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MATHEMATICAL LOGIC AND FOUNDATIONS

On BPI Restricted to Boolean Algebras of Size Continuum

by

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Summary. We show:

(i) The statement $\mathbf{P}(\omega)$ = "every partition of $\mathbb{R}$ has size $\leq |\mathbb{R}|$" is equivalent to the proposition $\mathbf{R}(\omega)$ = "for every subspace Y of the Tychonoff product $\mathbf{2}^{\mathcal{P}(\omega)}$ the restriction $\mathcal{B}|Y = \{Y \cap B : B \in \mathcal{B}\}$ of the standard clopen base $\mathcal{B}$ of $\mathbf{2}^{\mathcal{P}(\omega)}$ to Y has size $\leq |\mathcal{P}(\omega)|$".
(ii) In **ZF**, $\mathbf{P}(\omega)$ does not imply "every partition of $\mathcal{P}(\omega)$ has a choice set".
(iii) Under $\mathbf{P}(\omega)$ the following two statements are equivalent:
   (a) For every Boolean algebra of size $\leq |\mathbb{R}|$ every filter can be extended to an ultrafilter.
   (b) Every Boolean algebra of size $\leq |\mathbb{R}|$ has an ultrafilter.

1. Notation and terminology. Let $\mathbf{X} = (X, T)$ be a topological space. We shall denote topological spaces by boldface letters and underlying sets by lightface letters.

$\mathbf{X}$ is said to be *compact* iff every open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$. Equivalently, $\mathbf{X}$ is compact iff every family $\mathcal{G}$ of closed subsets of $\mathbf{X}$ with the *finite intersection property*, fip for abbreviation, has a non-empty intersection.

A subset A of X is called a *regular open set* if $A = \mathrm{int}(A)$. It is known that the set of all regular open sets of $\mathbf{X}$ forms a Boolean algebra under the following set of operations:

- $1 = X$ and $0 = \emptyset$,
- $U \wedge V = U \cap V$,

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- $U \vee V = \mathrm{int}(\overline{U \cup V})$,
- $U' = X \setminus \overline{U}$.

Given a set X, we introduce the following notions and notations:

1. **BPI**(X): Every filter of X is included in an ultrafilter of X.
2. **UF**(X): There is a free ultrafilter on X.
3. **P**(X): Every partition of $\mathcal{P}(X)$ has size $\leq |\mathcal{P}(X)|$.
4. A family $\mathcal{A}$ of subsets of X is called *independent* iff for any two non-empty finite, disjoint subsets $\mathcal{C}, \mathcal{B} \subseteq \mathcal{A}$ the set $\bigcap \mathcal{C} \cap \bigcap \{B^c : B \in \mathcal{B}\}$ is infinite.
5. A family $\mathcal{A} = \{A_i : i \in I\} \subseteq \mathcal{P}(X)$ is *almost disjoint* iff each A_i is infinite and for all $i, j \in I$, $i \neq j$, $|A_i \cap A_j| < \aleph_0$.
6. $\mathbf{2}^X$ denotes the Tychonoff product of the discrete space $\mathbf{2}$ $(2 = \{0, 1\})$ and
$$\mathcal{B}(X) = \{[p] : p \in \mathrm{Fn}(X, 2)\},$$
where $\mathrm{Fn}(X, 2)$ is the set of all finite partial functions from X into 2, and
$$[p] = \{f \in 2^{\mathbb{R}} : p \subset f\}$$
will denote the standard (clopen) base for the product topology on $\mathbf{2}^X$.
7. $\mathcal{F}(X)$ will denote the set of all filters of X together with the topology $T_{\mathcal{F}}$ generated by the family
$$\mathcal{C}_X = \{[A] : A \in \mathcal{P}(X)\}, \quad \text{where} \quad [A] = \{\mathcal{F} \in \mathcal{F}(X) : A \in \mathcal{F}\}.$$
Since the function $H : \mathcal{P}(X) \to \mathcal{C}_X$, $H(A) = [A]$, is clearly $1:1$ and onto it follows that $|\mathcal{C}_X| = |\mathcal{P}(X)|$.
8. $S(X)$ will denote the *Stone space* of the Boolean algebra of all subsets of X, i.e., the set of all ultrafilters on X together with the topology it inherits as a subspace of $\mathcal{F}(X)$.
Even though $[A]$ for $A \in \mathcal{P}(X)$ may not be a closed set in $\mathcal{F}(X)$ (if $a \in A$, $b \in A^c$ and $C = \{a, b\}$ then $\mathcal{H}_C \notin [A]$ where $\mathcal{H}_C$ is the filter of all supersets of C; since for every basic neighborhood $[H]$ of $\mathcal{H}_C$, the filter $\mathcal{H}_{\{a\}}$ of all supersets of $\{a\}$ is in $[A] \cap [H]$, it follows that $[A]$ is not closed), it turns out that the restriction $[A] \cap S(X)$ is a closed subset of $S(X)$ and, in addition,
$$\mathcal{B}_X = \{[A] \cap S(X) : A \in \mathcal{P}(X)\}$$
is a (clopen) base for $S(X)$.
9. $S^*(X)$ will denote the subspace of all free ultrafilters of $S(X)$ and
$$\mathcal{B}_X^* = \{\langle A \rangle = [A] \cap S^*(X) : A \in \mathcal{P}(X)\}, \text{ where } \langle A \rangle = [A] \cap S^*(X),$$
is the restriction of the base $\mathcal{B}_X$ to $S^*(X)$. We point out here that $\neg$**UF**(X) implies $S^*(X) = \emptyset$ and consequently $\mathcal{B}_X^* = \emptyset$. Hence, $\mathcal{B}_X^* \neq \emptyset$ $\leftrightarrow$ **UF**(X).

10. $\sim$ will denote the equivalence relation on $\mathcal{P}(X)$ given by: $A \sim B$ iff $|A \triangle B| < \aleph_0$, where $\triangle$ denotes the operation of symmetric difference, and $\mathcal{P}(X)/\text{fin}$ stands for the quotient set of $\sim$. For every $A \in \mathcal{P}(X)$, (A) will denote the $\sim$ equivalence class of A, i.e., $(A) = \{B \in \mathcal{P}(X) : |A \triangle B| < \aleph_0\}$.
11. $\mathbf{BF}(X)$: For every $Y \subseteq \mathcal{F}(X), |\{[A] \cap Y : A \in \mathcal{P}(X)\}| \leq |\mathcal{P}(X)|$.
12. **BPI** (Boolean Prime Ideal Theorem, Form 14 in [6]): Every Boolean algebra has a prime ideal.

2. Introduction and preliminary results. There is a plethora of characterizations of **BPI** in several branches of mathematics. For most of these characterizations we refer the reader to the book by P. Howard and J. E. Rubin [6]. Well-known equivalents related to Boolean algebras are listed in the next theorem:

THEOREM 1. *The following are equivalent:*

(i) **BPI**.
(ii) *Every ideal J of a Boolean algebra $\mathcal{B}$ is a subset of a prime ideal I of $\mathcal{B}$.*
(iii) *Every Boolean algebra has an ultrafilter.*
(iv) *Every filter $\mathcal{H}$ of a Boolean algebra $\mathcal{B}$ is a subset of an ultrafilter $\mathcal{F}$ of $\mathcal{B}$.*
(v) *For every set X,* $\mathbf{BPI}(X)$.

We recall here that for the proof of (i)$\to$(ii) one applies **BPI** to the quotient $\mathcal{B}/J$ to get a prime ideal P of $\mathcal{B}/J$. Then the inverse image I of P under the canonical homomorphism is a prime ideal including J.

For $X = \omega$ the following characterizations of $\mathbf{BPI}(\omega)$ have been established in [3] and [8] respectively.

THEOREM 2 ([3]). *"$S(\omega)$ is compact" iff* $\mathbf{BPI}(\omega)$.

THEOREM 3 ([8]). *The following are equivalent:*

(i) $\mathbf{BPI}(\omega)$.
(ii) *The product $\mathbf{2}^{\mathbb{R}}$ is compact.*
(iii) *In a Boolean algebra $\mathcal{B}$ of size $\leq |\mathbb{R}|$ every filter extends to an ultrafilter.*

Proof. (iii)$\to$(i) is straightforward. For a proof of (ii)$\leftrightarrow$(i) different than the one given in [8] see [3].

To complete the proof of the theorem, fix a Boolean algebra $(\mathcal{B}, \mathbf{0}, \mathbf{1}, +, \cdot)$ of size $\leq |\mathbb{R}|$ (the operations $+$ and $\cdot$ of $\mathcal{B}$ denote symmetric difference and join, respectively). Let $\mathcal{H}$ be a filter of $\mathcal{B}$. By our hypothesis the Tychonoff product $\mathbf{2}^{\mathcal{B}}$ is compact. We claim that for every $b \in \mathcal{B}$ the set

$$G_b = \{f \in 2^{\mathcal{B}} : \mathbf{0} \notin f^{-1}(1) \wedge (\forall a, c \in f^{-1}(1),\ f(a.c) = 1) \\ \wedge (f(b) = 1 \vee f(\mathbf{1} + b) = 1) \wedge \mathcal{H} \subseteq f^{-1}(1)\}$$

is closed. Indeed, fix $h \in G_b^c$. We consider the following cases:

- $h(\mathbf{0}) = 1$. Clearly, $V = [\{(\mathbf{0}, 1)\}]$ is a neighborhood of h missing G_b.
- $\exists a, c \in h^{-1}(1)$, $h(a \cdot c) = 0$. Clearly, $V = [\{(a, 1), (b, 1), (a \cdot c, 0)\}]$ is a neighborhood of h disjoint from G_b.
- $h(b) = 0$ and $h(\mathbf{1}+b) = 0$. It is easy to see that $V = [\{(b, 0), (\mathbf{1}+b, 0)\}]$ is a neighborhood of h avoiding G_b.
- There exists $a \in \mathcal{H}$ with $h(a) = 0$. In this case, $V = [\{(a, 0)\}]$ is a neighborhood of h with $V \cap G_b = \emptyset$.

It is straightforward to verify that the family $\mathcal{G} = \{G_b : b \in \mathcal{B}\}$ has the fip. Thus, by the compactness of $\mathbf{2}^{\mathcal{B}}$, $\bigcap \mathcal{G} \neq \emptyset$. Clearly, for every $g \in \bigcap \mathcal{G}$, $g^{-1}(1)$ is an ultrafilter of $\mathcal{B}$ including $\mathcal{H}$. ∎

In view of Theorems 1 and 3, the most natural question which pops up at this point, is the following question which was also asked in [8]:

Question 1. Can the statement: $\mathbf{WBPI}(\omega)$ = *Every Boolean algebra $\mathcal{B}$ of size $\leq |\mathbb{R}|$ has an ultrafilter* be added to the list of Theorem 3?

Remark 4. (i) In [5] it has been shown that there exists a **ZF** model $\mathcal{N}[\Gamma]$ which is an extension of the basic Cohen model $\mathcal{M}$ satisfying $\mathbf{UF}(\omega)$ but not $\mathbf{BPI}(\omega)$. Thus, in $\mathcal{N}[\Gamma]$ the Boolean algebra $\mathcal{B} = (\mathcal{P}(\omega), \triangle, \cap)$, has free ultrafilters but there is a filter of $\mathcal{B}$ which is not a subset of any ultrafilter of $\mathcal{B}$.

(ii) Regarding Question 1, we point out here that we cannot use the argument with the quotient Boolean algebra following Theorem 1. Indeed, $\mathcal{B}/J$ is a partition of $\mathcal{B}$ and since $|\mathcal{B}| \leq |\mathbb{R}|$, we may consider $\mathcal{B}/J$ as a partition of $\mathbb{R}$. Hence, if $\mathbf{P}(\omega)$ holds true, then $|\mathcal{B}/J| \leq |\mathbb{R}|$ and $\mathbf{BPI}(\omega)$ is equivalent to $\mathbf{WBPI}(\omega)$. However, $\mathbf{P}(\omega)$ is unprovable in **ZF** as the forthcoming Theorem 13 shows.

The research in this paper is motivated by Question 1. Using a different technique than the one outlined after Theorem 1, we will prove in Theorem 11 that under $\mathbf{BF}(\omega)$ the statements $\mathbf{BPI}(\omega)$ and $\mathbf{WBPI}(\omega)$ are equivalent. Surprisingly enough, we will see in Theorem 13 that $\mathbf{BF}(\omega)$ is just a disguised form of $\mathbf{P}(\omega)$.

3. Compactness of certain subspaces of $\mathcal{F}(\omega)$ in ZF. It is very well known that in **ZFC** the subspaces $S(\omega)$ and $S^*(\omega)$ of $\mathcal{F}(\omega)$ are compact. $S(\omega)$ is homeomorphic to the Čech–Stone compactification $\beta(\omega)$ of the discrete space ω, and $S^*(\omega) = S(\omega) \setminus \{\omega\}$ is a closed subspace of $S(\omega)$. However, in **ZF**, $S(\omega)$ need not be compact. So, one may ask whether it could be the case that in some model $\mathcal{M}$ of **ZF**, $S^*(\omega)$ is compact but $S(\omega)$ fails to be compact.

Surprisingly enough, the question has a trivial answer. Indeed, suppose ω has no free ultrafilters (that is, $\mathbf{UF}(\omega)$ fails), In this case, $S^*(\omega)$ is empty, hence compact. Furthermore, $\mathbf{BPI}(\omega)$ must fail in this case, so $S(\omega)$ is not compact, by Theorem 2. (One model of $\mathbf{ZF} + \neg\mathbf{UF}(\omega)$ is Feferman's model $\mathcal{M}2$ in [6].)

However, the next theorem says that the possibility of $S^*(\omega) = \emptyset$ is in fact the only impediment to the equivalence of compactness of $S^*(\omega)$ and $S(\omega)$.

THEOREM 5.

(i) *"$S^*(\omega)$ is compact" iff $\neg\mathbf{UF}(\omega) \vee$ "$S(\omega)$ is compact".*
(ii) *$\mathbf{UF}(\omega) \wedge$ "$S^*(\omega)$ is compact " iff "$S(\omega)$ is compact".*
(iii) *$\mathbf{UF}(\omega)$ does not imply "$S^*(\omega)$ is compact".*
(iv) *"$S^*(\omega)$ is compact" $\nrightarrow \mathbf{UF}(\omega)$.*

Proof. (i) We show ($\rightarrow$) as the converse is straightforward (recall that $S^*(\omega)$ is a closed subspace of $S(\omega)$). If $\mathbf{UF}(\omega)$ fails there is nothing to show. Assume $\mathbf{UF}(\omega) \wedge$ "$S^*(\omega)$ is compact". We shall prove that "$S(\omega)$ is compact". To this end, it suffices by Theorem 2 to show that $\mathbf{BPI}(\omega)$ holds true. Let $\mathcal{H}$ be a free filter of ω. Clearly, $\mathbf{UF}(\omega)$ implies that

$$\{\langle H\rangle : H \in \mathcal{H}\}$$

is a family of non-empty sets of the clopen base $\mathcal{B}_\omega$ with the fip. Hence, by the compactness of $S^*(\omega)$,

$$W = \bigcap\{\langle H\rangle : H \in \mathcal{H}\} \neq \emptyset.$$

It is easy to see that every $\mathcal{F} \in W$ is an ultrafilter of ω extending $\mathcal{H}$.

(ii) ($\rightarrow$) is immediate from (i). For ($\leftarrow$), use Theorem 2, which implies that "$S(\omega)$ is compact" implies $\mathbf{UF}(\omega)$.

(iii) It is known that in the model $\mathcal{N}[\Gamma]$ (see Remark 4), $\mathbf{UF}(\omega)$ holds but $\mathbf{BPI}(\omega)$ fails. Hence, by (i) and Theorem 2, "$S^*(\omega)$ is compact" fails in $\mathcal{N}[\Gamma]$.

(iv) Note that in any model $\mathcal{M}$ of **ZF** and $\neg\mathbf{UF}(\omega)$, "$S^*(\omega)$ is compact" holds true. ■

4. The size of $|\mathcal{B}^*_\omega|$ and $|\mathcal{P}(\omega)/\mathrm{fin}|$ in ZF. If $\mathbf{UF}(\omega)$ fails then $\mathcal{B}^*_\omega = \emptyset$ and there is nothing to say about $|\mathcal{B}^*_\omega|$. Regarding $\mathcal{P}(\omega)/\mathrm{fin}$ however, we observe that in **ZF**,

$$|\mathcal{P}(\omega)/\mathrm{fin}| \geq |\mathbb{R}|. \tag{1}$$

Indeed, if $\mathcal{A} = \{A_i : i \in \mathbb{R}\}$ is an almost disjoint family of ω (one can easily define such families in **ZF**), then, for all distinct $i, j \in \mathbb{R}$, we have $(A_i) \neq (A_j)$ and the function $T : \mathbb{R} \to \mathcal{P}(\omega)/\mathrm{fin}$ given by $T(i) = (A_i)$ is easily seen to be $1 : 1$.

We point out here that the inequality given in (1) can consistently be strict in **ZF**. Indeed, A. Blass has shown in [1, Proposition 3.2, p. 745 and Proposition 3.7, p. 748] that in his model $\mathcal{M}[\mathbb{R}_0]$, $\mathbf{UF}(\omega)$ fails but $\mathcal{P}(\omega)/\mathrm{fin}$ has a free ultrafilter. Since $\mathbf{UF}(\omega)$ is equivalent to saying that $\mathbb{R}$ has a free ultrafilter (see [5] for a proof), we see that possibly $|\mathcal{P}(\omega)/\mathrm{fin}| \neq |\mathbb{R}|$. Thus, in view of (1), $\mathcal{M}[\mathbb{R}_0]$ satisfies $|\mathcal{P}(\omega)/\mathrm{fin}| > |\mathbb{R}|$. With no free ultrafilters on ω in this model, $|\mathcal{B}^*_\omega| = 0$. In particular,

$$\mathcal{M}[\mathbb{R}_0] \models |\mathcal{B}^*_\omega| < |\mathbb{R}| < |\mathcal{P}(\omega)/\mathrm{fin}|.$$

In $\mathbf{ZF} + \mathbf{UF}(\omega)$ things are different. We observe that for every $A \in \mathcal{P}(\omega)$, $\langle A \rangle \neq \emptyset$, and for every $A, B \in \mathcal{P}(\omega)$,

$$\langle A \rangle = \langle B \rangle \quad \text{iff} \quad (A) = (B) \quad (\text{iff } |A \triangle B| < \aleph_0). \tag{2}$$

To see $(\to)$ assume that $\langle A \rangle = \langle B \rangle$ but $|A \triangle B| = \aleph_0$. Then either $|A \setminus B| = \aleph_0$ or $|B \setminus A| = \aleph_0$. Assume that $|A \setminus B| = \aleph_0$ and let, by $\mathbf{UF}(\omega)$, $\mathcal{U}$ be a free ultrafilter of $A \setminus B$. Clearly, the filter $\mathcal{F}$ of ω generated by $\mathcal{U}$ is easily seen to be a free ultrafilter on ω such that $\mathcal{F} \in \langle A \rangle \setminus \langle B \rangle$, a contradiction. Thus, $|A \triangle B| < \aleph_0$.

To see $(\leftarrow)$ assume that $|A \triangle B| < \aleph_0$ and $\langle A \rangle \neq \langle B \rangle$. Clearly, either $\langle A \rangle \setminus \langle B \rangle \neq \emptyset$ or $\langle B \rangle \setminus \langle A \rangle \neq \emptyset$. Assume $\langle A \rangle \setminus \langle B \rangle \neq \emptyset$ and fix $\mathcal{F} \in \langle A \rangle \setminus \langle B \rangle$. Clearly, $A, B^c \in \mathcal{F}$. Hence, $A \cap B^c \in \mathcal{F}$ and since $\mathcal{F}$ is free, it follows easily that $|A \cap B^c| = \aleph_0$. Thus, $|A \triangle B| = \aleph_0$, a contradiction.

Hence, by (2), the function $f : \mathcal{P}(\omega)/\mathrm{fin} \to \mathcal{B}^*_\omega$ given by the formula $f(G) = \bigcup\{\langle A \rangle : A \in G\}$ is well defined, $1 : 1$ and onto. Thus, $\mathbf{UF}(\omega)$ implies $|\mathcal{B}^*_\omega| = |\mathcal{P}(\omega)/\mathrm{fin}|$. The converse also holds, since if $|\mathcal{B}^*_\omega| \neq 0$ then $\mathbf{UF}(\omega)$. Thus

$$\mathbf{UF}(\omega) \quad \text{iff} \quad |\mathcal{B}^*_\omega| = |\mathcal{P}(\omega)/\mathrm{fin}|. \tag{3}$$

Hence, in view of (3) and (1), we have a proof of part (i) of the next theorem.

Theorem 6.

(i) $|\mathcal{P}(\omega)/\mathrm{fin}| \leq |\mathbb{R}| \wedge \mathbf{UF}(\omega)$ *iff* $|\mathcal{B}^*_\omega| = |\mathbb{R}|$.
(ii) "$\mathcal{P}(\omega)/\mathrm{fin}$ *is well-orderable*" *iff* "$\mathcal{P}(\omega)$ *is well-orderable*". *Hence,* "$\mathcal{P}(\omega)/\mathrm{fin}$ *is well-orderable*" *implies* $|\mathcal{B}^*_\omega| = |\mathcal{P}(\omega)/\mathrm{fin}| = |\mathbb{R}|$.

Proof. (ii) $(\to)$ If $\mathcal{P}(\omega)/\mathrm{fin}$ is well-orderable, then by (1), $\mathbb{R}$ is well-orderable. Hence, $\mathcal{P}(\omega)/\mathrm{fin}$ has a choice set, and $\mathbf{UF}(\omega)$ holds true.

$(\leftarrow)$ This is straightforward. ∎

REMARK 7. Since $\mathcal{P}(\omega)/\mathrm{fin}$ is a partition of $\mathcal{P}(\omega)$, it follows that if "$\mathcal{P}(\omega)/\mathrm{fin}$ has a choice set" then $|\mathcal{P}(\omega)/\mathrm{fin}| \le |\mathbb{R}|$. Thus, "$\mathcal{P}(\omega)/\mathrm{fin}$ has a choice set" $\wedge$ $\mathbf{UF}(\omega) \to |\mathcal{P}(\omega)/\mathrm{fin}| = |\mathcal{B}_\omega^*| = |\mathbb{R}|$.

Next we show that the size of each of the bases $\mathcal{C}_\omega$ and $\mathcal{B}_\omega$ is equal to $|\mathbb{R}|$.

THEOREM 8. $|\mathcal{C}_\omega| = |\mathcal{B}_\omega| = |\mathbb{R}|$.

Proof. The functions $T : \mathcal{P}(\omega) \to \mathcal{C}_\omega$, $T(A) = [A]$, and $H : \mathcal{P}(\omega) \to \mathcal{B}_\omega$, $H(A) = [A] \cap S(\omega)$, are clearly $1 : 1$ and onto. Indeed, if $A \ne B$ then either $A \setminus B \ne \emptyset$ or $B \setminus A \ne \emptyset$. Assume $A \setminus B \ne \emptyset$. Then the filter $\mathcal{F}$ of all supersets of $A \setminus B$ is in $[A] \setminus [B]$, and the fixed ultrafilter $\mathcal{F}_x$ generated by any element $x \in A \setminus B$ is in $[A] \cap S(\omega)$ but not in $[B] \cap S(\omega)$. Hence, $T(A) \ne T(B)$ and $H(A) \ne H(B)$. ∎

QUESTION 2.

(a) Does $\mathbf{UF}(\omega)$ imply $|\mathcal{B}_\omega^*| \le |\mathbb{R}|$?
(b) What is the status of the implications between "$|\mathcal{B}_\omega^*| = |\mathbb{R}|$" and "$\mathcal{P}(\omega)/\mathrm{fin}$ has a choice set"?

REMARK 9. Regarding Question 2(a), we note that $|[\mathbb{R}]^\omega| = |\mathbb{R}|$ (Form 368 in [6]) implies the inequality $|\mathcal{B}_\omega^*| \le |\mathbb{R}|$. However, the status of the implication between $\mathbf{UF}(\omega)$ and "$|[\mathbb{R}]^\omega| = |\mathbb{R}|$" is unknown to us. It is also indicated as unknown in [6].

In [4] it has been shown, in $\mathbf{ZF}$, that the function

$$T : S(\omega) \to \mathbf{2}^{\mathcal{P}(\omega)}, \qquad T(\mathcal{F}) = \chi_{\mathcal{F}},$$

is $1 : 1$, onto, continuous and such that for every $A \in \mathcal{P}(\omega)$,

$$T([A]) = [\{(A,1)\}] \cap T(S(\omega)).$$

If $\mathbf{UF}(\omega)$ holds true, then for every $A \in \mathcal{P}(\omega)$, $\langle A \rangle \ne \emptyset$ and consequently the restriction $T^* : S^*(\omega) \to \mathbf{2}^{\mathcal{P}(\omega)}$ of T to $S^*(\omega)$ is an embedding such that

$$T^*(\langle A \rangle) = [\{(A,1)\}] \cap T^*(S^*(\omega)).$$

Hence,

$$|\mathcal{B}_\omega^*| \le |\mathbb{R}| \quad \text{iff} \quad |\{[p] \cap T^*(S^*(\omega)) : p \in \mathrm{Fn}(\mathcal{P}(\omega), 2)\}| \le |\mathbb{R}|.$$

Thus, $\mathbf{UF}(\omega)$ and $\mathbf{R}(\omega)$ (= For every $H \subset 2^{\mathcal{P}(\omega)}$, $|\{[p] \cap H : p \in \mathrm{Fn}(\mathcal{P}(\omega), 2)\}| \le |\mathbb{R}|$) together imply $|\mathcal{B}_\omega^*| = |\mathbb{R}|$.

We shall come back again to $\mathbf{R}(\omega)$ in Section 6, where we will show that $\mathbf{R}(\omega)$, $\mathbf{BF}(\omega)$ and $\mathbf{P}(\omega)$ are all equivalent.

5. $\mathbf{BPI}(\omega)$ and $\mathbf{WBPI}(\omega)$ are equivalent in $\mathbf{ZF} + \mathbf{BF}(\omega)$. Before we state and prove the main result of this section, we need to establish some auxiliary results.

PROPOSITION 10.

(i) $\mathcal{C}_\omega(=\{[A] : A \in \mathcal{P}(\omega)\})$ *is a base for* $T_{\mathcal{F}}$ *of size* $\mathcal{P}(\omega)$.
(ii) *Let* $A \in \mathcal{P}(\omega)$. *Then* $\mathcal{F} \in \overline{[A]}$ *iff* $\mathcal{F} \cup \{A\}$ *has the fip.*
(iii) *For all* $A \in \mathcal{P}(\omega)$, $[A]$ *is a regular open set.*

Proof. (i) Clearly $\mathcal{C}_\omega$ is closed under finite intersections ($\mathcal{F} \in [A] \cap [B]$ iff $A, B \in \mathcal{F}$ iff $A \cap B \in \mathcal{F}$ iff $\mathcal{F} \in [A \cap B]$) and covers $\mathcal{F}(\omega)$. The second assertion follows from Theorem 8.

(ii) Fix $A \in \mathcal{P}(\omega)$. We have: $\mathcal{F} \in \overline{[A]}$ iff $\forall F \in \mathcal{F}, [F] \cap [A] \neq \emptyset$ iff $\forall F \in \mathcal{F}, [F \cap A] \neq \emptyset$ iff $\forall F \in \mathcal{F}, F \cap A \neq \emptyset$ ($\rightarrow$ of the last equivalence is straightforward, and for the other implication note that the filter $\mathcal{G}$ generated by $\mathcal{F} \cup \{A\}$ satisfies $\mathcal{G} \in [A \cap F]$; hence $[A \cap F] \neq \emptyset$) iff $\mathcal{F} \cup \{A\}$ has the fip.

(iii) Fix $A \in \mathcal{P}(\omega)$. Obviously, $[A] \subseteq \operatorname{int}(\overline{[A]})$. To show $\operatorname{int}(\overline{[A]}) \subseteq [A]$, let $[B]$ be any basic open set such that $[B] \subseteq \overline{[A]}$. It suffices to show $[B] \subseteq [A]$. Suppose $[B] \nsubseteq [A]$. Then $B \nsubseteq A$, so let $\mathcal{F}$ be the filter generated by $\{B \setminus A\}$. Then $\mathcal{F} \in [B]$, but by (ii), $\mathcal{F} \notin \overline{[A]}$, a contradiction. ∎

THEOREM 11. *Assume* $\mathbf{BF}(\omega)$. *The following are equivalent:*

(i) *In every Boolean algebra of size* $\leq |\mathbb{R}|$ *every filter can be extended to an ultrafilter.*
(ii) *Every Boolean algebra of size* $\leq |\mathbb{R}|$ *has an ultrafilter.*
(iii) $\mathbf{BPI}(\omega)$.

Proof. In view of Theorem 3 and the obvious implication (i)$\rightarrow$(ii), it suffices to show that (ii) implies (iii). Fix a free filter $\mathcal{H}$ of ω and consider the subspace $\mathcal{H}(\omega) = \{\mathcal{F} \in \mathcal{F}(\omega) : \mathcal{H} \subseteq \mathcal{F}\}$ of $\mathcal{F}(\omega)$.

CLAIM. *For every* $A \in \mathcal{P}(\omega)$, $[A] \cap \mathcal{H}(\omega)$ *is a regular open subset of* $\mathcal{H}(\omega)$.

Proof of the Claim. Fix $A \in \mathcal{P}(\omega)$. Clearly, $\operatorname{int}_{\mathcal{H}(\omega)}(\overline{[A] \cap \mathcal{H}(\omega)}) = \operatorname{int}_{\mathcal{H}(\omega)}(\overline{[A]} \cap \mathcal{H}(\omega))$. We show that this set is equal to $[A] \cap \mathcal{H}(\omega)$. Since $[A] \cap \mathcal{H}(\omega) \subseteq \overline{[A]} \cap \mathcal{H}(\omega)$, it follows that

$$[A] \cap \mathcal{H}(\omega) \subseteq \operatorname{int}_{\mathcal{H}(\omega)}(\overline{[A]} \cap \mathcal{H}(\omega)). \tag{4}$$

For the reverse inclusion, we relativize the proof of Proposition 10(iii). Assume $[B] \cap \mathcal{H}(\omega) \subseteq \overline{[A]} \cap \mathcal{H}(\omega)$. Let $\mathcal{F} \in [B] \cap \mathcal{H}(\omega)$, and suppose toward a contradiction that $\mathcal{F} \notin [A]$. Since $A \notin \mathcal{F}$, the collection $\mathcal{F} \cup \{B \setminus A\}$ has the fip, so it generates a filter $\mathcal{G}$. But then $\mathcal{G} \in [B] \cap \mathcal{H}(\omega)$ and $\mathcal{G} \notin \overline{[A]}$, a contradiction. Thus $\mathcal{F} \in [A] \cap \mathcal{H}(\omega)$, and

$$\operatorname{int}_{\mathcal{H}(\omega)}(\overline{[A]} \cap \mathcal{H}(\omega)) \subseteq [A] \cap \mathcal{H}(\omega). \tag{5}$$

From (4) and (5) we have $\operatorname{int}_{\mathcal{H}(\omega)}(\overline{[A] \cap \mathcal{H}(\omega)}) = [A] \cap \mathcal{H}(\omega)$ as required.

By the claim, $\mathcal{A} = \{[A] \cap \mathcal{H}(\omega) : A \in \mathcal{P}(\omega)\}$ is a family of regular open sets of $\mathcal{H}(\omega)$ and by $\mathbf{BF}(\omega)$, $|\mathcal{A}| \leq |\mathbb{R}|$.

Let $\mathcal{B}$ be the Boolean algebra of all regular open sets of $\mathcal{H}(\omega)$ generated by the family $\mathcal{A}$. Clearly, $|\mathcal{B}| \le |\mathbb{R}|$. Let, by our hypothesis, $\mathcal{U}$ be an ultrafilter of $\mathcal{B}$ and put

$$\mathcal{F} = \{A \in \mathcal{P}(\omega) : [A] \cap \mathcal{H}(\omega) \in \mathcal{U}\}.$$

To complete the proof of the theorem it suffices to show:

Claim. *$\mathcal{F}$ is an ultrafilter of ω and $\mathcal{H} \subseteq \mathcal{F}$.*

Proof of the Claim. Since, for every $H \in \mathcal{H}$, $[H] \cap \mathcal{H}(\omega) = \mathcal{H}(\omega)$ and $\mathcal{U}$ is a filter, it follows that $H \in \mathcal{F}$, and consequently $\mathcal{H} \subseteq \mathcal{F}$. Since $\omega \in \mathcal{H}$, it follows that $\omega \in \mathcal{F}$. Furthermore, it is trivially true that $\emptyset \notin \mathcal{F}$.

We next show that $\mathcal{F}$ is a filter. Fix $A, B \in \mathcal{F}$. Then, $[A] \cap \mathcal{H}(\omega), [B] \cap \mathcal{H}(\omega) \in \mathcal{U}$, and consequently $[A] \cap \mathcal{H}(\omega) \cap [B] \cap \mathcal{H}(\omega) = [A \cap B] \cap \mathcal{H}(\omega) \in \mathcal{U}$. Thus, $A \cap B \in \mathcal{F}$.

Fix $A \in \mathcal{F}$ and $B \in \mathcal{P}(\omega)$ with $A \subseteq B$. We show that $B \in \mathcal{F}$. Clearly, $[A] \cap \mathcal{H}(\omega) \in \mathcal{U}$ and $[A] \cap \mathcal{H}(\omega) \subseteq [B] \cap \mathcal{H}(\omega)$. Thus, $[B] \cap \mathcal{H}(\omega) \in \mathcal{U}$ and $B \in \mathcal{F}$ as required.

Next we show that $\mathcal{F}$ is maximal. Fix $A \in \mathcal{P}(\omega)$. For every filter $\mathcal{G} \in \mathcal{H}(\omega)$, one of $\mathcal{G} \cup \{A\}$ or $\mathcal{G} \cup \{A^c\}$ has the fip, so $\mathcal{G} \in \overline{[A]} \cup \overline{[A^c]}$. It follows that $\overline{([A] \cap \mathcal{H}(\omega)) \cup ([A^c] \cap \mathcal{H}(\omega))} = \mathcal{H}(\omega)$, and hence $[A] \vee [A^c] = \mathbf{1}$ in the regular open algebra $\mathcal{B}$. Thus either $[A] \in \mathcal{U}$ or $[A^c] \in \mathcal{U}$, and either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. ∎

Remark 12. Let $\mathcal{H}(\omega)$ and $\mathcal{A}$ be as in the proof of Theorem 11. Clearly, for every $A, B \in \mathcal{P}(\omega)$ satisfying

$$\exists H \in \mathcal{H} \text{ such that } H \cap (A \setminus B) = H \cap (B \setminus A) = \emptyset \tag{6}$$

we have $[A] \cap \mathcal{H}(\omega) = [B] \cap \mathcal{H}(\omega)$. Furthermore, the binary relation $\sim$ on $\mathcal{P}(\omega)$ given by

$$A \sim B \quad \text{iff} \quad \exists H \in \mathcal{H} \text{ satisfying (6)}$$

is easily seen to be an equivalence relation on $\mathcal{P}(\omega)$. Hence, if $\mathbf{P}(\omega)$ holds true, then $|\mathcal{A}| = |\mathcal{P}(\omega)/{\sim}| \le |\mathbb{R}|$, and the conclusion of Theorem 11 goes through if we replace the hypothesis $\mathbf{BF}(\omega)$ with $\mathbf{P}(\omega)$.

6. Some equivalents of $\mathbf{BF}(\omega)$. Remarks 9 and 12 indicate that the statements $\mathbf{BF}(\omega)$, $\mathbf{R}(\omega)$ and $\mathbf{P}(\omega)$ might be equivalent. We show next that this is the case.

Theorem 13. *The following are equivalent:*

(a) $\mathbf{BF}(\omega)$.
(b) $\mathbf{R}(\omega)$: *For every $H \subset 2^{\mathcal{P}(\omega)}$, $|\{[p] \cap H : p \in \mathrm{Fn}(\mathcal{P}(\omega), 2)\}| \le |\mathcal{P}(\omega)|$.*
(c) $\mathbf{P}(\omega)$: *Every disjoint family of subsets of $\mathcal{P}(\omega)$ has size $\le |\mathcal{P}(\omega)|$.*

In particular, "every partition of $\mathbb{R}$ has a selector" implies $\mathbf{BF}(\omega)$.

Proof. (a)$\to$(b). Fix an independent family $\mathcal{A}$ of subsets of ω of size $|\mathbb{R}|$ (such a family is known to exist in **ZF**) and let $H \subseteq 2^{\mathcal{A}}$. We will show that $|\{H \cap [p] : p \in \mathrm{Fn}(\mathcal{A}, 2)\}| \leq |\mathbb{R}|$.

Let $S = \{\mathcal{F}_h : h \in H\}$ where, for every $h \in H$, $\mathcal{F}_h$ is the filter on ω generated by the filterbase

$$\mathcal{W}_h = \{B_0 \cap \cdots \cap B_{n-1} : 0 < n < \omega \text{ and } \forall i \in n\ (h(B_i) = 1 \text{ or } h(B_i^c) = 0)\}.$$

It is straightforward to verify that for all $h, f \in H$, $f \neq h \leftrightarrow \mathcal{F}_f \neq \mathcal{F}_h$. Hence, $|H| = |S|$.

For every $p \in \mathrm{Fn}(\mathcal{A}, 2)$ let

$$A_p = \bigcap \{B \in \mathcal{A} : p(B) = 1 \text{ or } p(B^c) = 0\}.$$

By our hypothesis,

$$(7) \qquad |\{S \cap [A_p] : p \in \mathrm{Fn}(\mathcal{A}, 2)\}| \leq |\{S \cap [A] : A \in \mathcal{P}(\omega)\}| \leq |\mathbb{R}|.$$

Claim. *For every $h \in H$ and $p \in \mathrm{Fn}(\mathcal{A}, 2)$, $\mathcal{F}_h \in [A_p] \leftrightarrow h \in [p]$.*

Proof of the Claim. To see ($\to$) we assume that $\mathcal{F}_h \in [A_p]$ but $h \notin [p]$. This means that there exists $A \in \mathrm{Dom}(p)$ such that $h(A) \neq p(A)$. We consider the following cases:

- $h(A) = 1$ and $p(A) = 0$. Since $\mathcal{F}_h \in [A_p]$ and $h(A) = 1$ we have $A_p \in \mathcal{F}_h$ and $A \in \mathcal{F}_h$. Since $p(A) = 0$ we infer that $A^c \supseteq A_p$, hence $A^c \in \mathcal{F}_h$, a contradiction.
- $h(A) = 0$ and $p(A) = 1$. Since $h(A) = 0$ we have $A^c \in \mathcal{F}_h$. Since $p(A) = 1$ we see that $A \supseteq A_p$, hence $A \in \mathcal{F}_h$, a contradiction.

Hence, $h \in [p]$.

To see ($\leftarrow$) assume that $h \in [p]$. Clearly, $A_p \in \mathcal{W}_h$, and consequently $A_p \in \mathcal{F}_h$. Hence, $\mathcal{F}_h \in [A_p]$ as required, finishing the proof of the claim.

In view of the claim, it follows that for every $p, q \in \mathrm{Fn}(\mathcal{A}, 2)$,

$$S \cap [A_p] = S \cap [A_q] \ \leftrightarrow\ H \cap [p] = H \cap [q].$$

Hence, the function $f : \{H \cap [p] : p \in \mathrm{Fn}(\mathcal{A}, 2)\} \to \{S \cap [A_p] : p \in \mathrm{Fn}(\mathcal{A}, 2)\}$ given by

$$f(H \cap [p]) = S \cap [A_p]$$

is well defined and $1:1$. Hence, by (7), $|\{H \cap [p] : p \in \mathrm{Fn}(\mathcal{A}, 2)\}| \leq |\mathbb{R}|$ as required.

(b)$\to$(c). Fix a partition $\mathcal{P} = \{P_i : i \in I\}$ of $\mathcal{P}(\omega)$ and let $S = \{\chi_{P_i} : i \in I\}$. Let $B = \{p \in \mathrm{Fn}(\mathcal{P}(\omega), 2) : \mathrm{Dom}(p) = p^{-1}(1) \subset P_i$ for some $i \in I\}$. Clearly, for every $p \in B$, $\mathrm{Dom}(p) \subset P_i$, $[p] \cap S = \{\chi_{P_i}\}$ and, by our hypothesis, $|\mathcal{P}| = |S| = |\{[p] \cap S : p \in B\}| \leq |\{[p] \cap S : p \in \mathrm{Fn}(\mathcal{P}(\omega), 2)\}| \leq |\mathbb{R}|$ as required.

(c)$\to$(a). Fix $S \subseteq \mathcal{F}(\omega)$. It is easy to see that the binary relation $\approx$ on $\mathcal{C}_\omega$ given by $P \approx Q$ iff $P \cap S = Q \cap S$ is an equivalence relation. Since $\mathcal{C}_\omega/{\approx}$ is a partition of $\mathcal{C}_\omega$ and $\mathcal{C}_\omega$ in view of Theorem 8(i) has size $|\mathbb{R}|$, it follows by our hypothesis that $|\mathcal{C}_\omega/{\approx}| \leq |\mathbb{R}|$. Since $|\{[A] \cap S : A \in \mathcal{P}(\omega)\}| = |\mathcal{C}_\omega/{\approx}|$, the conclusion of $\mathbf{BF}(\omega)$ for the set $\{[A] \cap S : A \in \mathcal{P}(\omega)\}$ is satisfied. ∎

Remark 14. (i) We point out here that for every infinite set X, $\mathbf{BF}(X)$ and $\mathbf{P}(X)$ are equivalent.

To see $\mathbf{BF}(X) \to \mathbf{P}(X)$, fix a partition $P = \{A_i : i \in I\}$ of X and let $Y = \{\mathcal{F}_i : i \in I\}$, where for every $i \in I$, $\mathcal{F}_i$ is the filter generated by $\{A_i\}$. By $\mathbf{BF}(X)$ we have $|\{[A] \cap Y : A \in \mathcal{P}(X)\}| \leq |X|$. Since $[A_i] \cap Y = \{\mathcal{F}_i\}$ for every $i \in I$, it follows that the function $f : I \to \{[A] \cap Y : A \in \mathcal{P}(X)\}$, $f(i) = \{\mathcal{F}_i\}$, is $1:1$. Thus, $|P| \leq |X|$.

To see that $\mathbf{P}(X) \to \mathbf{BF}(X)$, fix $Y \subseteq \mathcal{F}(X)$ and define an equivalence relation $\sim$ on $\mathcal{P}(X)$ by requiring: $A \sim B$ iff $[A] \cap Y = [B] \cap Y$. Clearly, $|\{[A] \cap Y : A \in \mathcal{P}(X)\}| = |\mathcal{P}(X)/{\sim}| \leq |X|$.

(ii) We do not know whether for every infinite set X, $\mathbf{R}(X)$ and $\mathbf{P}(X)$ are equivalent. Under the extra assumption $\mathbf{LIF}(X)$ = "X has an independent family of size $|\mathcal{P}(X)|$" the proof of Theorem 13 goes through with X in place of ω. However, it is consistent with $\mathbf{ZF}$ that there exist sets having no independent families. e.g., sets which do not split into two infinite sets. For the relative strength of $\forall X, \mathbf{LIF}(X)$ we refer the reader to [2].

7. Independence results

Theorem 15.

(i) $\mathbf{P}(\omega)$ *implies "every family* $\mathcal{A} = \{A_i : i \in \mathbb{R}\}$ *of 2-element sets of* $\mathcal{P}(\mathbb{R})$ *has a choice set". In particular,* $\mathbf{P}(\omega)$ *is not provable in* $\mathbf{ZF}$.

(ii) $\mathbf{UF}(\omega)$ *does not imply* $\mathbf{P}(\omega)$. *In particular,* $\mathbf{UF}(\omega)$ *does not imply "every partition of* $\mathbb{R}$ *has a choice set" (Form 203 in [6]).*

(iii) $\mathbf{P}(\omega)$ *does not imply "every partition of* $\mathbb{R}$ *has a choice set".*

Proof. (i) Fix a family $\mathcal{A} = \{A_i : i \in \mathbb{R}\}$ of 2-element sets of $\mathcal{P}(\mathbb{R})$. Without loss of generality we may assume that $\bigcap A_i = \emptyset$ for all $i \in \mathbb{R}$ (if $A, B \in A_i$ and $A \subseteq B$ then we choose A, otherwise if $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$ then we replace A by $A \setminus B$ and B with $B \setminus A$). Fix a $1:1$ and onto function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and for every $i \in \mathbb{R}$, let $f_i : \mathbb{R} \to \mathbb{R} \times \{i\}$ be the function given by $f_i(x) = (x, i)$. Clearly, $\{\{f_i(X) : X \in A_i\} : i \in \mathbb{R}\}$ is a family of subsets of $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ such that $\bigcup\{\{f_i(X) : X \in A_i\} : i \in \mathbb{R}\}$ is a family of disjoint subsets of $\mathbb{R} \times \mathbb{R}$. Hence, $\mathcal{H} = \{f(f_i(X)) : i \in \mathbb{R}, X \in A_i\}$ is a family of disjoint subsets of $\mathbb{R}$. Thus, by our hypothesis, we can identify $\mathcal{H}$ with a subset of $\mathbb{R}$ and consequently we may consider $\mathcal{A}$ as a family of 2-element subsets of $\mathbb{R}$. Hence, we may choose from each member of $\mathcal{A}$ its largest element.

The second assertion follows from the fact that the statement "every family $\mathcal{A} = \{A_i : i \in \mathbb{R}\}$ of 2-element sets of $\mathcal{P}(\mathbb{R})$ has a choice set" fails in the second Cohen Model, Model $\mathcal{M}7$ in [6]. So, $\mathbf{P}(\omega)$ fails in $\mathcal{M}7$.

(ii) It is shown in [5] that there is a model of $\mathbf{ZF} + \mathbf{UF}(\omega)$ in which there is a family of 2-element members of $\mathcal{P}(\mathbb{R})$ with no choice set. Thus $\mathbf{P}(\omega)$ is false in this model by (i).

(iii) Let $\mathcal{N}$ denote the Basic Cohen Model. We recall that $\mathcal{N}$ is a symmetric model obtained by adding first a countable number of generic reals along with the set A containing them to a ground model $\mathcal{M}$ of $\mathbf{ZFC} + \mathbf{CH}$ and then retracting to a model $\mathcal{N} \subset \mathcal{M}[G]$ which contains the set A but no well-ordered enumeration of any infinite subset of A. We recall the following additional facts about $\mathcal{N}$, $\mathcal{M}[G]$ and the set A:

(a) $\mathcal{M}$ and $\mathcal{M}[G]$ have the same cardinal numbers. In particular, in $\mathcal{N}$ we have $\aleph_1 < |\mathbb{R}|$ ($\aleph_1 = |\mathcal{P}(\omega)^{\mathcal{M}}|$ and $\mathcal{P}(\omega)^{\mathcal{M}} \subset \mathcal{P}(\omega)^{\mathcal{N}}$ imply $\aleph_1 < |\mathcal{P}(\omega)^{\mathcal{N}}| = |\mathbb{R}|$).

(b) For any $X \in \mathcal{N}$, there is an ordinal α and a function $f \in \mathcal{N}$ such that $f \colon X \to [A]^{<\omega} \times \alpha$ is one-to-one (see Lemma 5.25 in [7]).

(c) The set A is dense in $\mathbb{R}$.

Clearly, in view of (c), $\mathcal{P} = \{P_n \cap A : n \in \mathbb{N}\} \cup \{\mathbb{R} \setminus A\}$, where $P_n = (n, n+1) \cap A$, is a (countable) partition of $\mathbb{R}$ without a choice set.

We show next that every partition of $\mathbb{R}$ in $\mathcal{N}$ has size $\le |\mathbb{R}|$. To see this, fix some $\mathcal{P} \in \mathcal{N}$ which is a partition of $\mathbb{R}$. By (b), let k be the least well-ordered cardinal number α for which there is a $1:1$ function $f \in \mathcal{N}$, $f \colon P \to \alpha \times [A]^{<\omega}$. In $\mathcal{M}[G]$, where $\mathcal{P}$ has a choice function, we have $|\mathcal{P}| \le |\mathbb{R}| = \aleph_1$ by (a). Thus there is no onto function from $\mathcal{P}$ to $\aleph_2$, in $\mathcal{M}[G]$ or in $\mathcal{N}$. It follows that $k \le \aleph_1$, and so in $\mathcal{N}$, $|\mathcal{P}| \le |\aleph_1 \times [A]^{<\omega}|$. Since $(\aleph_1 < |\mathbb{R}|)^{\mathcal{N}}$ by (a) and $|[A]^{<\omega}| \le |\mathbb{R}|$, we have $|\mathcal{P}| \le |\mathbb{R}|$. ∎

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