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Weighted Estimates for the Maximal Operator of a Multilinear Singular Integral

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Summary. An improved multiple Cotlar inequality is obtained. From this result, weighted norm inequalities for the maximal operator of a multilinear singular integral including weak and strong estimates are deduced under the multiple weights constructed recently.

1. Introduction. Grafakos and Torres [4] systematically studied multilinear Calderón–Zygmund singular integral operators $T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ with some boundedness properties, defined by

$$T(f_1, \ldots, f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m,$$

where $K(x, y_1, \ldots, y_m)$ is a locally integrable function supported away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ and satisfies

(i) (Size estimate)

$$|K(x, y_1, \ldots, y_m)| \le \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \tag{1.1}$$

for some $A > 0$ and all $(x, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \neq y_j$ for some j;

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(ii) (Smoothness estimates)

$$|K(x,y_1,\dots,y_m)-K(x',y_1,\dots,y_m)| \leq \frac{A|x-x'|^\epsilon}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\epsilon}} \tag{1.2}$$

for some $\epsilon>0$ whenever $|x-x'|\leq \frac{1}{2}\max_{1\leq j\leq n}|x-y_j|$, and also, for each j,

$$|K(x,y_1,\dots,y_j,\dots,y_m)-K(x,y_1,\dots,y_j',\dots,y_m)| \leq \frac{A|y_j-y_j'|^\epsilon}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\epsilon}} \tag{1.3}$$

for some $\epsilon>0$ whenever $|y_j-y_j'|\leq \frac{1}{2}\max_{1\leq j\leq n}|x-y_j|$.

In [5], the authors considered the corresponding maximal operator defined as

$$T^*(f_1,\dots,f_m)(x)=\sup_{\delta>0}|T_\delta(f_1,\dots,f_m)(x)|,$$

where T_δ, the truncated operator of T, is

$$T_\delta(f_1,\dots,f_m)(x) = \int\limits_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta^2} K(x,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m.$$

Similarly to the linear setting, *Cotlar's inequality*, for all $\eta>0$,

$$T^*(f_1,\dots,f_m)(x)\leq C\Big(M_\eta(T(f_1,\dots,f_m))(x)+\prod_{i=1}^m Mf_i(x)\Big), \tag{1.4}$$

where $M_\eta(f)(x)=\sup_{Q\ni x}|Q|^{-1}\int_Q|f|^\eta)^{1/\eta}$ was employed to show

THEOREM 1.1 (Boundedness of T^*, [5]). *Assume that* $1/p=1/p_1+\cdots+1/p_m$.

(i) *If* $1<p_1,\dots,p_m\leq\infty$ *and* $p<\infty$, *then*
$$T^*:L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n).$$

(ii) *If* $1\leq p_1,\dots,p_m\leq\infty$ *and* $p<\infty$, *then*
$$T^*:L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n).$$

By generalizing Coifman and Fefferman's good-λ inequality of [1], Grafakos and Torres proved a weighted norm inequality for T^*:

THEOREM 1.2 (Weighted boundedness of T^*, [5]). *Assume that* $1<p_1,\dots,p_m<\infty$, $p_0=\min(p_1,\dots,p_m)$ *and* $1/p=1/p_1+\cdots+1/p_m$. *If* $\omega\in A_{p_0}$, *then*
$$T^*:L^{p_1}(\omega)\times\cdots\times L^{p_m}(\omega)\to L^p(\omega).$$

Recently, Lerner, Ombrosi, Pérez, Torres and Trujillo-González [6] constructed a new theory of multiple $A_{\vec{p}}$ weights.

Definition 1.1 (Multiple weights, [6]). Let $1 \le p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$. Given $\vec{\omega} = (\omega_1, \dots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. By definition, $\vec{\omega} \in A_{\vec{p}}$ if and only if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{\omega}} \right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{1/p_i'} < \infty.$$

When $p_i = 1$, $(|Q|^{-1} \int_Q \omega_i^{1-p_i'})^{1/p_i'}$ is understood as $(\inf_Q \omega_i)^{-1}$.

A more subtle multilinear maximal operator $\mathcal{M}$ which is defined as

$$\mathcal{M}(f_1, \dots, f_m)(x) = \sup_Q \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i|$$

was investigated to characterize the multiple weights in [6]. The authors showed that $\vec{\omega} \in A_{\vec{p}}$ is equivalent to either of the two weighted estimates for $\mathcal{M}$:

(i) If $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$, then

$$L^{p_1}(\omega_1) \times \dots \times L^{p_m}(\omega_m) \to L^p(\nu_{\vec{\omega}}). \tag{1.5}$$

(ii) If $1 \le p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$, then

$$L^{p_1}(\omega_1) \times \dots \times L^{p_m}(\omega_m) \to L^{p,\infty}(\nu_{\vec{\omega}}). \tag{1.6}$$

Further, it was proved in [6] that T satisfies both (1.5) and (1.6) by using unweighted boundedness, Fefferman–Stein inequalities and a sharp estimate. For more details, the readers are referred to [6].

It is natural to ask whether (1.5) and (1.6) hold for T^*. We will give a positive answer in this note. Instead of the good-λ inequality we will employ Cotlar's inequality as in [2, p. 147]. However, $\prod_{i=1}^m Mf_i$ fails to satisfy either (1.5) or (1.6) (see [6]), which makes us improve (1.4) by replacing the m-fold product of M with $\mathcal{M}$. After the modification, we shall obtain not only strong type bounds but also weak endpoint estimates.

Theorem 1.3 (Weighted estimates for T^*). *Assume that* $1/p = 1/p_1 + \dots + 1/p_m$ *and* $\vec{\omega} \in A_{\vec{p}}$. *Then both* (1.5) *and* (1.6) *hold for* T^*.

Throughout this article, we write $\vec{f} = (f_1, \dots, f_m)$, $\vec{y} = (y_1, \dots, y_m)$ and $\int_{Q^m} \prod_{i=1}^m f_i(y_i)\,dy_i = \int_Q \cdots \int_Q f_1(y_1) \cdots f_m(y_m)\,dy_1 \cdots dy_m$ for convenience.

2. Weighted norm inequalities. The multiple $A_{\vec{p}}$ weights are appropriate for the maximal function $\mathcal{M}(\vec{f})$ which is more refined than $\prod_{i=1}^m Mf_i$. Hence, we improve Cotlar's inequality as follows.

LEMMA 2.1 (Improved Cotlar inequality). *For any $\eta > 0$, there is a $C > 0$ depending on η such that*

$$T^*(\vec{f})(x) \le C(M_\eta(T(\vec{f}))(x) + \mathcal{M}(\vec{f})(x)), \tag{2.1}$$

where $M_\eta(f)(x) = \sup_{Q\ni x}(|Q|^{-1}\int_Q |f|^\eta)^{1/\eta}$.

Proof. The basic idea is due to [5] and [6].

Fix $x \in \mathbb{R}^n$, $0 < \eta < 1/m$ and $\delta > 0$. Denote by $Q(x,\delta)$ the cube of center x and edge length 2δ with sides parallel to the axes, and set $U_\delta(x) = \{\vec{y} \in (Q(x,\delta))^m : \sum_{i=1}^m |x-y_i|^2 > \delta^2\}$. It is clear that

$$|T_\delta(\vec{f})(x)| \le \Big|\int_{U_\delta(x)} K(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\Big| + \Big|\int_{((Q(x,\delta))^m)^c} K(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\Big|. \tag{2.2}$$

By invoking the size condition (1.1), the first term on the right hand side of (2.2) can be estimated as follows:

$$\begin{aligned}\Big|\int_{U_\delta(x)} K(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\Big| &\le \int_{U_\delta(x)} \frac{A}{(\sum_{i=1}^m |y_i-x|)^{mn}}\prod_{i=1}^m |f_i(y_i)|\,dy_i\\ &\le \int_{U_\delta(x)} \frac{C}{\delta^{mn}}\prod_{i=1}^m |f_i(y_i)|\,dy_i\\ &\le \prod_{i=1}^m \frac{C}{(2\delta)^n}\int_{Q(x,\delta)} |f_i(y_i)|\,dy_i\\ &\le C\mathcal{M}(\vec{f})(x).\end{aligned}$$

We now estimate the second term in (2.2). Pick $z \in Q(x,\delta/2)$ and set $\vec{f_0} = (f_1\chi_{Q(x,\delta)},\dots,f_m\chi_{Q(x,\delta)})$. Then

$$\int_{((Q(x,\delta))^m)^c} K(z,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i = T(\vec{f})(z) - T(\vec{f_0})(z),$$

which means

$$\begin{aligned}&\Big|\int_{((Q(x,\delta))^m)^c} K(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\Big|\\ &\quad\le \Big|\int_{((Q(x,\delta))^m)^c} K(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i - \int_{((Q(x,\delta))^m)^c} K(z,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\Big|\\ &\qquad + |T(\vec{f})(z) - T(\vec{f_0})(z)|.\end{aligned}$$

In virtue of the smoothness condition (1.2), we can deduce that

$$\Big| \int\limits_{((Q(x,\delta))^m)^c} K(x,\vec{y}) \prod_{i=1}^m f_i(y_i)\,dy_i - \int\limits_{((Q(x,\delta))^m)^c} K(z,\vec{y}) \prod_{i=1}^m f_i(y_i)\,dy_i \Big|$$
$$\le \int\limits_{((Q(x,\delta))^m)^c} \frac{C|x-z|^\epsilon}{(\sum_{i=1}^m |x-y_i|)^{mn+\epsilon}} \prod_{i=1}^m |f_i(y_i)|\,dy_i$$
$$\le \sum_{i_1,\dots,i_l} \sum_{k=0}^\infty C\delta^\epsilon \prod_{i\in\{i_1,\dots,i_l\}}$$
$$\times \int\limits_{Q(x,\delta)} |f_i(y_i)|\,dy_i \int\limits_{(Q(x,2^{k+1}\delta))^{m-l}\setminus(Q(x,2^k\delta))^{m-l}} \frac{\prod_{i\notin\{i_1,\dots,i_l\}} |f_i(y_i)|\,dy_i}{(\sum_{i=1}^m |x-y_i|)^{mn+\epsilon}}$$
$$\le \sum_{i_1,\dots,i_l} \sum_{k=0}^\infty C\delta^\epsilon \prod_{i\in\{i_1,\dots,i_l\}}$$
$$\times \int\limits_{Q(x,\delta)} |f_i(y_i)|\,dy_i \int\limits_{(Q(x,2^{k+1}\delta))^{m-l}} \frac{\prod_{i\notin\{i_1,\dots,i_l\}} |f_i(y_i)|\,dy_i}{(2^k\delta)^{mn+\epsilon}}$$
$$\le \sum_{k=0}^\infty \frac{C}{2^{k\epsilon}} \prod_{i=1}^m \frac{1}{(2^{k+2}\delta)^n} \int\limits_{Q(x,2^{k+1}\delta)} |f_i(y_i)|\,dy_i$$
$$\le C\mathcal{M}\vec{f}(x),$$

where $\emptyset \ne \{i_1,\dots,i_l\} \subsetneq \{1,\dots,m\}$. This implies

$$\Big| \int\limits_{((Q(x,\delta))^m)^c} K(x,\vec{y}) \prod_{i=1}^m f_i(y_i)\,dy_i \Big| \le C\mathcal{M}\vec{f}(x) + |T(\vec{f}\,)(z)| + |T(\vec{f_0})(z)|. \tag{2.3}$$

Raising (2.3) to the power η, integrating over $z \in Q = Q(x,\delta/2)$ and dividing by $|Q|$, we conclude that

$$\Big| \int\limits_{((Q(x,\delta))^m)^c} K(x,\vec{y}) \prod_{i=1}^m f_i(y_i)\,dy_i \Big|^\eta \le C(\mathcal{M}\vec{f}(x))^\eta + M(|T(\vec{f}\,)|^\eta)(x) + \frac{1}{|Q|} \int\limits_Q |T(\vec{f_0})(z)|^\eta\,dz.$$

Finally, the proof can be finished by using the arguments in [5] which proved $|Q|^{-1}\int_Q |T(\vec{f_0})|^\eta \le C(\prod_{i=1}^m |Q|^{-1}\int_Q |f_i|)^\eta$. ∎

It is well known that M is bounded from $L^p(\omega)$ to $L^{p,\infty}(\omega)$ when $\omega \in A_p$ and $p \ge 1$. Similar to the proof in [2, p. 135], we get the following lemma to show the weak estimates in the main theorem.

LEMMA 2.2. *If $\omega \in A_p$ and $p \geq 1$, then M maps $L^{p,\infty}(\omega)$ to $L^{p,\infty}(\omega)$.*

Proof. For any cube Q and $\omega \in A_p$,

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q \omega\Big)\Big(\frac{\lambda}{|Q|}\int_{Q\cap\{|f|>\lambda\}}\Big)^p \\ &\leq \Big(\frac{1}{|Q|}\int_Q \omega\Big)\Big(\frac{\lambda^p}{|Q|}\int_{Q\cap\{|f|>\lambda\}}\omega\Big)\Big(\frac{1}{|Q|}\int_Q \omega^{1-p'}\Big)^{p-1}\\ &\leq C\frac{\lambda^p}{|Q|}\int_{Q\cap\{|f|>\lambda\}}\omega.\end{aligned}$$

When $p=1$, $(|Q|^{-1}\int_Q \omega^{1-p'})^{p-1}$ is understood as $(\inf_Q \omega)^{-1}$. Then we obtain

$$\begin{aligned}\int_Q \omega &\leq C\Big(\frac{\lambda}{|Q|}\int_{Q\cap\{|f|>\lambda\}}\Big)^{-p}\Big(\lambda^p\int_{Q\cap\{|f|>\lambda\}}\omega\Big)\\ &\leq C\Big(\frac{1}{|Q|}\int_Q |f|\Big)^{-p}\Big(\lambda^p\int_{Q\cap\{|f|>\lambda\}}\omega\Big).\end{aligned}$$

A Calderón–Zygmund decomposition for f at height $4^{-n}\lambda$ yields a sequence of cubes $\{Q_k\}$ such that $4^{-n}\lambda < |Q_k|^{-1}\int_{Q_k} f$. Additionally, we have $\{Mf > \lambda\} \subset \bigcup_k 3Q_k$ as in [2]. Since the function $\omega \in A_p$ is doubling, it is immediate that

$$\begin{aligned}\int_{\{Mf>\lambda\}}\omega \leq \sum_k \int_{3Q_k}\omega \leq C3^{np}\sum_k\int_{Q_k}\omega\\ &\leq C3^{np}\sum_k\Big(\frac{1}{|Q_k|}\int_{Q_k}|f|\Big)^{-p}\Big(\lambda^p\int_{Q_k\cap\{|f|>\lambda\}}\omega\Big) \leq C12^{np}\int_{\{|f|>\lambda\}}\omega,\end{aligned}$$

which means $\|Mf\|_{L^{p,\infty}(\omega)} \leq C\|f\|_{L^{p,\infty}(\omega)}$. ∎

Proof of Theorem 1.3. Before the final proof, we should recall another fact from [6]: if $\vec{\omega} \in A_{\vec{p}}$, then $\nu_{\vec{\omega}} \in A_{mp}$.

When $1 \leq p_1, \ldots, p_m < \infty$, we have a weak type result. Let $\eta \leq 1/m$. The improved Cotlar inequality (2.1), Lemma 2.2 and the weighted estimates (1.6) for $\mathcal{M}$ and T in [6] imply

$$\begin{aligned}\|T^*(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})} &\leq C(\|M_\eta(T(\vec{f}))\|_{L^{p,\infty}(\nu_{\vec{\omega}})} + \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})})\\ &= C(\|M(|T(\vec{f})|^\eta)\|^{1/\eta}_{L^{p/\eta,\infty}(\nu_{\vec{\omega}})} + \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})})\\ &\leq C(\|\,|T(\vec{f})|^\eta\|^{1/\eta}_{L^{p/\eta,\infty}(\nu_{\vec{\omega}})} + \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})})\\ &= C(\|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})} + \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})}) \leq C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}.\end{aligned}$$

The proof of the strong case is similar. ∎

3. Further results. X. T. Duong, R. Gong, L. Grafakos, J. Li and L. Yan [3] studied the maximal operator of a multilinear singular integral with non-smooth kernel. Together with the corresponding Cotlar inequalities with $\prod_{i=1}^{m} Mf_i$ and unweighted bounds, they obtained the counterpart of Theorem 1.2 in that case. However, weighted norm inequalities with new multiple weights for non-smooth operators as we did for Calderón–Zygmund operators have not been proved yet.

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