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PARTIAL DIFFERENTIAL EQUATIONS

An Elliptic Neumann Problem with Subcritical Nonlinearity

by

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Summary. We establish the existence of a solution to the Neumann problem in the half-space with a subcritical nonlinearity on the boundary. Solutions are obtained through the constrained minimization or minimax. The existence of solutions depends on the shape of a boundary coefficient.

1. Introduction. Let $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0,\infty)$. For a point $x \in \mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0,\infty)$ we use the notation $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N > 0$. In this paper we consider a semilinear Neumann problem in $H^1(\mathbb{R}^N_+)$, $N > 2$,

$$
\begin{cases} -\Delta u + u = 0 \quad \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u(x',0)}{\partial x_N} = b(x')u^{p-1}(x',0) \quad \text{on } \mathbb{R}^{N-1}, \quad u > 0 \quad \text{on } \mathbb{R}^N_+, \end{cases} \tag{1.1}
$$

where $p \in (2, 2(N-1)/(N-2))$ and $b \in L^\infty(\mathbb{R}^{N-1})$. It is well known that the trace embedding of the Sobolev space $H^1(\mathbb{R}^N_+)$ into $L^p(\mathbb{R}^{N-1})$, $p \in (2, 2(N-1)/(N-2))$ is continuous but not compact. The norm in $H^1(\mathbb{R}^N_+)$ is defined by

$$\|u\|^2 = \int_{\mathbb{R}^N_+} (|\nabla u|^2 + u^2)\,dx.$$

It is assumed that $\lim_{|x'|\to\infty} b(x') = b_\infty > 0$.

In this paper we prove existence when (i) $b(x') > b_\infty$ on $\mathbb{R}^{N-1}$ or (ii) $b(x') > m^{-(p-2)/2} b_\infty$ on $\mathbb{R}^{N-1}$, provided that b is invariant with respect to a finite

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subgroup of $O(\mathbb{R}^{N-1})$ of cardinality m acting freely on $\mathbb{R}^{N-1}\setminus\{0\}$. We also consider the case when the above penalty condition is reversed: $b(x') < b_\infty$ on $\mathbb{R}^{N-1}$. However, in this case we only present a partial result (see Theorem 1.4) which depends on the convexity of $b(x')$.

The main results of this paper are the following:

THEOREM 1.1. *Suppose that $b(x')$ is a $\mathbb{Z}^{N-1}$-periodic function. Then problem* (1.1) *admits a solution.*

THEOREM 1.2. *Suppose that $b \in L^\infty(\mathbb{R}^{N-1})$ and that $b_\infty < b(x')$ on $\mathbb{R}^{N-1}$. Then problem* (1.1) *admits a solution.*

THEOREM 1.3. *Suppose that $b(x')$ is invariant with respect to a finite subgroup $G \subset O(\mathbb{R}^{N-1})$ of cardinality m acting freely on $\mathbb{R}^{N-1}\setminus\{0\}$ and that*

$$b(x') > m^{-(p-2)/2} b_\infty \quad \text{for } x' \in \mathbb{R}^{N-1}. \tag{1.2}$$

Then problem (1.1) *admits a G-invariant solution.*

The proofs of Theorems 1.1 and 1.2 are standard. Solutions are obtained as multiples of minimizers of the constrained minimization problem

$$c_b = \inf_{u\in H^1(\mathbb{R}^N_+),\, \int_{\mathbb{R}^{N-1}} b(x')|u(x',0)|^p\,dx'=1} \int_{\mathbb{R}^N_+} (|\nabla u|^2 + u^2)\,dx. \tag{1.3}$$

In the case of the proof of Theorem 1.3 the space $H^1(\mathbb{R}^N_+)$ in the above minimization problem will be replaced by a subspace of G-invariant functions in x'. Similar results are known for the equation

$$-\Delta u + u = |u|^{p-2}u \quad \text{on } \mathbb{R}^N,$$

where $1 < p < 2N/(N-2)$ (see [4], [6]).

THEOREM 1.4. *Assume that $b \in L^\infty(\mathbb{R}^{N-1})$ is such that*

$$b(x') < b_\infty \quad \text{for } x \in \mathbb{R}^{N-1}. \tag{1.4}$$

Then there exists a finite set $Y \subset \mathbb{Z}^{N-1}$ and $c_{y'} \in [0,1]$, $y' \in Y$, $\sum_Y c_{y'} = 1$, such that problem (1.1) *with $b^Y(x') = \sum_Y c_{y'} b(x'-y')$ in place of $b(x')$ has a solution.*

Note that $b^Y(x') < b^Y_\infty = b_\infty$. We do not know if existence holds for every b, or whether convexity is essential for the existence. If b is radially symmetric, problem (1.1) admits a solution radially symmetric in the variables x' obtained as a multiple of a minimizer of the problem

$$\inf_{\int_{\mathbb{R}^{N-1}} b(x')|u(x',0)|^p\,dx'=1,\, u\in H^1_{\mathrm{r}}(\mathbb{R}^N_+)} \int_{\mathbb{R}^N_+} (|\nabla u|^2 + u^2)\,dx,$$

where $H^1_{\mathrm{r}}(\mathbb{R}^N_+)$ is a subspace of $H^1(\mathbb{R}^N_+)$ consisting of functions radially symmetric in x'. The existence of a minimizer follows from the compactness of the trace embedding of $H^1_{\mathrm{r}}(\mathbb{R}^N_+)$ into a subspace of radially symmetric functions in $L^p(\mathbb{R}^{N-1})$, $p \in (2, 2^{2(N-1)/(N-2)})$.

2. Global compactness. Theorem 2.2 below is a particular case of the functional-analytic global compactness theorem from [5], applied to the Sobolev space $H^1(\mathbb{R}^N_+)$, $N > 2$, with the norm $\|\cdot\|$ and the dislocations defined by shifts $u \mapsto u(\cdot - y', \cdot)$, $y' \in \mathbb{Z}^{N-1}$. The derivation of this particular case is completely analogous to the case of $H^1(\mathbb{R}^N)$ with shifts by $y \in \mathbb{Z}^N$ elaborated in [5], once one takes into account the following statement, close to the one from [3], which deals with convergence in $L^p(\mathbb{R}^N)$.

LEMMA 2.1. *Let u_k be a bounded sequence in $H^1(\mathbb{R}^N_+)$ and let $p \in (2, 2(N-1)/(N-2))$. Then $u_k(\cdot + y'_k, \cdot) \rightharpoonup 0$ for all $y'_k \in \mathbb{Z}^{N-1}$ implies $\|u_k\|_{L^p(\mathbb{R}^{N-1})} \to 0$.*

Proof. Assume that $u_k(\cdot + y'_k, \cdot) \rightharpoonup 0$ for any $y'_k \in \mathbb{Z}^{N-1}$. Consider a unit cube $Q := (0,1)^{N-1}$. By the trace inequality for bounded domains, there is a $C > 0$ such that

$$\tag{2.1} \int_{Q+y'} |u_k(x',0)|^p\,dx' \le C\|u_k\|^2_{H^1((Q+y')\times(0,\infty))}\Big(\int_{Q+y'} |u_k(x',0)|^p\,dx'\Big)^{1-2/p}$$

for all $y' \in \mathbb{Z}^{N-1}$. By adding (2.1) over $y' \in \mathbb{Z}^{N-1}$, and noting that the union $\bigcup_{y'\in\mathbb{Z}^{N-1}}(Q+y')$ is $\mathbb{R}^{N-1}$ up to a set of measure zero, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} |u_k(x',0)|^p\,dx' &\le C\|u_k\|^2 \sup_{y'\in\mathbb{Z}^{N-1}}\Big(\int_Q |u_k(x'+y',0)|^p\,dx'\Big)^{1-2/p}\\ &\le 2C\|u_k\|^2\Big(\int_Q |u_k(x'+y'_k,0)|^p\,dx'\Big)^{1-2/p}\end{aligned}$$

where $y'_k \in \mathbb{Z}^{N-1}$ is any sequence satisfying

$$\tag{2.2} \Big(\int_Q |u_k(x'+y'_k,0)|^p\,dx'\Big)^{1-2/p} \ge \frac{1}{2}\sup_{y'\in\mathbb{Z}^{N-1}}\Big(\int_Q |u_k(x'+y',0)|^p\,dx'\Big)^{1-2/p}.$$

It remains to note that by compactness of the trace of $H^1(Q\times(0,\infty))$ into $L^p(Q)$, one has $u_k(\cdot + y'_k, 0) \to 0$ in $L^p(\mathbb{R}^{N-1})$, so that the assertion of the lemma follows from (2.2). ∎

THEOREM 2.2. *Let $\{u_k\} \subset H^1(\mathbb{R}^N_+)$ be a bounded sequence. Then there exist $w^{(n)} \in H^1(\mathbb{R}^N_+)$, $y_k^{(n)'} \in \mathbb{Z}^{N-1}$, $y_k^{(1)'} = 0$, with $k, n \in \mathbb{N}$, such that for a* *relabelled subsequence,*

$$w^{(n)} = \underset{k\to\infty}{w\text{-lim}}\, u_k(\cdot + y_k^{(n)'}, \cdot), \tag{2.3}$$

$$|y_k^{(n)'} - y_k^{(m)'}| \to 0 \qquad \text{as } k \to \infty \text{ for } n \neq m, \tag{2.4}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^2 \leq \limsup \|u_k\|^2, \tag{2.5}$$

$$u_k - \sum_{n\in\mathbb{N}} w^{(n)}(\cdot - y_k^{(n)'}, \cdot) \to 0 \qquad \text{in } L^p(\mathbb{R}^{N-1}) \quad \text{as } k \to \infty,\ p \in \left(2, \frac{2(N-1)}{N-2}\right), \tag{2.6}$$

where the series $\sum_{n\in\mathbb{N}} w^{(n)}(\cdot - y_k^{(n)'}, \cdot)$ *converges uniformly in* k*.*

The following lemma is a variant of the Brézis–Lieb lemma from [1].

LEMMA 2.3. *Let* $b \in L^\infty(\mathbb{R}^{N-1})$ *and assume that* $b(x') \to b_\infty \in \mathbb{R}$ *as* $|x'| \to \infty$*. Let* u_k*,* $w^{(n)}$*, and* $y_k^{(n)'}$ *be as in Theorem* 2.2*. Then for every* $p \in (2, 2(N-1)/(N-2))$*,* $y' \in \mathbb{Z}^{N-1}$*,*

$$\lim_{k\to\infty} \int_{\mathbb{R}^{N-1}} b(x')|u_k(x'+y', 0)|^p\, dx' = \int_{\mathbb{R}^{N-1}} b(x')|w^{(1)}(x'+y', 0)|^p\, dx' + \sum_{n\geq 2} \int_{\mathbb{R}^{N-1}} b_\infty |w^{(n)}(x', 0)|^p\, dx' \tag{2.7}$$

and the convergence is uniform in y'*.*

Proof. First we note that the statement easily reduces to the case $y = 0$ due to the convergence of $b(x')$ to b_∞ as $|x'| \to \infty$, once one considers the left hand side of (2.7) as $\lim_{k\to\infty} \int_{\mathbb{R}^{N-1}} b(x'-y')|u_k(x', 0)|^p\, dx'$. For the case $y = 0$ we give a sketch of the proof only, since similar statements have been proved several times elsewhere. In view of Lemma 2.1 we may assume that $u_k = \sum_{n\in\mathbb{N}} w^{(n)}(\cdot - y_k^{(n)'}, \cdot)$. Since the series is absolutely convergent and $u \mapsto \int_{\mathbb{R}^{N-1}} b(x')|u(x', 0)|^p\, dx'$ is continuous in $H^1(\mathbb{R}^N_+)$, it suffices to prove the lemma if the sum has finitely many terms. By density of $C_0^\infty(\mathbb{R}^N)|_{\mathbb{R}^N_+}$ in $H^1(\mathbb{R}^N_+)$, it suffices to prove the lemma when $w^{(n)} \in C_0^\infty(\mathbb{R}^N)|_{\mathbb{R}^N_+}$. Since $|y_k^{(n)'} - y_k^{(m)'}| \to \infty$ for $m \neq n$, there is a k_0 such that for all $k \geq k_0$ all the functions $w^{(n)}(\cdot - y_k^{(n)'}, \cdot)$ have disjoint supports. In this case

$$\int_{\mathbb{R}^{N-1}} b(x')|u_k(x', 0)|^p\, dx' = \sum_{n\geq 1} \int_{\mathbb{R}^{N-1}} b(x' + y_k^{(n)'})|w^{(n)}(x', 0)|^p\, dx' \to \int_{\mathbb{R}^{N-1}} b(x')|w^{(1)}(x', 0)|^p\, dx' + \sum_{n\geq 2} \int_{\mathbb{R}^{N-1}} b_\infty |w^{(n)}(x', 0)|^p\, dx'. \tag{2.8}$$ ■

3. Proofs of Theorems 1.1–1.3. The results of Section 2 will now be applied to prove Theorems 1.1–1.3.

Proof of Theorem 1.1. Let $\{u_k\} \subset H^1(\mathbb{R}^N_+)$ be a minimizing sequence for the constant c_b with $\int_{\mathbb{R}^{N-1}} b(x')|u_k(x',0)|^p\,dx' = 1$ for each k. We apply Theorem 2.2 with dislocations $g_{y'_k} : u \mapsto u(\cdot + y'_k, \cdot)$, $y'_k \in \mathbb{Z}^{N-1}$. Let $\{u_k\}$, $\{w^{(n)}\}$ and $\{{y'_k}^{(n)}\}$ be subsequences generated by Theorem 2.2. According to Theorem 2.2, since $b(x')$ is periodic, we have

$$1 = \int_{\mathbb{R}^{N-1}} b(x')|u_k(x',0)|^p\,dx' = \sum_n \int_{\mathbb{R}^{N-1}} b(x')|w^{(n)}(x',0)|^p\,dx'. \tag{3.1}$$

It follows from (2.5) that

$$\sum_n \|w^{(n)}\|^2 \le c_b. \tag{3.2}$$

We now set $\int_{\mathbb{R}^{N-1}} |w^{(n)}(x',0)|^p\,dx' = t_n$. Obviously we have $\|t_n^{-1/p} w^{(n)}\| \ge c_b$, which yields $\|w^{(n)}\|^2 \ge c_b t_n^{2/p}$. Applying this to (3.2), we get

$$\sum_n t_n^{2/p} \le 1. \tag{3.3}$$

On the other hand, we deduce from (3.1) that $\sum_n t_n = 1$. Since $2/p < 1$, the last relation and (3.3) can only hold if exactly one term t_n, say $t_{n_\circ}$, is nonzero and $t_n = 0$ for all $n \ne n_\circ$. This yields $\|w^{(n_\circ)}\|^2 = c_b$ and hence $w^{(n_\circ)}$ is a minimizer. ∎

Corollary 3.1. *Let $b(x') = 1$ on $\mathbb{R}^{N-1}$. Then there exists a minimizer for c_b.*

We now consider the case $b(x') > b_\infty$ on $\mathbb{R}^{N-1}$.

Proof of Theorem 1.2. Let $c_\infty = c_b$ with $b(x') \equiv b_\infty$. By Corollary 3.1 the constant c_∞ is attained on a positive function v. Hence

$$\begin{aligned} c_b &\le \frac{\int_{\mathbb{R}^N_+} (|\nabla v|^2 + v^2)\,dx}{(\int_{\mathbb{R}^{N-1}} b(x')|v(x',0)|^p\,dx')^{2/p}} \\ &< \frac{\int_{\mathbb{R}^N_+} (|\nabla v|^2 + v^2)\,dx}{(\int_{\mathbb{R}^{N-1}} b_\infty|v(x',0)|^p\,dx')^{2/p}} = c_\infty. \end{aligned} \tag{3.4}$$

Let $\{u_k\}$ be a minimizing sequence for c_b. We may assume that $u_k \rightharpoonup w$ in $H^1(\mathbb{R}^N_+)$ and also $u_k \rightharpoonup w$ in $L^p(\mathbb{R}^{N-1})$. Setting

$$a(u) = \int_{\mathbb{R}^N_+} (|\nabla u|^2 + u^2)\,dx \quad \text{and} \quad v_k = u_k - w$$

we can write

$$c_b = a(w) + a(v_k) + o(1)$$

up to a subsequence and by the Brézis–Lieb lemma [1] we also have

$$1 = \int_{\mathbb{R}^{N-1}} b(x')|u_k(x',0)|^p\,dx' = \int_{\mathbb{R}^{N-1}} b(x')|v_k(x',0)|^p\,dx' + \int_{\mathbb{R}^{N-1}} b(x')|w(x',0)|^p\,dx' + o(1).$$

We deduce from the last two relations that

$$\begin{aligned} c_b &\geq c_b\Big(\int_{\mathbb{R}^{N-1}} b(x')|w(x',0)|^p\,dx'\Big)^{2/p} \\ &\quad + c_b\Big(\int_{\mathbb{R}^{N-1}} b(x')|v_k(x',0)|^p\,dx'\Big)^{2/p} + o(1) \\ &= c_b\Big(\int_{\mathbb{R}^{N-1}} b(x')|w(x',0)|^p\,dx'\Big)^{2/p} \\ &\quad + \Big(1 - \int_{\mathbb{R}^{N-1}} b(x')|w(x',0)|^p\,dx'\Big)^{2/p} + o(1). \end{aligned}$$

We therefore have either

(i) $\int_{\mathbb{R}^{N-1}} b(x')|w(x',0)|^p\,dx' = 1$ or
(ii) $\int_{\mathbb{R}^{N-1}} b(x')|w(x',0)|^p\,dx' = 0$.

We show that (ii) cannot occur. Indeed, if $\int_{\mathbb{R}^{N-1}} b(x')|w(x',0)|^p\,dx' = 0$, then $u_k \rightharpoonup 0$ in $H^1(\mathbb{R}^N_+)$ and in $L^p(\mathbb{R}^{N-1})$ (in the sense of traces). Since $b(x') \to b_\infty$ as $|x'| \to \infty$, we get

$$1 = \int_{\mathbb{R}^{N-1}} b(x')|u_k(x',0)|^p\,dx' = \int_{\mathbb{R}^{N-1}} b_\infty|u_k(x',0)|^p\,dx' + o(1).$$

This yields $c_\infty \leq c_b$, which contradicts (3.4). Hence case (i) holds and w is a minimizer for c_b. ■

To prove Theorem 1.3, we introduce the subspace $H^1_G(\mathbb{R}^N_+)$ of $H^1(\mathbb{R}^N_+)$ defined by

$$H^1_G(\mathbb{R}^N_+) = \{u \in H^1(\mathbb{R}^N_+) : u \circ \gamma = u \text{ for all } \gamma \in G\}$$

and set

$$c_{b,G} = \sup_{\|u\|=1,\, u\in H^1_G(\mathbb{R}^N_+)} \int_{\mathbb{R}^{N-1}} b(x')|u(x',0)|^p\,dx'.$$

We also need

$$c_{\infty,G} = \sup_{\|u\|=1,\, u\in H^1_G(\mathbb{R}^N_+)} \int_{\mathbb{R}^{N-1}} b_\infty|u(x',0)|^p\,dx'.$$

Observe that

$$c_{\infty,G} = c_\infty := \sup_{\|u\|=1,\, u\in H^1(\mathbb{R}^N_+)} \int_{\mathbb{R}^{N-1}} b_\infty |u(x',0)|^p \, dx'. \tag{3.5}$$

Indeed, $c_{\infty,G} \le c_\infty$ since $H^1_G(\mathbb{R}^N_+) \subset H^1(\mathbb{R}^N_+)$. Moreover, the standard argument based on spherical decreasing rearrangements (with respect to the $\mathbb{R}^{N-1}$-variable) implies that c_∞ is attained on a radially symmetric function, that is, on $H^1_G(\mathbb{R}^N_+)$, and (3.5) is immediate. It then follows from (1.2) and (3.5) that

$$c_\infty < m^{(p-2)/2} c_{b,G}. \tag{3.6}$$

Proof of Theorem 1.3. Let $\{u_k\} \subset H^1_G(\mathbb{R}^N_+)$ be a maximizing sequence for the constant $c_{b,G}$. We apply Theorem 2.2 to the sequence $\{u_k \circ \gamma\}$, $\gamma \in G$. We have, by the G-invariance,

$$\operatorname*{w\text{-}lim}_{k\to\infty} u_k(\cdot + \gamma y_k^{(n)'}, \cdot) = \operatorname*{w\text{-}lim}_{k\to\infty} u_k(\gamma^{-1}\cdot + y_k^{(n)'}, \cdot) = w^{(n)} \circ \gamma^{-1}. \tag{3.7}$$

Let $n > 1$. Since G is a finite group whose nontrivial elements have no fixed points, $|\gamma y_k^{(n)'} - \gamma' y_k^{(n)'}| \to \infty$ whenever $\gamma \ne \gamma'$. Hence there are m distinct terms of the form $w^{(n)}(\cdot + \gamma, \cdot)$, $\gamma \in G$, in the expansion (2.6). Therefore (2.6) takes the form

$$u_k - w^{(1)} - \sum_{n>1,\,\gamma\in G} w^{(n)}(\cdot + \gamma y_k^{(n)}, \cdot) \to 0. \tag{3.8}$$

It is easy to see that

$$\|w^{(1)}\|^2 + m \sum_{n>1} \|w^{(n)}\|^2 \le 1 \tag{3.9}$$

and

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} b(x')|w^{(1)}(x',0)|^p \, dx' + m \sum_{n>1} \int_{\mathbb{R}^{N-1}} b_\infty |w^{(n)}(x',0)|^p \, dx' \\ = \lim_{k\to\infty} \int_{\mathbb{R}^{N-1}} b(x')|u(x',0)|^p \, dx' = c_{b,G}. \end{aligned} \tag{3.10}$$

Let $t_1 = \|w^{(1)}\|^2$ and $t_n = m\|w^{(n)}\|^2$ for $n > 1$. Then the relation (3.9) takes the form

$$\sum_{n\ge1} t_n \le 1. \tag{3.11}$$

On the other hand, using (3.6), the definitions of the quantities $c_{b,G}$, c_∞ and $c_{\infty,G}$, as well as (3.5) we derive the following inequality:

$$c_{b,G} \le c_{b,G} t_1^{p/2} + m c_{\infty,G} \sum_{n>1} t_n^{p/2} m^{-p/2} \le c_{b,G} t_1^{p/2} + c_{b,G} \sum_{n>1} t_n^{p/2}.$$

This yields

$$\sum_n t_n^{p/2} \geq 1,$$

which combined with (3.11) implies that only one term t_n is nonzero, say $t_{n_\circ}$. It follows from (3.9) and (3.10) that $n_\circ = 1$. ∎

4. Problem with the reverse penalty. In this section we prove Theorem 1.4. Let

$$c_b := \sup_{\|u\|\leq 1} \inf_{y' \in \mathbb{Z}^{N-1}} \int b(x')|u(x'-y', x_N)|^p\, dx. \tag{4.1}$$

Let u_k be a sequence satisfying, with some $y_k' \in \mathbb{Z}^{N-1}$, $\|u_k\| \leq 1$,

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} & b(x')|u_k(x'+y',0)|^p dx' \\ & \geq \int_{\mathbb{R}^{N-1}} b(x')|u_k(x'+y_k',0)|^p dx' \to c_b, \qquad y' \in \mathbb{Z}^{N-1}. \end{aligned} \tag{4.2}$$

We will call any such sequence a maximizing sequence. Note that $|u_k|$ is then also a maximizing sequence, and in what follows we assume that $u_k \geq 0$. Moreover, $u_k(\cdot - y_k', \cdot)$ is also a maximizing sequence corresponding to $y_k' = 0$, so without loss of generality we set $y_k' = 0$. Let us apply Theorem 2.2, noting that since $u_k \geq 0$, all translated weak limits $w^{(n)}$ are non-negative.

Passing to the limit in (4.2) with $y' = y_k^{(m)'} + z'$, $z' \in \mathbb{Z}^{N-1}$, we obtain from Lemma 2.3,

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} & (b(x') - b_\infty)|w^{(m)}(x'+z',0)|^p\, dx' \\ & \qquad + \sum_n \int_{\mathbb{R}^{N-1}} b_\infty |w^{(n)}(x',0)|^p\, dx' \\ \geq & \int_{\mathbb{R}^{N-1}} (b(x') - b_\infty)|w^{(1)}(x',0)|^p\, dx' + \sum_n \int_{\mathbb{R}^{N-1}} b_\infty |w^{(n)}(x',0)|^p\, dx' = c_b. \end{aligned} \tag{4.3}$$

Note that $w^{(1)} \neq 0$, for if it were zero, (4.3) would imply that $w^{(m)} = 0$ for every m, which yields $c_b = 0$. This is a contradiction. Note also that (4.3) with $m = 1$ implies

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} & b(x')|w^{(1)}(x',0)|^p\, dx' \\ & \leq \int_{\mathbb{R}^{N-1}} b(x')|w^{(1)}(x'+y',0))|^p\, dx', \qquad y' \in \mathbb{Z}^{N-1}. \end{aligned} \tag{4.4}$$

Let $Y \subset \mathbb{Z}^{N-1}$ be the set of y' for which equality holds in (4.4). Note that Y is finite, since

$$\lim_{|y'|\to\infty} \int_{\mathbb{R}^{N-1}} b(x')|w^{(1)}(x'+y',0)|^p = \int_{\mathbb{R}^{N-1}} b_\infty |w^{(1)}(x',0)|^p \, dx'$$
$$> \int_{\mathbb{R}^{N-1}} b(x')|w^{(1)}(x',0)|^p \, dx'.$$

Let $g_{y'}(u) = \int b(x'-y')|u(x',0)|^p \, dx' \in C^1(H^1(\mathbb{R}^N_+))$. Assume that the function $w^{(1)} \in H^1(\mathbb{R}^N_+)$ does not belong to the positive cone generated by $g'_{y'}(w^{(1)})$, $y' \in Y$. Then there exists a function $v \in C_0^\infty(\mathbb{R}^N)|_{\mathbb{R}^N_+}$ with $\|v\| = 1$ and an $\varepsilon > 0$ such that $(w^{(1)}, v) < -2\varepsilon$ and $(g'_{y'}(w^{(1)}), v) > 2\varepsilon$. Consider now a sequence $u_k + tv$, $t > 0$. Then $\|u_k + tv\|^2 \le \|u_k\|^2 + t^2 + 2t(u_k, v) \le 1 + t^2 - 4\varepsilon t \le 1$ if $t \le 4\varepsilon$ and for all t sufficiently small the functional $g_{y'}(u_k + tv)$ satisfies

$$\begin{aligned}
g_{y'}(u_k + tv) &= \int b(x'-y')|(u_k + tv)(x',0)|^p \, dx' \\
&= \int b(x'-y')|(w^{(1)} + tv)(x',0)|^p \, dx' + \sum_{n\ge 2} \int b_\infty |w^{(n)}(x',0)|^p \, dx' + o(1) \\
&\ge \sum_{n\ge 2} \int b_\infty |w^{(n)}(x',0)|^p \, dx' + \int b(x'-y')|w^{(1)}(x',0)|^p \, dx' + \varepsilon t + o(1) \\
&= c_b + \varepsilon t + o(1).
\end{aligned}$$

Hence there is a $t_0 > 0$ and a $k(t)$ such that for every $k \ge k(t)$ and $0 < t < t_0$,

$$g_{y'}(u_k + tv) \ge c_b + \frac{1}{2}\varepsilon t. \tag{4.5}$$

Suppose that $y' \notin Y$. Let

$$\delta := \inf_{y \in \mathbb{Z}^{N-1}\setminus Y} \int b(x'-y')|w^{(1)}(x',0)|^p \, dx' - \int b(x')|w^{(1)}(x',0)|^p \, dx'. \tag{4.6}$$

In view of (4.4), $\delta \ge 0$. Since

$$\begin{aligned}
\lim_{|y'|\to\infty} \int b(x'-y')|w^{(1)}(x',0)|^p \, dx' &= \int b_\infty |w^{(1)}(x',0)|^p \, dx' \\
&> \int b(x')|w^{(1)}(x',0)|^p \, dx',
\end{aligned} \tag{4.7}$$

the mapping $y' \mapsto \int b(x'-y')|w^{(1)}(x',0)|^p \, dx' - \int b(x')|w^{(1)}(x',0)|^p \, dx'$ has a point of minimum over $y' \in \mathbb{Z}^{N-1} \setminus Y$, and by definition of Y the minimal value cannot be zero.

Then

$$\begin{aligned} g_{y'}(u_k+tv) &= \int b(x'-y')|(u_k+tv)(x',0)|^p\,dx' \\ &= \int b(x'-y')|(w^{(1)}+tv)(x',0)|^p\,dx' + \sum_{n\geq 2}\int b_\infty|w^{(n)}(x',0)|^p\,dx' + o(1) \\ &\geq \sum_{n\geq 2}\int b_\infty|w^{(n)}(x',0)|^p\,dx' + \int b(x'-y')|w^{(1)}(x',0)|^p\,dx' + Ct + o(1) \\ &\geq c_b + \delta - Ct + o(1). \end{aligned}$$

Note that the $o(1)$ term is uniform in t and y (the latter due to Lemma 2.3), so that there is a $t > 0$ such that for every k sufficiently large, $g_{y'}(u_k+tv) > c_b + \frac{1}{2}\delta$ if $y' \notin Y$. Combining this with a similar estimate for $y' \in Y$, we deduce that for some k and t, $\inf_{y'\in\mathbb{Z}^{N-1}} g_{y'}(u_k+tv) > c_b$. This is a contradiction. Thus u is in the convex hull of $g'_{y'}$, which yields (1.3). ∎

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