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Fat P-sets in the Space ω^*

by

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Summary. We prove that—consistently—in the space ω^* there are no P-sets with the $\mathfrak{c}$-cc and any two fat P-sets with the $\mathfrak{c}^+$-cc are coabsolute.

1. Introduction. A closed subset $A \subseteq X$ of a topological space X is said to be a *P-set* if the inclusion $A \subseteq \operatorname{int} \bigcap R$ holds for any countable family R of open neighborhoods of A. A point $x \in X$ is a *P-point* if the one-element set $\{x\}$ is a P-set. Here, we consider the case of $X = \omega^* = \beta[\omega] \backslash \omega$, the remainder of the Stone–Čech compactification of ω (= the nonnegative integers with the discrete topology). Hence, we may assume that the family R mentioned above consists of open-closed neighborhoods of A. The existence of P-points follows e.g. from the continuum hypothesis (see [6]), but it is not provable in set theory. In fact, Shelah constructed a model in which $\mathfrak{c} = \omega_2$ and there are no P-points (see [7]).

Let κ be a cardinal. We say that a set A *has the κ-cc* (the κ (anti-)chain condition) if every disjoint family of relatively open subsets of A has power less than κ. In [3] a construction is presented of a model in which $\mathfrak{c} = \omega_2$ and there are no P-sets with the ω_1-cc in the remainder ω^* (see also [4] for all the references). In this paper we strengthen this theorem by describing a model in which $\mathfrak{c} = \omega_2$ and there are no P-sets with the $\mathfrak{c}$-cc. Moreover, in the constructed model any two fat (see below for the definition) P-sets with the $\mathfrak{c}^+$-cc are alike in the sense that they are coabsolute (i.e. their Gleason spaces are homeomorphic).

The case of non-fat sets requires a different forcing and will be considered elsewhere.

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2. Basic facts and definitions. If A, B are subsets of ω then $A \subseteq_* B$ means $A \setminus B$ is finite. Closed subsets of ω^* can be identified with filters on ω. Indeed, any closed set in ω^* can be written as

$$\bigcap\{\overline{A} \setminus \omega : A \in F\}$$

where F is a filter on ω ($\overline{A}$ is the closure of $A \subseteq \omega$ in $\beta[\omega]$), and vice versa. P-filters (filters corresponding to P-sets) are then characterized by the following condition:

For every family $\{A_i : i \in \omega\} \subseteq F$ there is an $A \in F$ such that $A \subseteq_* A_i$ for all $i \in \omega$.

Equivalently, P-ideals I (ideals dual to P-filters) have the following property:

For every $\{B_i : i \in \omega\} \subseteq I$ there is a $B \in I$ such that $B_i \subseteq_* B$ for all $i \in \omega$.

An ideal I (and its dual filter F) is called *fat* if it satisfies the following condition:

If $\{B_i : i \in \omega\} \subseteq I$ is such that $\lim_i \min B_i = \infty$, then there is an infinite $Z \subseteq \omega$ such that $\bigcup_{i \in Z} B_i \in I$.

In [5] it is proved that every P-set having the $\mathfrak{c}$-cc is fat.

For the rest of this section let us fix a fat P-filter F and its dual I.

The filter F determines a forcing $\mathbb{P} = \mathbb{P}(F)$ in the following way: Let $\mathcal{T}$ consist of all trees $t = (t, \leq_t)$, which are suborderings of $(\omega, \leq)$ whose domains t are infinite elements of I. Fix a tree t_0 order isomorphic to the Cantor tree $\mathcal{B}$ (i.e. a full binary tree of height ω). Moreover we can assume that $\min t_0 = 0$. So, $\mathcal{B}$ can be treated as the set of all finite zero-one sequences. Moreover we can assume that if $h : \{0,1\}^{<\omega} \to (t_0, \leq_{t_0})$ is the required isomorphism then the following condition holds:

If x_1, x_2 are different elements of $\mathcal{B}$, $\operatorname{length}(x_1 \cap x_2) = l$ *and $x_1(l) < x_2(l)$ then $h(x_1|_{\{0,1,\ldots,l\}}) < h(x_2|_{\{0,1,\ldots,l\}})$.*

We define the relation which partially orders the family $\mathcal{T}$ as follows: $(t, \leq_t) \leq (s, \leq_s)$ if and only if $(s, \leq_s)$ is a subordering of $(t, \leq_t)$ and each branch of $(t, \leq_t)$ contains cofinally a (unique) branch of $(s, \leq_s)$.

For any $t \in \mathcal{T}$ and $n \in \omega$ let $t^{(n)} = t \setminus \{0, \ldots, n\}$ denote the subtree of t obtained from t by deleting the elements $0, \ldots, n$.

Now define $\mathbb{P} = \mathbb{P}(F)$ as the set of all pairs $p = \langle t^p, f^p \rangle$, where $t^p \in \mathcal{T}$ and $t^p \leq t_0^{(n)}$ for some n and a tree t_0, and $f^p : t^p \to \{0,1\}$. The ordering on $\mathbb{P}$ is defined as follows:

$$p \leq q \equiv t^p \leq t^q \text{ and } f^p \supseteq f^q.$$

Thus, the sets of branching points of the trees are exactly the same. (In fact, these points are precisely the branching points of the fixed tree t_0.) It follows that each branch of the tree t^p is uniquely coded by a branch of t_0 (and $\mathcal{B}$).

To see how $\mathbb{P}$ works, suppose that $G \subseteq \mathbb{P}$ is a generic filter. For any branch b of t_0 in V let b^p be the branch of t^p which almost contains b cofinally; put $t^G = \bigcup_{p\in G} t^p$, $f^G = \bigcup_{p\in G} f^p$ and $b^G = \bigcup_{p\in G} b^p$. Clearly, t^G is a tree (but $t^G \notin I$), $f^G : t^G \to \{0,1\}$ and b^G is a branch of t^G extending b. Define

$$X_b = \{i \in \omega : i \in b^G \text{ and } f^G(i) = 1\}.$$

By assumption, the sets t^p belong to I and therefore the set $(\omega \setminus t^p) \cap A$ is infinite, for each $A \in F$ and $p \in \mathbb{P}$. It follows that the sets

$$E^{A,b}_{n,\varepsilon} = \{p \in \mathbb{P} : \exists i \geq n\, [i \in A \cap b^p \text{ and } f^p(i) = \varepsilon]\}$$

are dense for each $A \in F$, $n \in \omega$, $\varepsilon = 0, 1$ and any branch b of t_0 (in V). Hence, X_b intersects infinitely each set $A \in F$, and the complement $b^G \setminus X_b$ has the same property. In [3], it is proved that the countable product $\mathbb{P}^\omega(F) = \mathbb{P}(F) \times \mathbb{P}(F) \times \cdots$ is always ω-proper and ω^ω-*bounding* (i.e. each function $f : \omega \to \omega$ from $V[G]$ is majorized by a function $g : \omega \to \omega$ from V). Thus ω_1 is not collapsed and since distinct branches b^G are almost disjoint we may say that the forcing $\mathbb{P}$ adds uncountably many almost disjoint Gregorieff-like sets X_b. In particular, the Suslin number of the set $\bigcap\{\overline{A} \setminus \omega : A \in F\}$ determined by F will be uncountable in $V[G]$.

A similar notion of forcing is used in [3] to construct a model in which there are no closed ccc P-sets (in ω^*). Since ω_1 is not collapsed, to obtain an uncountable family of relatively open subsets of $\bigcap F$ it is enough to add a new element X_b for every branch b^V of $\mathcal{B}^V$ (i.e. every branch which belongs to the ground model). Since, starting with $\mathfrak{c} = \omega_1$, we iterate the forcing $\mathbb{P}^\omega(F)$ ω_2 times, in the resulting model we have $\mathfrak{c} = \omega_2$. So to ensure that each closed P-set is $\mathfrak{c}$-cc we have to add new elements of the type X_b for ω_2 (new) branches of $\mathcal{B}$.

Note that since the set t^G does not belong to the ground model its characteristic function is a new branch in the binary tree $\mathcal{B}$.

3. Construction of the model. Let us fix a ground model V in which $\mathfrak{c} = \omega_1$, $2^{\omega_1} = \omega_2$ and the diamond principle holds:

> *There is a sequence $\langle T_\alpha : \alpha < \omega_2$ and $\mathrm{cf}(\alpha) = \omega_1\rangle$ such that for every $Y \subseteq H(\omega_2)$ the set $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1$ and $Y \cap H_\alpha = T_\alpha\}$ is stationary.*

Here, $H(\omega_2)$ denotes the family of all sets of hereditary power less than ω_2; $H(\omega_2) = \bigcup_{\alpha<\omega_2} H_\alpha$ is a standard decomposition into a continuously increasing chain with $\mathrm{card}(H_\alpha) = \omega_1$. (As V we can take for example the constructible universe.)

Define by induction a countable support iteration $\langle \mathbb{P}_\alpha : \alpha < \omega_2 \rangle$ as follows:

- $\mathbb{P}_\alpha = \mathrm{Lim}_{\beta<\alpha} \mathbb{P}_\beta$ (limit with countable supports) for limit α,
- $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{P}^\omega(T_\alpha)$ if $\mathrm{cf}(\alpha) = \omega_1$ and $\mathbb{P}_\alpha \Vdash$ "T_α is a fat P-filter",
- $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$ in all remaining cases ($\mathbb{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name of an arbitrary forcing whose countable support iteration is proper and ω^ω-bounding).

We can assume that $\mathbb{P}_\alpha \subseteq H(\omega_2)$. In view of [7, Ch. V, Theorem 4.3] the forcing $\mathbb{P}_{\omega_2}$ is proper and ω^ω-bounding. Let G be a generic filter.

4. Main theorem. The model constructed in the previous section satisfies the following.

THEOREM 1. *In the model $V[G]$ there are no P-sets with the $\mathfrak{c}$-cc.*

Proof. Suppose the opposite and derive a contradiction. Thus, there is a fat P-filter $F \in V[G]$. Clearly, we have

$$V[G] \models \text{“}\mathfrak{c} = (\omega_2)^V\text{”} \quad \text{and} \quad V[G|\alpha] \models \text{“}\mathfrak{c} = \omega_1\text{”} \quad \text{for all } \alpha < \omega_2.$$

Set $F_\alpha = F \cap V[G|\alpha]$. We claim that

The set $C_1 = \{\alpha < \omega_2 : F_\alpha \in V[G|\alpha]\}$ is an ω_1-cub

(i.e. unbounded and closed under limits of length ω_1). Choose a canonical name $\underline{F} \subseteq H(\omega_2)$ of F consisting of some pairs $\langle x, p\rangle$ (in fact, *canonical names* for pairs, which are elements of $H(\omega_2)$), where x is a $\mathbb{P}_{\omega_2}$-name of a subset of ω and $p \in \mathbb{P}_{\omega_2}$. Denote by $\underline{F}(x)$ the set $\{p : \langle x, p\rangle \in \underline{F}\}$). Since $\mathrm{card}\, \underline{F}(x)$ is at most ω_1, the set

$$C_2 = \{\alpha < \omega_2 : \forall x\, [x \in V^{(\mathbb{P}_\alpha)} \to \underline{F}(x) \subseteq \mathbb{P}_\alpha]\}$$

is easily seen to be an ω_1-cub. Plainly, the set $\underline{F}_\alpha = \underline{F} \cap (V^{(\mathbb{P}_\alpha)} \times \mathbb{P}_\alpha)$ is a $\mathbb{P}_\alpha$-name and we have

$$\underline{F}_\alpha[G|\alpha] = \underline{F}[G] \cap V[G|\alpha] = F_\alpha,$$

whence $F_\alpha \in V[G|\alpha]$ for all $\alpha \in C_2$.

In addition we may assert that the F_α's are fat P-filters, because they are so on an ω_1-cub subset of C_1. Indeed, if $R = \{A_i : i \in \omega\} \subseteq F_\alpha$, then there is an $A \in F$ such that $A \subseteq_* A_i$ for all $i \in \omega$. There is a $\beta < \omega_2$ such that $A \in V[G|\beta]$ (cf. [7, Ch. V, 4.4]. Since there are at most $\mathfrak{c} = \omega_1$ countable subfamilies R of F_α in $V[G|\alpha]$, there must be an ordinal $\alpha^* \in C_1$ such that in $V[G|\alpha^*]$ we can find some lower bounds $A \in F$ for all such R's. Now define inductively $\alpha_0 = \alpha$, $\alpha_{\xi+1} = \alpha^*_\xi$ and $\alpha_\lambda = \sup_{\xi<\lambda} \alpha_\xi$ for limit λ. If $\beta = \sup_{\xi<\omega_1} \alpha_\xi$, then $\beta \in C_1$ and $F_\beta \in V[G|\beta]$ is a P-filter. Obviously, the set

$$C_1' = \{\beta < \omega_2 : F_\beta \text{ is a } P\text{-filter}\}$$

is ω_1-closed, which proves that C_1' is an ω_1-cub.

In a similar way we show that F_β is fat on an ω_1-cub subset $\subseteq C_1$, which proves the claim.

A standard reasoning also shows that the set

$$C_3 = \{\alpha < \omega_2 : \underline{F} \cap H_\alpha = \underline{F}_\alpha\}$$

is an ω_1-cub. Hence, $C = C_1 \cap C_3$ is also an ω_1-cub. By the diamond principle applied to $Y = \underline{F} \subseteq H(\omega_2)$ there is an $\alpha < \omega_2$, with $\mathrm{cf}(\alpha) = \omega_1$ such that T_α is a $\mathbb{P}_\alpha$-name of the fat P-filter F_α. By the definition of the iteration we force with $\mathbb{P}^\omega(F_\alpha)$ at stage α, i.e. $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * P^\omega(\underline{F}_\alpha)$.

To finish the proof fix a $p \in \mathbb{P}_{\omega_2}$. Then the αth term $p(\alpha)$ can be written as $p(\alpha) = \langle p_n(\alpha) : n \in \omega\rangle$. Let b be a branch in the binary tree $\mathcal{B}$ (in $V[G|\alpha]$) and denote by $b^G_{n,\alpha}$ the corresponding branch in $t^G_{n,\alpha} = \bigcup_{p\in G} t^p_{n,\alpha}$, where $p|_\alpha \Vdash$ "$p(\alpha) = \langle t^p_{n,\alpha}, f^p_{n,\alpha}\rangle$". Define

$$X^\alpha_{n,\alpha}(b) = \{k \in b^G_{n,\alpha} : f^G_{n,\alpha}(k) = 1\}.$$

Note that for any $\xi > \alpha$ with $\mathrm{cf}(\xi) = \omega_1$, $t^G_{m,\xi} = \bigcup_{p\in G} t^p_{m,\xi}$ determines a new branch in the binary tree and thus a new branch $b^{m,\xi}_{n,\alpha}$ in the tree $t^G_{n,\alpha}$. Indeed, we can identify the set $t^G_{n,\xi}$ with its characterictic function. Such a function is just a (new) branch in the binary tree, say b. Let $b^{m,\xi}_{n,\alpha}$ be the corresponding branch of the tree $t^G_{n,\alpha}$. (Of course, this branch does not belong to the model $V[G|\alpha]$.) Define

$$X^{m,\xi}_{n,\alpha} = \{k \in b^{m,\xi}_{n,\alpha} : f^G_{m,\alpha}(k) = 1\}.$$

(Note that the sets $X^{m,\xi}_{n,\alpha}$ for $\alpha \le \xi$ and $m \in \omega$ form a family of $\mathfrak{c}$ almost disjoint Gregorieff-like sets centered with F_α.)

Actually, what we will use is

LEMMA 1. *Assume that $\langle \xi_i : i \in \omega\rangle \in V$ is a sequence of ordinals greater then or equal to α, of cofinality ω_1. Let $\langle n_i : i \in \omega\rangle \in V[G|\alpha]$ and $\langle m_i : i \in \omega\rangle \in V[G|\sup \xi_i]$ be sequences of natural numbers* (*the first one is injective*). *Suppose that a sequence $\langle b_i : i \in \omega\rangle$ of branches of the binary tree $\mathcal{B}$ and a function $g : \omega \to \omega$ belong to $V[G|\alpha]$. Denote by X_i the set $X^\alpha_{n_i,\alpha}(b_i)$ if $\xi_i = \alpha$, and $X^{m_i,\xi}_{n_i,\alpha}$ otherwise. Then the set*

$$\bigcap_{i\in\omega} [(\omega \setminus X_i) \cup [0, g(i))] \text{ is in the ideal } I_\alpha \text{ (dual to } F_\alpha).$$

Proof. The proof consists of two cases:

CASE 1. Suppose that $\xi_i = \alpha$ for all $i \in \omega$. It is enough to show that the assertion of the lemma holds for V and the forcing $\mathbb{P}^\omega(F_\alpha)$. So suppose that

F is a fat P-filter in V. For a given $p = \langle (t_n, f_n) : n \in \omega \rangle$ there is a set B and integers $k_i \in \omega$ such that

$$t_{n_i} \setminus [0, k_i) \subseteq B \quad \text{ for all } i \in \omega.$$

We may assume that $g(i) < k_i < k_{i+1}$ and $y_i = [k_i, k_{i+1}) \setminus B \neq \emptyset$ for all i. Extend each t_{n_i} by adding to the branch b_i all elements of y_i and put $f_{n_i}(k) = 1$ for $k \in y_i$. Then q obtained in this way forces that

$$[(\omega \setminus X_i) \cup [0, g(i))] \cap y_i = \emptyset,$$

and hence

$$\bigcap_{i \in \omega} [(\omega \setminus X_i) \cup [0, g(i))] \cap \bigcup_{i \in \omega} y_i = \emptyset.$$

But $\bigcup_{i \in \omega} y_i = [k_{m_0}, \infty) \setminus B$ and thus

$$q \Vdash \text{“} \bigcap_{i \in \omega} (\omega \setminus X_i) \cup [0, g(i)) \subseteq_* B\text{”},$$

which proves the first case.

Case 2. Suppose that there are $\xi_i > \alpha$. Without losing generality we can treat $V[G|\alpha]$ as a ground model. Define y_i as above. Then first of all, for all $i \in \{j : \xi_j = \alpha\}$, extend conditions (t_{n_i}, f_{n_i}) by adding new elements to the trees $t_{n_i,\alpha}$, in the way described above.

Let $\xi_i > \alpha$. By definition, $t_{n_i} \le t_0^{(m)}$ for some $m \in \omega$. (In fact, $t_{n_i} \le t_0^{(m)}(\xi_i, n_i)$, since the choice of the tree t_0 depends on an ideal $I(\xi_i, n_i)$ considered at stage ξ_i and for the index n_i. We can assume that if $I(\xi_1, n_1) \subseteq I(\xi_2, n_2)$ for some $\xi_1 < \xi_2$ or $\xi_1 = \xi_2$ and $n_1 < n_2$, then $t_0(\xi_1, n_1) \subseteq t_0(\xi_2, n_2)$.) We choose a finite sequence of pairwise comparable conditions $p = p_0 \ge p_1 \ge \cdots \ge p_k$ which forces all properties listed below. To simplify notation we will write h_i, x_l, y_i etc. instead of their canonical names.

For a fixed isomorphism $h_i : \mathcal{B} \to t_0$ denote by c_l the images $h_i(x_l)$, where x_l is a sequence of length l and range $\{1\}$. Put

$$c_i = c_{l_0^i}, \quad \text{where} \quad l_0^i = \{l : c_l \in t_{n_i}\}.$$

Thus c_i is the least branching point of t_{n_i} which belongs to the branch $u_i = \bigcup_{l \in \omega} h_i(x_l)$.

We extend t_{n_i} as follows: For all $r \in y_i$ and $r < c_i$ we put $r \le_{t_{n_i}} c$. There exists the least $l_i \in \omega$ such that

$$y_i \setminus \{0, 1, \ldots, c_i\} \subset [c_i, c_{l_i}) \cap \omega.$$

Thus we add all elements of y_i to the branch corresponding to u_i and put $f_{n_i}(r) = 1$ for each of them. Moreover extend the condition $p(\xi_i)(m_i) = (t_{m_i,\xi_i}, f_{m_i,\xi_i})$ in such a way that

$$p|_{\xi_i} \Vdash \text{“}\{0, 1, \ldots, l_i\} \subset t_{m_i,\xi_i}.\text{”}$$

The condition p_k obtained in this way forces the same as the q defined in Case 1. So Lemma 1 is proved.

Now, we can finish the proof of Theorem 1. From the $\mathfrak{c}$-cc assumption it follows that for every $i \in \omega$ there is an $\eta_i < \omega_2$ such that $\omega \setminus X^{m_i,\xi}_{n_i,\alpha} \in F$ for all $\xi \geq \eta_i$ with $\mathrm{cf}(\xi) = \omega_1$ and $m_i \in \omega$. Let $\omega_2 > \xi_i > \eta_j$ with $\mathrm{cf}(\xi_i) = \omega_1$ and $m_i \in \omega$.

Since F is a P-set, there is an $A \in F$ and a function $g : \omega \to \omega$ such that

$$A \setminus [0, g(i)) \subseteq \omega \setminus X^{m_i,\xi_i}_{n_i,\alpha} \quad \text{for each } i \in \omega,$$

which implies $\bigcap_{i\in\omega}(\omega \setminus X^{m_i,\xi_i}_{n_i,\alpha}) \cup [0, g(i)) \in F_\alpha$. Since $\mathbb{P}_{\omega_2}$ is bounding we may assume that $g \in V$. This contradicts Lemma 1 and the proof is finished. ∎

5. Structure of a fat P-set. The coabsoluteness assertion follows easily from the following theorem, which clarifies the structure of any fat P-set.

THEOREM 2. *In the model $V^{\mathbb{P}_{\omega_2}}[G]$ every fat P-set F has a π-base tree of height ω, each vertex of which splits into $\mathfrak{c}$ elements.*

Proof. Let F denote also the corresponding filter. Keeping the notation from the preceding proof we use the sets X^i_γ to construct a dense tree in the factor algebra $P(\omega)/F$. Fix an enumeration of the branches of the binary tree $\{b_\gamma : \gamma < \omega_2\}$. Denote by X^i_γ the set $X^{m_i,\xi_i}_{i,\alpha}$ if b_γ is the characteristic function of the tree $t^G_{m_i,\xi_i}$ ($\xi_i > \alpha$), and the set $X^\alpha_{n_i,\alpha}(b_\gamma)$ if b_γ belongs to the model $V[G|\alpha]$. Notice first that for a given positive element $a = [A] > \mathbb{O}$ from $P(\omega)/F$, there is an $i \in \omega$ such that

$$a \cdot x^i_\gamma > \mathbb{O} \quad \text{for } \mathfrak{c}\text{-many } \gamma,$$

where $x^i_\gamma = [X^i_\gamma]$ denotes the equivalence class mod F of elements determined by the branch b_γ in the tree $t^G_{i,\alpha}$.

Indeed, suppose the opposite, i.e. for each $i \in \omega$ there is a β_i such that

$$A \cap X^i_\gamma \in I \quad \text{for every } \gamma \geq \beta_i.$$

Let $\gamma > \sup_{j\in\omega} \beta_j$ with $\mathrm{cf}(\gamma) = \omega_1$. We then have

$$A \setminus B_i \subseteq \omega \setminus X^i_\gamma \quad \text{for all } i \in \omega,$$

where the B_i's belong to I. Since I is a P-ideal there is a set $B \in I$ such that $B_i \subseteq_* B$ for all $i \in \omega$ and hence

$$A \setminus B \subseteq \bigcap_{i\in\omega} (\omega \setminus X^i_\gamma) \cup [0, g(i)),$$

where g may be assumed to be in V, as the forcing is bounding. Now, Lemma 1 implies that $A \setminus B$ and hence A are in I, which contradicts the positivity of $a = [A]$.

Clearly, deleting "small" sets (i.e. those from I) and renumbering we may assume that we are given a matrix

$$M = \{x_\gamma^i : i \in \omega \text{ and } \gamma < \mathfrak{c}\}$$

such that every element x_γ^i is positive in $P(\omega)/F$ and each positive element $a > \mathbb{O}$ intersects $\mathfrak{c}$ elements x_γ^i in some row i (depending on a). Obviously, each row in M is an antichain. Finally, note that any positive $a > \mathbb{O}$ splits into $\mathfrak{c}$ elements. To see this, we check that the filter

$$F_a = \{X \subseteq \omega : [a] \le [X]\} = \{X \subseteq \omega : A \setminus X \in I\}$$

is a fat P-filter, as F is, and then apply Theorem 1 to convince ourselves that the algebra $P(\omega)/F_a$ has an antichain of power $\mathfrak{c}$.

Now, define a tree T as follows. Extend, if necessary, the first row $\{x_\gamma^0 : \gamma < \mathfrak{c}\}$ to a maximal antichain $T_0 = \{y_\beta^0 : \beta < \mathfrak{c}\}$.

Assume that the levels $T_0, \ldots, T_{n-1}$ are already defined so that each $T_i = \{y_\beta^i : \beta < \mathfrak{c}\}$ is a maximal antichain. Extend each of the antichains $\{y_\beta^{n-1} \cap x_\gamma^n : \gamma < \mathfrak{c}\} \setminus \{\mathbb{O}\}$ to a maximal T_n^β. By the remark above we may assert that T_n^β always has $\mathfrak{c}$ elements. Let $T_n = \bigcup_{\beta<\mathfrak{c}} T_n^\beta$. Clearly, in the resulting tree $T = \bigcup_{n\in\omega} T_n$ each vertex splits into $\mathfrak{c}$ elements and every positive $a > \mathbb{O}$ intersects $\mathfrak{c}$ elements of some level T_n. Hence, to each $a > \mathbb{O}$ we may assign an element $y_a \in T$ such that $a \cdot y_a > \mathbb{O}$ and $y_a \ne y_b$ whenever $a \ne b$, as in [1]. Replace any such y_a by $a \cdot y_a$ if $y_a - a > \mathbb{O}$, and the same for all $y \le y_a$. Clearly, the tree T so modified becomes additionally a π-base. ∎

6. Final remarks. From Theorem 2 it follows immediately that for any two fat P-filters F_1 and F_2 in $V[G]$ the Boolean algebras $P(\omega)/F_1$ and $P(\omega)/F_2$ have order isomorphic dense subsets. This, in turn, implies that the completions of $P(\omega)/F_1$ and $P(\omega)/F_2$ are Boolean isomorphic, or, equivalently, the Gleasons $\mathbb{G}(F_1)$ and $\mathbb{G}(F_2)$ are homeomorphic.

Finally, let us comment on other consistent configurations of P-sets. For example, we can construct a model with $\mathfrak{c} > \omega_1$ in which there are no P-points and there are two fat P-sets, one with the ω_1-cc and the other with the ω_2-cc. In connection with Theorem 2 we note that it is consistent to have two fat P-sets having the $\mathfrak{c}^+$-cc with distinct Gleasons. The proof is more difficult and will be published elsewhere.

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