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CONVEX AND DISCRETE GEOMETRY

On-line Packing Squares into n Unit Squares

by

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Summary. If $n \geq 3$, then any sequence of squares of side lengths not greater than 1 whose total area does not exceed $\frac{1}{4}(n+1)$ can be on-line packed into n unit squares.

1. Introduction. Let $C, C_1, C_2, \dots$ be planar convex bodies. We say that (C_i) can be *packed* into C if there exist rigid motions σ_i such that $\bigcup \sigma_i C_i \subseteq C$ and $\sigma_i C_i$ have pairwise disjoint interiors. The *on-line* version of packing is the following: initially the first set C_1 is given without any information on the next bodies; then we find each successive set C_i only after the motion σ_{i-1} has been provided. The placement of each packed set $\sigma_i C_i$ cannot be changed afterwards. The survey of results concerning packings and on-line packings is given in [1], [2] and [5].

Moon and Moser [6] proved that any sequence of squares whose total area does not exceed $\frac{1}{2}$ can be packed into the unit square. The best known upper bound is smaller for the packing with the on-line restriction. In [3] it is shown that every sequence of squares with total area not greater than $\frac{1}{3}$ can be on-line packed into the unit square.

We propose the problem of on-line packing squares into a number of unit squares. Let $I_1, \dots, I_n$ be pairwise disjoint squares of sides of length 1 and let $J_n = I_1 \cup \dots \cup I_n$.

Observe that $n+1$ squares of side lengths greater than $\frac{1}{2}$, and consequently of total area greater than $\frac{1}{4}(n+1)$, cannot be packed into J_n. The reason is that the interior of any square of side length greater than $\frac{1}{2}$ packed into a unit square I_m contains the center of I_m.

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The aim of this paper is to show that any sequence of squares of side lengths not greater than 1 whose total area does not exceed $\frac{1}{4}(n+1)$ can be on-line packed into J_n provided $n \geq 3$.

The area of C is denoted by $|C|$.

2. Subbrics. In the next section the *method of the first free subbrick*, introduced in [4], will be used for packing small squares.

Let k be a non-negative integer. By a *brick of size* $(3,k)$ we mean a rectangle of side lengths $1/(3 \cdot 2^k)$ and $1/(2 \cdot 2^k)$. By a *brick of size* $(4,k)$ we mean a rectangle of side lengths $1/(4 \cdot 2^k)$ and $1/(3 \cdot 2^k)$.

We can dissect any brick of size $(3,k)$ into two congruent bricks, called *subbrics*, of size $(4,k)$. Furthermore, we can dissect any brick of size $(4,k)$ into two congruent bricks, called *subbrics*, of size $(3,k+1)$. Consequently, any square I_m can be dissected into $6 \cdot 4^k$ subbricks of size $(3,k)$ and into $12 \cdot 4^k$ subbricks of size $(4,k)$. Bricks of size $(3,0)$ are also called *subbricks*.

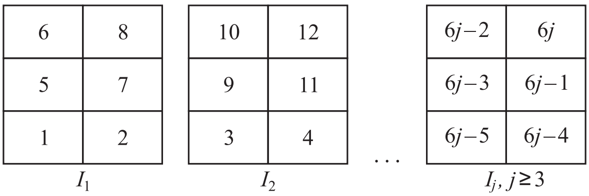

Fig. 1

Without loss of generality we can assume that the unit squares are parallel as in Fig 1.

We number all subbricks of J_n of size $(3,0)$ by integers from 1 to $6n$ as in Fig. 1. Bricks numbered 1, 2, 5, 6, 7 and 8 are contained in I_1. Bricks numbered 3, 4, 9, 10, 11 and 12 are contained in I_2. Bricks numbered $6j-5$, $6j-4, \ldots, 6j$ are contained in I_j, for $j = 3, \ldots, n$.

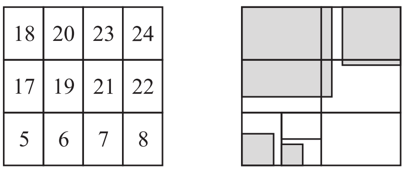

Fig. 2

For each positive integer k we number all $6n \cdot 4^k$ subbricks of J_n of size $(3,k)$ by integers from 1 to $6n \cdot 4^k$; also, for each non-negative integer k we number all $12n \cdot 4^k$ subbricks of J_n of size $(4,k)$ by integers from 1 to $12n \cdot 4^k$ so that the following two conditions are fulfilled:

(1) The numbers $2m-1$ and $2m$ are assigned to the subbricks of size $(4,k)$ of the subbrick of size $(3,k)$ numbered m so that the subbrick numbered $2m-1$ is to the left of the subbrick numbered $2m$. The only exception is the numbering of the four subbricks of size $(4,0)$ contained in two subbricks of size $(3,0)$ numbered 9 and 10 (see Fig. 2, where all subbricks of size $(4,0)$ contained in I_2 are shown).

(2) The numbers $2l-1$ and $2l$ are assigned to the subbricks of size $(3,k+1)$ of the subbrick of size $(4,k)$ numbered l so that the subbrick number $2l-1$ is situated lower than the subbrick numbered $2l$.

The subbrick of size $(3,k)$ numbered t is denoted by $(3,k,t)$. The subbrick of size $(4,k)$ numbered u is denoted by $(4,k,u)$.

3. Packing algorithm. Let (S_i) be a sequence of squares of side lengths not greater than 1. Denote by s_i the side length of S_i. If $s_i \le \frac{1}{3}$, then S_i is *small*, otherwise S_i is *big*.

If $1/(4\cdot 2^k) < s_i \le 1/(3\cdot 2^k)$, then

$$|S_i| = s_i^2 > \frac{1}{(4\cdot 2^k)^2} > \frac{1}{3}\cdot\frac{1}{3\cdot 2^k}\cdot\frac{1}{2\cdot 2^k}.$$

If $1/(3\cdot 2^{k+1}) < s_i \le 1/(4\cdot 2^k)$, then

$$|S_i| = s_i^2 > \frac{1}{(6\cdot 2^k)^2} = \frac{1}{3}\cdot\frac{1}{4\cdot 2^k}\cdot\frac{1}{3\cdot 2^k}.$$

Consequently, for each small square S_i there is a brick $B_i \supset S_i$ such that $|S_i| > \frac{1}{3}|B_i|$.

Let $i \ge 1$ be an integer. We will define B_i' once the square S_i is packed.

Packing of small squares. If S_i is small, then by a *free i-subbrick* we mean a subbrick congruent to B_i whose interior is disjoint from any set B_j' for $j < i$. Denote by P_i the free i-subbrick of J_n with the smallest possible number.

If (a) P_i is either the subbrick $(3,0,13)$ or $(3,0,14)$, then we pack S_i into P_i so that $\sigma_i S_i$ contains a vertex of I_3. We set $P_i = B_i'$.

If (b) there are small squares S_p and S_q of side lengths greater than $\frac{1}{4}$ such that S_p is packed into the subbrick $(3,0,13)$ and S_q is packed into the subbrick $(3,0,14)$, and if i is the smallest integer greater than q such that $\frac{1}{4} < s_i \le \frac{1}{3}$, then S_i is an *extra-square*. We pack S_i into the union of two subbricks $(3,0,13)$ and $(3,0,14)$ between the squares $\sigma_p S_p$ and $\sigma_q S_q$. We set $B_i' = \emptyset$.

If neither (a) nor (b) holds, then we pack S_i into P_i and we set $P_i = B_i'$.

Packing of big squares. If S_i is big, then we find the smallest integer l such that it is possible to pack S_i into I_l so that one vertex of $\sigma_i S_i$ is a vertex of I_l. We pack S_i into I_l so that one vertex of $\sigma_i S_i$ is a vertex of I_l and so that $\sigma_i S_i$ contains a subbrick of size $(4,0)$ with the greatest possible number (we pack big squares starting from the top of unit squares in Figs. 1 and 2).

Now we define B'_i.

If $s_i > \frac{2}{3}$, then we set $B'_i = I_l$. Obviously, $|S_i| > \frac{4}{9} > \frac{1}{3}|B'_i|$.

If $\frac{1}{2} < s_i \le \frac{2}{3}$, then the packed square $\sigma_i S_i$ is contained in the union of four subbricks of size $(3,0)$. Let B'_i be that union. Obviously, $|S_i| > \frac{1}{4} > \frac{1}{3}\cdot\frac{2}{3} = \frac{1}{3}|B'_i|$.

If $\frac{1}{3} < s_i \le \frac{1}{2}$, then $\sigma_i S_i$ is contained in the union of two subbricks of size $(3,0)$. Let B'_i be that union. Obviously, $|S_i| > \frac{1}{9} = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{3}|B'_i|$.

4. Efficiency of the packing algorithm. First we show how effective our method is for packing of big squares.

LEMMA. *If $n \ge 1$ and if a sequence of big squares cannot be on-line packed into $J_n = I_1 \cup \cdots \cup I_n$ by the method described in Section 3, then the total area of the squares exceeds $\frac{1}{4}(n+1)$.*

Proof. Let (S_i) be a sequence of big squares. Assume that they cannot be packed into J_n by the method presented in Section 3.

Denote by S_z the first square from the sequence which cannot be packed into J_n. Furthermore, denote by K the set of integers $k \in \{1,\dots,n\}$ such that at most three big squares are packed into I_k.

If $K = \emptyset$ (i.e., if four big squares are packed into each I_k), then the total area of the squares is greater than $\frac{4}{9}n + |S_z| > \frac{4}{9}n + \frac{1}{9} > \frac{1}{4}(n+1)$.

Consider the case when $K \ne \emptyset$. There is at most one $j \in K$ such that only squares of side lengths not greater than $\frac{1}{2}$ are packed into I_j. If there is $j \in K$ such that only squares of side lengths not greater than $\frac{1}{2}$ are packed into I_j, then let S_m be a square packed into I_j such that $s_m + s_z > 1$. Otherwise, let j be an integer from K and let S_m be a square packed into I_j such that $s_m + s_z > 1$. It is easy to verify that $s_m^2 + s_z^2 > \frac{1}{2}$. The total area of the squares packed into I_k is greater than $\frac{1}{4}$ for each $k \in \{1,\dots,n\}$, $k \ne j$. This implies that

$$\sum_{i=1}^{z} |S_i| > \frac{1}{4}(n-1) + s_m^2 + s_z^2 > \frac{1}{4}(n-1) + \frac{1}{2} = \frac{1}{4}(n+1). \ \blacksquare$$

THEOREM. *If $n \ge 3$, then any sequence of squares of side lengths not greater than 1 whose total area does not exceed $\frac{1}{4}(n+1)$ can be on-line packed into J_n.*

Proof. Let $n \ge 3$ and let (S_i) be a sequence of squares of side lengths not greater than 1 whose total area does not exceed $\frac{1}{4}(n+1)$.

We pack the squares from the sequence by the method described in Section 3.

Suppose that, contrary to the statement, it is impossible to pack S_1, $S_2, \ldots$ into J_n by this method. Let S_z be the square which stops the packing process and let

$$\zeta = \sum_{i=1}^{z} |S_i|.$$

We show that this leads to the false inequality

$$\zeta > \frac{1}{4}(n+1).$$

Obviously, if $i < z$, then $|S_i| > \frac{1}{3}|B_i'|$. Consider four cases.

CASE 1: *S_z is small.*

SUBCASE 1A: $s_z \le \frac{1}{4}$. Since S_z cannot be packed, it follows that there is no free z-subbrick of J_n. This implies that the total area of all free subbricks is smaller than

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)|B_z| = |B_z|.$$

Hence

$$\zeta > \frac{1}{3}\Big(\sum_{i=1}^{z-1} |B_i'| + |B_z|\Big) \ge \frac{1}{3}|J_n| = \frac{1}{3}n.$$

It is easy to verify that $\frac{1}{3}n \ge \frac{1}{4}(n+1)$ for $n \ge 3$. Consequently, $\zeta > \frac{1}{4}(n+1)$.

SUBCASE 1B: $s_z > \frac{1}{4}$. If a square of side length not greater than $\frac{1}{4}$ is packed into $I_3 \cup \cdots \cup I_n$, then we argue as in Subcase 1A.

Assume than no square of side length not greater than $\frac{1}{4}$ is packed into $I_3 \cup \cdots \cup I_n$. The total area of the free subbricks is smaller than $\frac{3}{2}|B_z|$ (now it can happen that two subbricks of size $(4,0)$: $(4,0,19)$ and $(4,0,20)$ are free).

Denote by U the union of the subbricks $(3,0,13)$ and $(3,0,14)$.

Since $\frac{1}{4} < s_z \le \frac{1}{3}$, it cannot be the case that exactly two small squares of side length greater than $\frac{1}{4}$ are packed into U.

If three small squares of side length greater than $\frac{1}{4}$ are packed into U, then the total area of the free subbricks is smaller than $\frac{3}{2}|B_z|$ but, on the other hand, the area of the extra-square is greater than $\frac{1}{16}$. Consequently,

$$\begin{aligned}\zeta &> \frac{1}{3}\Big(\sum_{i=1}^{z-1} |B_i'| + |B_z|\Big) + \frac{1}{16} \ge \frac{1}{3}\left(n - \frac{3}{2}|B_z| + |B_z|\right) + \frac{1}{16} \\ &= \frac{1}{3}\left(n - \frac{1}{12}\right) + \frac{1}{16} > \frac{1}{4}(n+1).\end{aligned}$$

If at most one small square of side length greater than $\frac{1}{4}$ is packed into U, then the total area of the (small and big) squares packed into I_3 is greater than $\frac{1}{3}+\frac{1}{16}$. Consequently,

$$\zeta > \frac{1}{3}\left(n-\frac{1}{2}|B_z|\right)+\frac{1}{16}=\frac{1}{3}\left(n-\frac{1}{12}\right)+\frac{1}{16}>\frac{1}{4}(n+1).$$

Case 2: *S_z is big and no small square is packed into $I_2\cup\cdots\cup I_n$.*

Subcase 2A: *all small packed squares are contained in the union of the subbricks* $(3,0,1)$ *and* $(3,0,2)$. If no big square is packed into I_1, then all big packed squares (and S_z) have sides longer than $\frac{2}{3}$, and consequently $\zeta>\frac{4}{9}n>\frac{1}{4}(n+1)$.

Assume that at least one big square is packed into I_1.

If either four or three big squares are packed into I_1, then the total area of the squares packed into I_1 is greater than $3\cdot\frac{1}{9}$. By the Lemma we know that the total area of the big squares packed into $I_2\cup\cdots\cup I_n$ plus $|S_z|$ is greater than $\frac{1}{4}(n-1+1)$. Consequently, $\zeta>3\cdot\frac{1}{9}+\frac{1}{4}n>\frac{1}{4}(n+1)$.

If a small square is packed outside the subbrick $(4,0,1)$, i.e., if the total area of the packed small squares is greater than $\frac{1}{3}\cdot\frac{1}{12}$ and if two big squares are packed into I_1, then

$$\zeta>2\cdot\frac{1}{9}+\frac{1}{3}\cdot\frac{1}{12}+\frac{1}{4}n=\frac{1}{4}(n+1).$$

If one big square is packed into I_1 or if two big squares are packed into I_1 and all small squares are contained in $(4,0,1)$, then arguing as in the proof of the Lemma we obtain $\zeta>\frac{1}{4}(n+1)$.

Subcase 2B: *a small square is packed into I_1 outside the union of the subbricks* $(3,0,1)$ *and* $(3,0,2)$. This implies that there is no free subbrick of size $(3,0)$ contained in I_2. Hence the total area of the squares packed into I_2 is greater than $\frac{4}{9}$. Moreover, the total area of the small squares is greater than $\frac{1}{3}\cdot\frac{1}{3}$. Consequently, by the Lemma,

$$\zeta>\frac{1}{9}+\frac{4}{9}+\frac{1}{4}(n-2+1)\geq\frac{1}{4}(n+1).$$

Case 3: *S_z is big and a small square is packed into $I_3\cup\cdots\cup I_n$.* Denote by s the greatest integer such that a small square is packed into I_s. Obviously, $s\geq 3$.

If all small squares packed into I_s are contained in the union of the subbricks $(3,0,6s-4)$ and $(3,0,6s-5)$, then we argue as in Subcase 2A; the total area of the squares packed into $I_s\cup\cdots\cup I_n$ plus $|S_z|$ is greater than $\frac{1}{4}(n-s+1+1)$. Arguing as in Case 1 we deduce that the total area of the squares packed into $I_1\cup\cdots\cup I_{s-1}$ is greater than $\frac{1}{3}(s-1-\frac{3}{2}\cdot\frac{1}{6})$.

Consequently,

$$\zeta > \frac{1}{3}\left(s - 1 - \frac{1}{4}\right) + \frac{1}{4}(n - s + 2) > \frac{1}{4}(n + 1).$$

If a small square is packed into I_s outside the union of the subbricks $(3, 0, 6s - 4)$ and $(3, 0, 6s - 5)$, then the total area of the squares packed into $I_1 \cup \cdots \cup I_s$ is greater than $\frac{1}{3}(s - 1 - \frac{1}{12} + \frac{1}{3})$. Consequently, by the Lemma,

$$\zeta > \frac{1}{3}\left(s - \frac{3}{4}\right) + \frac{1}{4}(n - s + 1) \geq \frac{1}{4}(n + 1).$$

Case 4: *S_z is big and at least one small square is packed into I_2 and no small square is packed into $I_3 \cup \cdots \cup I_n$.*

Subcase 4A: *a small square is packed into I_2 outside the union of the subbricks $(3, 0, 3)$ and $(3, 0, 4)$.* By the considerations of Case 1, the total area of the squares packed into I_1 plus the total area of the small squares packed into I_2 is greater than $\frac{4}{9}$.

If there is a big square packed into I_2, then the total area of the squares packed into $I_1 \cup I_2$ is greater than $\frac{5}{9}$. Consequently, by the Lemma,

$$\zeta > \frac{5}{9} + \frac{1}{4}(n - 2 + 1) > \frac{1}{4}(n + 1).$$

Denote by W the union of four subbricks: $(3, 0, 3)$, $(3, 0, 4)$, $(4, 0, 17)$ and $(4, 0, 18)$. If there is a small square packed into I_2 outside W, then the total area of squares packed into $I_1 \cup I_2$ is greater than $\frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$. Hence, by the Lemma,

$$\zeta > \frac{1}{2} + \frac{1}{4}(n - 2 + 1) = \frac{1}{4}(n + 1).$$

If no big square is packed into I_2 and if no small square is packed into I_2 outside W, then all big squares packed into I_j, for $j \in \{3, \ldots, n\}$ (and S_z) have side lengths greater than $\frac{2}{3}$. Hence $\zeta > \frac{4}{9} + \frac{4}{9}(n - 1) > \frac{1}{4}(n + 1)$.

Subcase 4B: *all small squares packed into I_2 are contained in the union of the subbricks $(3, 0, 3)$ and $(3, 0, 4)$.* If there is a big square S_u packed into I_1 and a big square S_v packed into I_2 such that $s_u + s_v > 1$, then, by the Lemma,

$$\zeta > s_u^2 + s_v^2 + \frac{1}{4}(n - 2 + 1) > \frac{1}{2} + \frac{1}{4}(n - 1) = \frac{1}{4}(n + 1).$$

Consider the opposite case.

If no big square is packed into I_2, then the side length of each big square packed into $I_3 \cup \cdots \cup I_n$ (and the side length of S_z) is greater than $\frac{2}{3}$. Moreover, the total area of the squares packed into $I_1 \cup I_2$ is greater than $\frac{1}{9}$. This implies that

$$\zeta > \frac{1}{9} + \frac{4}{9}(n - 1) \geq \frac{1}{4}(n + 1).$$

Assume that at least two big squares are packed into I_2.

Denote by ξ the total area of the squares packed into $I_1 \cup I_2$. We show that $\xi > \frac{1}{2}$. If either three or four big squares are packed into I_1, then $\xi > 3 \cdot \frac{1}{9} + \frac{2}{9} > \frac{1}{2}$. If two big squares are packed into I_1, then the total area of the small squares is greater than $\frac{1}{3} \cdot 2 \cdot \frac{1}{6}$. Consequently, $\xi > 2 \cdot \frac{1}{9} + \frac{1}{9} + \frac{2}{9} > \frac{1}{2}$. If one big square is packed into I_1, then a small square is packed into I_1 outside the union of the subbricks $(3, 0, 1)$ and $(3, 0, 2)$. Consequently, the total area of the small squares is greater than $\frac{2}{9}$ and $\xi > \frac{1}{9} + \frac{2}{9} + \frac{2}{9} > \frac{1}{2}$. If no big square is packed into I_1, then the total area of the small squares is greater than $\frac{1}{3} \cdot 5 \cdot \frac{1}{6}$ and $\xi > \frac{5}{18} + \frac{2}{9} = \frac{1}{2}$. By the Lemma we deduce that

$$\zeta > \xi + \frac{1}{4}(n - 2 + 1) > \frac{1}{4}(n + 1).$$

Finally, assume that exactly one big square S_v is packed into I_2.

If $s_v \geq \frac{2}{3}$, then the total area of the squares packed into $I_1 \cup I_2$ is greater than $\frac{1}{9} + \frac{4}{9}$ and, by the Lemma, $\zeta > \frac{5}{9} + \frac{1}{4}(n - 2 + 1) > \frac{1}{4}(n + 1)$.

Assume that $s_v < \frac{2}{3}$. This implies that the total area of the squares packed into $I_1 \cup I_2$ is greater than $\frac{1}{4} + s_v^2$ (if no big square is packed into I_1, then the total area of the small squares is greater than $\frac{1}{3}(4 \cdot \frac{1}{6} + \frac{1}{12}) = \frac{1}{4}$).

If $s_v \geq \frac{1}{2}$, then

$$\zeta > \frac{1}{4} + s_v^2 + \frac{1}{4}(n - 2 + 1) \geq \frac{1}{4}(n + 1).$$

If $s_v < \frac{1}{2}$, then the side length of each big square packed into $I_3 \cup \cdots \cup I_n$ (and the side length of S_z) is greater than $1 - s_v$. It is easy to verify that

$$s_v^2 + (n - 1)(1 - s_v)^2 \geq \frac{1}{4}n.$$

The total area of the squares packed into I_1 plus the total area of the small squares packed into I_2 is greater than $\frac{1}{4}$.

Consequently,

$$\zeta > \frac{1}{4} + \frac{1}{4}n = \frac{1}{4}(n + 1). \ \blacksquare$$

It remains an open question whether $n \geq 3$ can be replaced by $n \geq 1$ in the statement of the Theorem.

References

[1] K. Böröczky, Jr., *Finite Packing and Covering*, Cambridge Tracts in Math. 154, Cambridge Univ. Press, Cambridge, 2004.

[2] G. Fejes Tóth and W. Kuperberg, *Packing and covering with convex sets*, in: Handbook of Convex Geometry, P. M. Gruber and J. M. Wills (eds.), North-Holland, 1993, 799–860.

[3] X. Han, K. Iwama and G. Zhang, *Online removable square packing*, Theory Comput. Syst. 43 (2008), 38–55.

[4] J. Januszewski and M. Lassak, *On-line packing sequences of cubes in the unit cube*, Geom. Dedicata 67 (1997), 285–293.

[5] M. Lassak, *A survey of algorithms for on-line packing and covering by sequences of convex bodies*, in: Intuitive Geometry (Budapest, 1995), Bolyai Soc. Math. Stud. 6, János Bolyai Math. Soc., Budapest, 1997, 129–157.

[6] J. W. Moon and L. Moser, *Some packing and covering theorems*, Colloq. Math. 17 (1967), 103–110.

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