrho}{\partial z_j}(z_0) w_j = 0\}$.

Theorem 4.2. *Let* Ω *be a bounded, strongly pseudoconvex domain in* $\mathbb{C}^n$ *with smooth boundary. Let* $f \in C^\infty(\overline{\Omega} \times \mathbb{R})$ *be a strictly positive function which is increasing in the second variable and satisfies assumptions* **A**, **B** *and* **C**. *For boundary points* $z_0 \in \partial\Omega$ *let* $\Pi(z_0)$ *be the number described in Definition* 4.1. *Then any solution* u *to Problem* (1) *satisfies*

$$\lim_{z \to z_0} \frac{\Psi_n(u(z))}{d_\Omega(z)} = 4^{1/(n+1)} h(z_0)^{1/(n+1)} \Pi(z_0)^{-1/(n+1)}$$

for any $z_0 \in \partial\Omega$.

We can now prove the following proposition.

Proposition 4.3. *Let* Ω *be a bounded strongly pseudoconvex domain with smooth boundary and assume that* $f \in C^\infty(\overline{\Omega} \times \mathbb{R})$ *is a strictly positive function, increasing in the second variable and satisfying* **A**, **B** *and* **C**. *Assume also that*

$$\Psi_n(t)/\Psi_n'(t)$$

is bounded for large t. *If* u *and* v *are plurisubharmonic solutions of Problem* (1) *then* $u \equiv v$.

REMARK 4.4. The assumption that $\Psi_n(t)/\Psi_n'(t)$ is bounded for large t is fulfilled when f_1 has exponential growth but not when it has only polynomial growth.

Proof of Proposition 4.3. Assume that we have two distinct plurisubharmonic solutions u and v of Problem (1). Assume for the moment that we know that $\lim_{z\to z_0}(u(z)-v(z))=0$ for all $z_0\in\partial\Omega$. We shall return to this claim later to finish the proof. Assume that $\sup(u(z)-v(z);\, z\in\Omega)=K>0$. Then there is a $p\in\Omega$ such that $u(p)-v(p)=K$. At p we have $\det(u_{j\bar{k}}(p))\le\det(v_{j\bar{k}}(p))$. However, since $u(p)>v(p)$ we see that

$$\det(u_{j\bar{k}}(p))=f(p,u(p))>f(p,v(p))=\det(v_{j\bar{k}}(p)),$$

which is a contradiction. Hence $u(z)-v(z)\le 0$ in Ω. Arguing in the same way we also see that $v(z)-u(z)\le 0$ in Ω. This proves uniqueness.

It remains to prove our claim that $\lim_{z\to z_0}(u(z)-v(z))=0$. We know that for all $z_0\in\partial\Omega$ we have

$$\lim_{z\to z_0}\frac{\Psi_n(u(z))}{d_\Omega(z)}=\lim_{z\to z_0}\frac{\Psi_n(v(z))}{d_\Omega(z)}=C(z_0)$$

where $C(z_0)$ is the constant given in Theorem 4.2. Given $\varepsilon>0$, for z close to z_0 we have

$$(C(z_0)-\varepsilon)d_\Omega(z)\le\Psi_n(u(z))\le(C(z_0)+\varepsilon)d_\Omega(z)$$

and

$$(C(z_0)-\varepsilon)d_\Omega(z)\le\Psi_n(v(z))\le(C(z_0)+\varepsilon)d_\Omega(z).$$

This gives

$$\Psi_n^{-1}((C(z_0)+\varepsilon)d_\Omega(z))\le u(z)\le\Psi_n^{-1}((C(z_0)-\varepsilon)d_\Omega(z))$$

and

$$-\Psi_n^{-1}((C(z_0)-\varepsilon)d_\Omega(z))\le -v(z)\le -\Psi_n^{-1}((C(z_0)+\varepsilon)d_\Omega(z)).$$

We get

$$\begin{aligned}u(z)-v(z)&\le\Psi_n^{-1}((C(z_0)-\varepsilon)d_\Omega(z))-\Psi_n^{-1}((C(z_0)+\varepsilon)d_\Omega(z))\\&=-2\varepsilon d_\Omega(z)(\Psi_n^{-1})'(\eta(z))\end{aligned}$$

for some $\eta(z)\in[(C(z_0)-\varepsilon)d_\Omega(z),(C(z_0)+\varepsilon)d_\Omega(z)]$ by the mean-value theorem. Hence

$$\begin{aligned}u(z)-v(z)&\le-2\varepsilon d_\Omega(z)(\Psi_n^{-1})'(\eta(z))=-\frac{2\varepsilon d_\Omega(z)}{\eta(z)}\,\eta(z)\,\frac{1}{\Psi_n'(\Psi_n^{-1}(\eta(z)))}\\&=-\frac{2\varepsilon d_\Omega(z)}{\eta(z)}\,\frac{\Psi_n(\Psi_n^{-1}(\eta(z)))}{\Psi_n'(\Psi_n^{-1}(\eta(z)))}\le-\frac{2\varepsilon}{C(z_0)-\varepsilon}\,\frac{\Psi_n(\Psi_n^{-1}(\eta(z)))}{\Psi_n'(\Psi_n^{-1}(\eta(z)))}.\end{aligned}$$

The assumption that $\Psi_n(t)/\Psi_n'(t)$ is bounded for large t lets us conclude that, since ε is arbitrary, $\lim_{z\to z_0}(u(z)-v(z))=0$. ∎

Acknowledgements. I would like to thank Tobias Ekholm, Christer Kiselman and Jerk Matero for their help during the writing of this paper. Also I would like to thank Frank Kutzschebauch for making it possible for me to visit the University of Bern.

References

[1] Z. Błocki, *The $C^{1,1}$ regularity of the pluricomplex Green function*, Michigan Math. J. 47 (2000), 211–215.

[2] L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, II. Complex Monge–Ampère, and uniformly elliptic, equations*, Comm. Pure Appl. Math. 38 (1985), 209–252.

[3] S.-Y. Cheng and S.-T. Yau, *On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math. 33 (1980), 507–544.

[4] B. Guan, *The Dirichlet problem for complex Monge–Ampère equations and regularity of the pluri-complex Green function*, Comm. Anal. Geom. 6 (1998), 687–703.

[5] —, *A correction to "The Dirichlet problem for complex Monge–Ampère equations and regularity of the pluri-complex Green function"*, ibid. 8 (2000), 213–218.

[6] B. Ivarsson, *Interior regularity of solutions to a complex Monge–Ampère equation*, Ark. Mat. 40 (2002), 275–300.

[7] B. Ivarsson and J. Matero, *The blow-up rate of solutions to boundary blow-up problems for the complex Monge–Ampère operator*, submitted, 2005, 18 pp.

[8] S. Krantz, *Function Theory of Several Complex Variables*, Wiley, 1982.

[9] M. Taylor, *Partial Differential Equations: Basic Theory*, Springer, 1996.

Björn Ivarsson
Mathematisches Institute
Universität Bern
Sidlerstrasse 5
CH-3012 Bern, Switzerland
E-mail: bjoern.ivarsson@math.unibe.ch

Received November 30, 2005;
received in final form June 1, 2006 (7491)