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FUNCTIONAL ANALYSIS

Open Subsets of LF-spaces

by

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Summary. Let $F = \operatorname{ind}\lim F_n$ be an infinite-dimensional LF-space with density $\operatorname{dens} F = \tau$ $(\geq \aleph_0)$ such that some F_n is infinite-dimensional and $\operatorname{dens} F_n = \tau$. It is proved that every open subset of F is homeomorphic to the product of an $\ell_2(\tau)$-manifold and $\mathbb{R}^\infty = \operatorname{ind}\lim \mathbb{R}^n$ (hence the product of an open subset of $\ell_2(\tau)$ and $\mathbb{R}^\infty$). As a consequence, any two open sets in F are homeomorphic if they have the same homotopy type.

1. Introduction. A locally convex topological linear space F is called an *LF-space* if it is the strict inductive limit of Fréchet spaces (1). More precisely, F has a tower $F_1 \subsetneq F_2 \subsetneq \cdots$ of linear subspaces being Fréchet and a local basis consisting of balanced (circled) convex sets V such that $V \cap F_n$ is a neighborhood of 0 in F_n for each $n \in \mathbb{N}$. Then we write $F = \operatorname{ind}\lim F_n$. Given countably many Fréchet spaces F_n, $n \in \mathbb{N}$, we define $\sum_{n=1}^{\infty} F_n = \operatorname{ind}\lim \prod_{i=1}^{n} F_i$, where each $\prod_{i=1}^{n} F_i$ is identified with the subspace $\prod_{i=1}^{n} F_i \times \{0\}$ of $\prod_{i=1}^{n+1} F_i$. For LF-spaces, we refer to [8, Ch. II, §6], [12, Ch. 13], etc.

In this paper, we also consider the (topological) direct limit of a tower $X_1 \subset X_2 \subset \cdots$ of (topological) spaces which is denoted by $\varinjlim X_n$, that is, $\varinjlim X_n = \bigcup_{n\in\mathbb{N}} X_n$ with the topology such that U is open in $\varinjlim X_n$ if and only if $U \cap X_n$ is open in X_n for each $n \in \mathbb{N}$. Even if each X_n is a topological linear space, $\varinjlim X_n$ is not in general. If the addition of $\varinjlim X_n$ is continuous, then it is a topological linear space (2). In this case, if every X_n

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(1) A *Fréchet* space is a locally convex completely metrizable topological linear space.

(2) The scalar multiplication is continuous because $\mathbb{R} \times \varinjlim X_n = \varinjlim(\mathbb{R} \times X_n)$. However, $\varinjlim X_n \times \varinjlim X_n \neq \varinjlim(X_n \times X_n)$ in general, so the continuity of the addition is a problem.

is locally convex then so is $\varinjlim X_n$ (cf. [12, Problem 13-1-5]). For the tower $\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$, the direct limit $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ is a topological linear space, hence $\mathbb{R}^\infty$ is an LF-space, i.e., $\mathbb{R}^\infty = \operatorname{ind}\lim \mathbb{R}^n$. The Hilbert space with density τ ($\geq \aleph_0$) is denoted by $\ell_2(\tau)$, where $\ell_2 = \ell_2(\aleph_0)$.

The topological classification problem for LF-spaces is completely solved by the results of Mankiewicz [6, Theorem 2.14] and Toruńczyk [11, Theorem 6.1]: every LF-space $F = \operatorname{ind}\lim F_n$ is homeomorphic to ($\approx$) one of the spaces $\mathbb{R}^\infty$, $\ell_2(\tau) \times \mathbb{R}^\infty$ or $\sum_{n=1}^\infty \ell_2(\tau_n)$, where $\tau = \operatorname{dens} F$ and $\tau_1 < \tau_2 < \cdots$ with $\sup \tau_i = \operatorname{dens} F$. In fact, (1) $F \approx \mathbb{R}^\infty$ if $\dim F_n < \infty$ for each $n \in \mathbb{N}$; (2) $F \approx \ell_2(\tau) \times \mathbb{R}^\infty$ if some F_n is infinite-dimensional and $\operatorname{dens} F_n = \operatorname{dens} F = \tau$; (3) $F \approx \sum_{i=1}^\infty \ell_2(\tau_i)$ if $\operatorname{dens} F_n < \operatorname{dens} F$ for every $n \in \mathbb{N}$.

Given a space E (called a model space), a paracompact Hausdorff space M is called an *E-manifold* if it is locally homeomorphic to E, that is, each point of M has an open neighborhood homeomorphic to an open set in E. Although the theory of $\mathbb{R}^\infty$-manifolds has been well developed (cf. [3], [7], etc.), that of $\ell_2 \times \mathbb{R}^\infty$-manifolds has not. Not much is known about $\ell_2(\tau) \times \mathbb{R}^\infty$- or $\sum_{i=1}^\infty \ell_2(\tau_i)$-manifolds.

In the following,

let F be an LF-space such that $F \approx \ell_2(\tau) \times \mathbb{R}^\infty$, where $\tau \geq \aleph_0$.

In this paper, we show the following:

Main Theorem. *For each open set U in F, there exists an $\ell_2(\tau)$-manifold M such that $U \approx M \times \mathbb{R}^\infty$.*

We have the following corollaries. The first one follows from the classification theorem for $\ell_2(\tau)$-manifolds [5], [4] (cf. [2, Ch. IX, Theorem 7.3]): any two $\ell_2(\tau)$-manifolds with the same homotopy type are homeomorphic.

Corollary 1 (Classification). *Two open subsets of F are homeomorphic if they have the same homotopy type.*

Due to the stability theorem for $\ell_2(\tau)$-manifolds [9] (cf. [2, Ch. IX, Theorem 4.1]), $M \times \ell_2(\tau) \approx M$ for every $\ell_2(\tau)$-manifold M, hence we have the following:

Corollary 2 (Stability). *Every open set U in F is homeomorphic to $U \times F$.*

For each connected $\ell_2(\tau)$-manifold M, there exists a locally finite-dimensional simplicial complex K with $\operatorname{card} K^{(0)} \leq \tau$ such that $M \approx |K| \times \ell_2$, where $|K|$ admits the metric topology, by the triangulation theorem for $\ell_2(\tau)$-manifolds [4]. Thus, the following holds:

Corollary 3 (Triangulation). *Each open subset of F is homeomorphic to $|K| \times F$ for some locally finite-dimensional simplicial complex K with* $\operatorname{card} K^{(0)} \leq \tau$.

2. Outline of the proof. Let $\mathbf{I} = [0,1]$ and $\mathbb{R}_+ = [0,\infty)$. Since $\ell_2(\tau) \times \mathbb{R}_+^n \approx \ell_2(\tau) \times \mathbf{I}^n \approx \ell_2(\tau)$ for every $n \in \mathbb{N}$ (cf. [10]), it follows from the stability theorem for $\ell_2(\tau)$-manifolds [9] that $X \times \mathbb{R}_+^n \approx X \times \mathbf{I}^n \approx X$ for each $\ell_2(\tau)$-manifold X and $n \in \mathbb{N}$. Moreover, $\mathbb{R}^\infty \approx \mathbb{R}_+^\infty = \varinjlim \mathbb{R}_+^n$, the direct limit of the tower $\mathbb{R}_+ \subset \mathbb{R}_+^2 \subset \mathbb{R}_+^3 \subset \cdots$ (cf. [7]), where each $\mathbb{R}_+^n$ is identified with $\mathbb{R}_+^n \times \{0\} \subset \mathbb{R}_+^{n+1}$. Therefore, $F \approx \ell_2(\tau) \times \mathbb{R}_+^\infty$. Thus, we can consider U as an open set in $\ell_2(\tau) \times \mathbb{R}_+^\infty$. One should note that

$$\ell_2(\tau) \times \mathbb{R}_+^\infty = \ell_2(\tau) \times \varinjlim \mathbb{R}_+^n \neq \varinjlim(\ell_2(\tau) \times \mathbb{R}_+^n) \text{ as spaces.}$$

A closed set A in a space X is called a *Z-set* if for each open cover $\mathcal{U}$ of X there is a map $f : X \to X \setminus A$ which is $\mathcal{U}$-close to id, that is, every $\{x, f(x)\}$ is contained in some $U \in \mathcal{U}$. It is known that if an $\ell_2(\tau)$-manifold A is a Z-set in an $\ell_2(\tau)$-manifold X then A is *collared* in X, that is, there is an open embedding $\psi : A \times [0,1) \to X$ (called a *collar*) such that $\psi(x,0) = x$ for every $x \in A$.

For each $n \in \mathbb{N}$, let $U_n = U \cap (\ell_2(\tau) \times \mathbb{R}_+^n)$. As is easily observed, each U_n is an $\ell_2(\tau)$-manifold which is a Z-set in U_{n+1}. Note that $U_1 \subset U_2 \subset \cdots$ and $U = \bigcup_{n\in\mathbb{N}} U_n$. We define

$$M = \bigcup_{n\in\mathbb{N}} [n-1,n] \times U_n \subset \bigcup_{n\in\mathbb{N}} \mathbb{R}_+ \times \ell_2(\tau) \times \mathbb{R}_+^n = \mathbb{R}_+ \times \ell_2(\tau) \times \mathbb{R}_+^\infty.$$

Now, each $[n-1,n] \times U_n$ is an $\ell_2(\tau)$-manifold and

$$([n-1,n] \times U_n) \cap ([n,n+1] \times U_{n+1}) = \{n\} \times U_n,$$

where $\{n\} \times U_n$ is collared not only in $[n-1,n] \times U_n$ but also in $[n,n+1] \times U_{n+1}$ because it is a Z-set in the $\ell_2(\tau)$-manifold $[n,n+1] \times U_{n+1}$. It follows that M is a separable $\ell_2(\tau)$-manifold. Since $\mathbb{R}^\infty \approx [0,1)^\infty = \varinjlim [0,1)^n$, we shall show that $M \times [0,1)^\infty \approx U$.

Let $\Psi = (\psi_i)_{i\in\mathbb{N}}$ be a sequence of collars $\psi_i : U_i \times [0,1) \to U_{i+1}$. By the natural embedding

$$\psi_n \times \mathrm{id} : U_n \times [0,1) \times [0,1)^\infty \to U_{n+1} \times [0,1)^\infty,$$

we regard $U_n \times [0,1) \times [0,1)^\infty = U_n \times [0,1)^\infty$ as an open set in $U_{n+1} \times [0,1)^\infty$. Let U_Ψ be the direct limit of the following open tower:

$$U_1 \times [0,1)^\infty \underset{\psi_1 \times \mathrm{id}}{\subset} U_2 \times [0,1)^\infty \underset{\psi_2 \times \mathrm{id}}{\subset} \cdots.$$

Since each $U_n \times [0,1)^\infty$ is an open set in $\ell_2(\tau) \times \mathbb{R}_+^n \times [0,1)^\infty \approx \ell_2(\tau) \times \mathbb{R}^\infty$, it follows that U_Ψ is an $\ell_2(\tau) \times \mathbb{R}^\infty$-manifold. Since $U_n \times [0,1)^k \subset U_{n+k}$ for each $n, k \in \mathbb{N}$, we can regard $U_\Psi = \bigcup_{n\in\mathbb{N}} U_n$ as sets but the topology of U_Ψ depends on the sequence $\Psi = (\psi_i)_{i\in\mathbb{N}}$. The first step of the proof is to find Ψ so that $U_\Psi \approx U$.

Observe that U_Ψ is also the direct limit of the following open tower:

$$U_1 \times [0,1/2)^\infty \underset{\psi_1\times\mathrm{id}}{\subset} U_2 \times [0,2/3)^\infty \underset{\psi_2\times\mathrm{id}}{\subset} \cdots .$$

On the other hand, for each $n \in \mathbb{N}$, let

$$M_n^\infty = \left(\bigcup_{i=1}^{n} [i-1,n) \times U_i\right) \times \left[0, \frac{n}{n+1}\right)^\infty .$$

Then $M_1^\infty \subset M_2^\infty \subset \cdots$ are open sets in $M \times [0,1)^\infty$ and $M \times [0,1)^\infty = \bigcup_{n\in\mathbb{N}} M_n^\infty$. In the second step, we construct homeomorphisms

$$h_n : M_n^\infty \to U_n \times \left[0, \frac{n}{n+1}\right)^\infty, \qquad n \in \mathbb{N},$$

so that the following diagram commutes:

$$\begin{array}{ccc} M_n^\infty & \xrightarrow{\subset} & M_{n+1}^\infty \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle h_{n+1}} \\ U_n \times \left[0, \frac{n}{n+1}\right)^\infty & \xrightarrow[\psi_n\times\mathrm{id}]{\subset} & U_{n+1} \times \left[0, \frac{n+1}{n+2}\right)^\infty \end{array}$$

This implies that $M \times [0,1)^\infty \approx U_\Psi$.

To complete the proof, we use two more results on $\ell_2(\tau)$-manifolds. The following is proved in [5]:

THEOREM 1. *Let M and N be $\ell_2(\tau)$-manifolds. Every homotopy equivalence $f : M \to N$ is homotopic to ($\simeq$) a homeomorphism.*

We call an embedding $f : X \to Y$ a *Z-embedding* if $f(X)$ is a Z-set in Y. The following easily follows from the Z-set unknotting theorem [1]:

THEOREM 2. *Let $f : M \to N$ be a homeomorphism between $\ell_2(\tau)$-manifolds and $g : A \to N$ a Z-embedding of a Z-set A in M. If g is homotopic to the restriction $f|A$ then g extends to a homeomorphism $\tilde{g} : M \to M$ which is isotopic to f.*

3. The first step of the proof. For simplicity, we use the following notation:

$$\prod_{i=k}^{n<\omega} [0,a_i] = \bigcup_{n\ge k} \prod_{i=k}^{n} [0,a_i] \quad \text{for } a_i > 0,\ i \ge k.$$

For a subset $N \subset \ell_2(\tau) \times \mathbb{R}_+^n$ and a map $\alpha : N \to (0,1)$, we define

$$N(\alpha) = \{(x,t) \in N \times \mathbb{R}_+ \mid t < \alpha(x)\} \subset \ell_2(\tau) \times \mathbb{R}_+^{n+1}.$$

For each $n \in \mathbb{N}$, let $U_n = U \cap (\ell_2(\tau) \times \mathbb{R}_+^n)$. Then U_n is an $\ell_2(\tau)$-manifold. For a sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ of maps $\alpha_k : U_k \to (0,1)$ satisfying the condition

$U_k(\alpha_k) \subset U_{k+1}$, we can inductively define

$$\begin{aligned} U_n(\alpha_n, \ldots, \alpha_k) &= U_n(\alpha_n, \ldots, \alpha_{k-1})(\alpha_k) \\ &\subset U_k(\alpha_k) \subset U_{k+1} \quad \text{for each } k > n. \end{aligned}$$

Then, for each $n \in \mathbb{N}$,

$$U_n(\alpha_n) \subset U_n(\alpha_n, \alpha_{n+1}) \subset U_n(\alpha_n, \alpha_{n+1}, \alpha_{n+2}) \subset \cdots .$$

Let $U_n^\alpha = \bigcup_{k \geq n} U_n(\alpha_n, \ldots, \alpha_k) \subset U$. Thus, we have a tower $U_1^\alpha \subset U_2^\alpha \subset U_3^\alpha \subset \cdots$ with $U = \bigcup_{n \in \mathbb{N}} U_n^\alpha$. If each U_n^α is open in U then $U = \varinjlim U_n^\alpha$.

LEMMA 1. *There exists a sequence $\alpha = (\alpha_k)_{k \in \mathbb{N}}$ of maps $\alpha_k : U_k \to (0,1)$ such that $U_k(\alpha_k) \subset U_{k+1}$ for every $k \in \mathbb{N}$ and each U_n^α is open in U, hence $U = \varinjlim U_n^\alpha$. Moreover, for each $x \in U_k$ there is a neighborhood V of x in U_k and $a_i > 0$, $i > k$, such that $\inf_{y \in V} \alpha_k(y) > 0$ and*

$$\inf \left\{ \alpha_n(y) \;\middle|\; y \in V \times \textstyle\prod_{i=k+1}^n [0, a_i] \right\} > 0 \quad \textit{for every } n > k.$$

Proof. For each $k \in \mathbb{N}$, let $\mathcal{V}_k$ be a locally finite open cover of U_k and let $a_{V,i} \in (0,1]$, $i > k$, be such that

$$\operatorname{cl} V \times \prod_{i=k+1}^{n<\omega} [0, a_{V,i}] \subset U \quad \text{for each } k \in \mathbb{N} \text{ and } V \in \mathcal{V}_k.$$

We define $\beta_k : U_k \to \mathbf{I}$ as follows:

$$\beta_k(x) = \max \left\{ a_{V,k+1} \;\middle|\; V \in \mathcal{V}_j,\ j \leq k,\ x \in \operatorname{cl} V \times \prod_{i=j+1}^{k} [0, a_{V,i}] \right\},$$

where $\operatorname{cl} V \times \prod_{i=j+1}^k [0, a_{V,i}] = \operatorname{cl} V$ if $j = k$. Then β_k is upper semicontinuous because

$$\{(x,t) \in U_k \times \mathbf{I} \mid t \leq \beta_k(x)\} = \bigcup_{j \leq k} \bigcup_{V \in \mathcal{V}_j} \operatorname{cl} V \times \prod_{i=j+1}^{k+1} [0, a_{V,i}]$$

is closed in $U_k \times \mathbb{R}_+$. Choose an open set U'_{k+1} in U_{k+1} so that

$$\{(x,t) \in U_k \times \mathbf{I} \mid t \leq \beta_{k+1}(x)\} \subset U'_{k+1} \subset \operatorname{cl} U'_{k+1} \subset U_{k+1}.$$

Then we have a lower semicontinuous function $\gamma_k : U_k \to \mathbf{I}$ defined by

$$\gamma_k(x) = \sup\{t \in \mathbf{I} \mid \{x\} \times [0,t] \subset U'_{k+1}\}.$$

Since $\beta_k < \gamma_k$, there exists a continuous map $\alpha_k : U_k \to (0,1)$ such that $\beta_k < \alpha_k < \gamma_k$. Thus, $U_k(\alpha_k) \subset U_{k+1}$ for every $k \in \mathbb{N}$.

By the definition, for each $V \in \mathcal{V}_k$ and $n \geq k$,

$$\operatorname{cl} V \times \prod_{i=k+1}^{n+1} [0, a_{V,i}] \subset U_k(\alpha_k, \ldots, \alpha_n),$$

which implies $\inf_{y\in V}\alpha_k(y) \geq a_{V,k+1} > 0$ and

$$\inf\left\{\alpha_n(y)\;\middle|\; y \in V \times \prod_{i=k+1}^{n} [0,a_i]\right\} \geq a_{V,n+1} > 0 \qquad \text{for every } n > k.$$

To show that each U_n^α is open in U, let $x \in U_n^\alpha$. Choose $k \geq n$ so that $x \in U_n(\alpha_n,\dots,\alpha_k) \subset U_{k+1}$. Then x has the following open neighborhood in U:

$$W \times \prod_{i=k+2}^{m+1<\omega} [0,a_{V,i}),$$

where $W = V \cap U_n(\alpha_n,\dots,\alpha_k)$ and $V \in \mathcal{V}_{k+1}$. Now, by induction on $m > k$, we shall show that

$$W \times \prod_{i=k+2}^{m+1} [0,a_{V,i}] \subset U_n(\alpha_n,\dots,\alpha_m).$$

To this end, take an arbitrary

$$y = (z, t_{k+2},\dots,t_{m+1}) \in W \times \prod_{i=k+2}^{m+1} [0,a_{V,i}].$$

By the inductive assumption, it follows that

$$y' = (z, t_{k+2},\dots,t_m) \in W \times \prod_{i=k+2}^{m} [0,a_{V,i}] \subset U_n(\alpha_n,\dots,\alpha_{m-1}).$$

Since $t_{m+1} < a_{V,m+1} < \alpha_m(y')$, it follows that

$$y \in U_n(\alpha_n,\dots,\alpha_{m-1})(\alpha_m) = U_n(\alpha_n,\dots,\alpha_m).$$

Thus, we have

$$W \times \prod_{i=k+2}^{m+1<\omega} [0,a_{V,i}] = \bigcup_{m>k} W \times \prod_{i=k+2}^{m} [0,a_{V,i}] \subset U_n^\alpha.$$

Therefore, U_n^α is open in U. ■

Now, we shall construct a sequence $\Psi = (\psi_i)_{i\in\mathbb{N}}$ of collars $\psi_i : U_i \times [0,1) \to U_{i+1}$, $i \in \mathbb{N}$, so that $U_\Psi \approx U$. Recall U_Ψ is the direct limit of the following open tower:

$$U_1 \times [0,1)^\infty \underset{\psi_1\times\mathrm{id}}{\subset} U_2 \times [0,1)^\infty \underset{\psi_2\times\mathrm{id}}{\subset} \cdots,$$

where we regard $U_n \times [0,1)^\infty$ as an open set in $U_{n+1} \times [0,1)^\infty$ by the embedding

$$\psi_n \times \mathrm{id} : U_n \times [0,1)^\infty = U_n \times [0,1) \times [0,1)^\infty \to U_{n+1} \times [0,1)^\infty.$$

LEMMA 2. *There exists a sequence $\Psi = (\psi_n)_{n\in\mathbb{N}}$ of collars $\psi_n : U_n \times [0,1) \to U_{n+1}$ such that $U_\Psi \approx U$.*

Proof. Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a sequence of maps $\alpha_n : U_n \to (0,1)$ obtained by Lemma 1. Then U_n^α is open in U and $U = \varinjlim U_n^\alpha$. For each $n \in \mathbb{N}$, we define a collar $\psi_n : U_n \times [0,1) \to U_{n+1}$ by $\psi_n(x,t) = (x, \alpha_n(x)t)$. For every $k \in \mathbb{N}$, we inductively define $\delta_{n,k} : U_n \times [0,1)^k \to [0,1)$ as follows:

$$\begin{aligned}&\delta_{n,k}(x, t_{n+1}, \dots, t_{n+k})\\ &\qquad = \alpha_{n+k-1}(x, \delta_{n,1}(x, t_{n+1}), \dots, \delta_{n,k-1}(x, t_{n+1}, \dots, t_{n+k-1}))t_{n+k},\end{aligned}$$

where $\delta_{n,1}(x,t) = \alpha_n(x)t$. Then we have the following equation:

$$(*) \qquad \delta_{n,k}\left(x, \frac{s_{n+1}}{\alpha_n(x)}, \dots, \frac{s_{n+k}}{\alpha_{n+k-1}(x, s_{n+1}, \dots, s_{n+k-1})}\right) = s_{n+k}.$$

Define $h_n : U_n \times [0,1)^\infty \to U_n^\alpha$ and $g_n : U_n^\alpha \to U_n \times [0,1)^\infty$ as follows:

$$h_n(x, t_{n+1}, t_{n+2}, \dots) = (x, \delta_{n,1}(x, t_{n+1}), \delta_{n,2}(x, t_{n+1}, t_{n+2}), \dots),$$

$$g_n(x, s_{n+1}, s_{n+2}, \dots) = \left(x, \frac{s_{n+1}}{\alpha_n(x)}, \frac{s_{n+2}}{\alpha_{n+1}(x, s_{n+1})}, \frac{s_{n+3}}{\alpha_{n+2}(x, s_{n+1}, s_{n+2})}, \dots\right).$$

It is easily observed that $g_n \circ h_n = \mathrm{id}_{U_n \times [0,1)^\infty}$. By $(*)$, we have $h_n \circ g_n = \mathrm{id}_{U_n^\alpha}$. Thus, g_n is a bijection with $h_n = g_n^{-1}$. Moreover, $(\psi_n \times \mathrm{id}) \circ g_n = g_{n+1}|U_n^\alpha$ for all $n \in \mathbb{N}$, that is, the following diagram commutes:

$$\begin{array}{ccc} U_n^\alpha & \subset & U_{n+1}^\alpha \\ {\scriptstyle g_n}\big\downarrow & & \big\downarrow{\scriptstyle g_{n+1}} \\ U_n \times [0,1)^\infty & \xrightarrow[\psi_n \times \mathrm{id}]{\subset} & U_{n+1} \times [0,1)^\infty \end{array}$$

Indeed, for each $(x, s_{n+1}, s_{n+2}, \dots) \in U_n^\alpha$,

$$\begin{aligned}(\psi_n \times \mathrm{id}) \circ g_n(x, s_{n+1}, s_{n+2}, \dots) &= (\psi_n \times \mathrm{id})\left(x, \frac{s_{n+1}}{\alpha_n(x)}, \frac{s_{n+2}}{\alpha_{n+1}(x, s_{n+1})}, \dots\right)\\ &= \left(\psi_n\left(x, \frac{s_{n+1}}{\alpha_n(x)}\right), \frac{s_{n+2}}{\alpha_{n+1}(x, s_{n+1})}, \dots\right)\\ &= \left((x, s_{n+1}), \frac{s_{n+2}}{\alpha_{n+1}(x, s_{n+1})}, \dots\right)\\ &= g_{n+1}((x, s_{n+1}), s_{n+2}, \dots).\end{aligned}$$

We shall show that h_n and g_n are all continuous, which means that g_n is a homeomorphism. Then we shall have

$$U = \varinjlim U_n^\alpha \approx \varinjlim U_n \times [0,1)^\infty = U_\Psi.$$

To see the continuity of h_n at $x \in U_n \times [0,1)^\infty$, let V be a neighborhood of $h_n(x)$ in U_n^α. Then x is contained in some $U_n \times [0,1)^k$, which implies that $h_n(x) \in U_n(\alpha_1, \dots, \alpha_{k-1})$. We can find a neighborhood V' of $h_n(x)$ in

$U_n \times [0,1)^k$ and $0 < r_{n+k+i} < 1$, $i \in \mathbb{N}$, such that

$$h_n(x) \in V' \times \prod_{i=1}^{j<\omega} [0, r_{n+k+i}] \subset V.$$

Since $\delta_{n,1}, \ldots, \delta_{n,k}$ are continuous, it follows that $h_n|U_n \times [0,1)^k$ is continuous, hence x has a neighborhood W in $U_n \times [0,1)^k$ such that $h_n(W) \subset V'$. Then $W \times \prod_{i=1}^{j<\omega} [0, r_{n+k+i}]$ is a neighborhood of x in $U_n \times [0,1)^\infty$ and

$$h_n\Big(W \times \prod_{i=1}^{j<\omega} [0, r_{n+k+i}]\Big) \subset V' \times \prod_{i=1}^{j<\omega} [0, r_{n+k+i}] \subset V,$$

which implies that h_n is continuous at x.

To see the continuity of g_n at $x \in U_n^\alpha$, for each neighborhood V of $g_n(x)$ in $U_n \times [0,1)^\infty$ choose an open set W in U_n and $r_{n+i} > 0$, $i \in \mathbb{N}$, so that

$$g_n(x) \in W \times \prod_{i=1}^{j<\omega} [0, r_{n+i}] \subset V.$$

Due to Lemma 1, it can be assumed that $\inf_{y \in W} \alpha_n(y) > 0$ and

$$\inf\Big\{\alpha_{n+k}(y) \;\Big|\; y \in W \times \prod_{i=1}^{k} [0, r_{n+i}]\Big\} > 0 \quad \text{for every } k \in \mathbb{N}.$$

Hence, we can find $0 < q_{n+i} \le r_{n+i}$, $i \in \mathbb{N}$, such that

$$(y, s_{n+1}, \ldots, s_{n+j-1}) \in W \times \prod_{i=1}^{j} [0, r_{n+i}], \qquad s_{n+j} < q_{n+j},$$
$$\Rightarrow \quad \frac{s_{n+j}}{\alpha_{n+j-1}(y, s_{n+1}, \ldots, s_{n+j-1})} < r_{n+j}.$$

Then it follows that

$$g_n\Big(W \times \prod_{i=1}^{j<\omega} [0, q_{n+i}]\Big) \subset W \times \prod_{i=1}^{j<\omega} [0, r_{n+i}] \subset V,$$

which implies that g_n is continuous at x. ∎

Remark 1. Let $M_1 \subset M_2 \subset \cdots$ be a closed tower of $\ell_2(\tau)$-manifolds such that each M_i is a Z-set (hence collared) in M_{i+1}. Then $M_\infty = \bigcup_{i \in \mathbb{N}} M_n$ has a topology such that M_∞ is an $\ell_2(\tau) \times \mathbb{R}^\infty$-manifold and each M_i is a subspace of M_∞. Indeed, given a sequence $\Psi = (\psi_i)_{i \in \mathbb{N}}$ of collars $\psi_i : M_i \times [0,1) \to M_{i+1}$, we regard $M_n \times [0,1)^\infty$ as an open set in $M_{n+1} \times [0,1)^\infty$ by the natural embedding

$$\psi_n \times \mathrm{id} : M_n \times [0,1)^\infty = M_n \times [0,1) \times [0,1)^\infty \to M_{n+1} \times [0,1)^\infty.$$

Let M_Ψ be the direct limit of the open tower

$$M_1 \times [0,1)^\infty \underset{\psi_1\times\mathrm{id}}{\subset} M_2 \times [0,1)^\infty \underset{\psi_2\times\mathrm{id}}{\subset} \cdots .$$

Since every separable $\ell_2(\tau)$-manifold can be embedded into $\ell_2(\tau)$ as an open set by the open embedding theorem for $\ell_2(\tau)$-manifolds [5] (cf. [4]), each $M_n \times [0,1)^\infty$ is homeomorphic to an open set in $\ell_2(\tau)\times[0,1)^\infty \approx \ell_2(\tau)\times\mathbb{R}^\infty$. Then M_Ψ is an $\ell_2(\tau)\times\mathbb{R}^\infty$-manifold. Since $M_n \times [0,1)^k \subset M_{n+k}$ for each $n,k\in\mathbb{N}$, we can regard $M_\Psi = M_\infty$ as sets but the topology of M_Ψ depends on the sequence $\Psi = (\psi_i)_{i\in\mathbb{N}}$. One should note that $M_\Psi \neq \varinjlim M_n$. In fact, the topology of $\varinjlim M_n$ is finer than that of M_Ψ.

4. The second step of the proof. By Lemma 2, we have a sequence $\Psi = (\psi_i)_{i\in\mathbb{N}}$ of collars $\psi_i : U_i \times \mathbf{I} \to U_{i+1}$ such that U is homeomorphic to the direct limit U_Ψ of the following open tower:

$$U_1 \times [0,1)^\infty \underset{\psi_1\times\mathrm{id}}{\subset} U_2 \times [0,1)^\infty \underset{\psi_2\times\mathrm{id}}{\subset} \cdots .$$

The Main Theorem is reduced to the following:

LEMMA 3. *$M \times [0,1)^\infty \approx U_\Psi$.*

Proof. Here, we regard U_Ψ as the direct limit of the following open tower:

$$U_1 \times [0,1/2)^\infty \underset{\psi_1\times\mathrm{id}}{\subset} U_2 \times [0,2/3)^\infty \underset{\psi_2\times\mathrm{id}}{\subset} \cdots .$$

Recall we can write $M\times[0,1)^\infty = \bigcup_{n\in\mathbb{N}} M_n^\infty$, where $M_1^\infty \subset M_2^\infty \subset \cdots$ are open sets in $M\times[0,1)^\infty$ defined as follows:

$$M_n^\infty = \left(\bigcup_{i=1}^n [i-1,n)\times U_i\right) \times \left[0,\frac{n}{n+1}\right)^\infty .$$

To show that $M\times[0,1)^\infty \approx U_\Psi$, it suffices construct homeomorphisms

$$h_n : M_n^\infty \to U_n \times \left[0,\frac{n}{n+1}\right)^\infty, \qquad n\in\mathbb{N},$$

so that the following diagram commutes:

$$\begin{tikzcd} M_n^\infty \arrow[r, "\subset"] \arrow[d, "h_n"'] & M_{n+1}^\infty \arrow[d, "h_{n+1}"] \\ U_n \times \left[0,\frac{n}{n+1}\right)^\infty \arrow[r, "\subset", "\psi_n\times\mathrm{id}"'] & U_{n+1} \times \left[0,\frac{n+1}{n+2}\right)^\infty \end{tikzcd}$$

For each $n\in\mathbb{N}$, we define

$$M_n = \left(\bigcup_{i=1}^n [i-1,n)\times U_i\right) \times \left[0,\frac{n}{n+1}\right)^n .$$

Then it follows that

$$M_n^\infty = M_n \times \left[0, \frac{n}{n+1}\right)^\infty \quad \text{and} \quad M_n \times \left[0, \frac{n}{n+1}\right) \subset M_{n+1}.$$

If we could construct homeomorphisms

$$f_n : M_n \to U_n \times \left[0, \frac{n}{n+1}\right)^\infty, \qquad n \in \mathbb{N},$$

so that the following diagram commutes:

$$\begin{tikzcd} M_n \times \left[0, \frac{n}{n+1}\right) \arrow[r, "\subset"] \arrow[d, "f_n \times \mathrm{id}"'] & M_{n+1} \arrow[d, "f_{n+1}"] \\ U_n \times \left[0, \frac{n}{n+1}\right)^2 \arrow[r, "\psi \times \mathrm{id}"'] & U_{n+1} \times \left[0, \frac{n+1}{n+2}\right) \end{tikzcd}$$

then the desired homeomorphism h_n could be defined as follows:

$$h_n = f_n \times \mathrm{id} : M_n^\infty = M_n \times \left[0, \frac{n}{n+1}\right)^\infty \to U_n \times \left[0, \frac{n}{n+1}\right)^\infty.$$

To construct f_n inductively, let

$$\begin{aligned} \overline{M}_n &= \left(\bigcup_{i=1}^n [i-1, n] \times U_i\right) \times \left[0, \frac{n}{n+1}\right]^n, \\ \partial M_n &= \overline{M}_n \setminus M_n \\ &= \{n\} \times U_n \times \left[0, \frac{n}{n+1}\right]^n \\ &\qquad \cup \left(\bigcup_{i=1}^n [i-1, n] \times U_i\right) \times \left(\left[0, \frac{n}{n+1}\right]^n \setminus \left[0, \frac{n}{n+1}\right)^n\right). \end{aligned}$$

Similarly to M, we can see that these are $\ell_2(\tau)$-manifolds. Note that ∂M_n is a Z-set in $\overline{M}_n$. Let $p_n : \overline{M}_n \to U_n$ be the projection and $i_n : U_n \to \partial M_n \subset \overline{M}_n$ the injection defined by $i_n(x) = (n, x, v_n)$, where

$$v_n = \left(\frac{n}{n+1}, \dots, \frac{n}{n+1}\right) \in \left[0, \frac{n}{n+1}\right]^n.$$

Then $i_n(U_n) = U_n \times \{v_n\}$ is a strong deformation retract of both $\overline{M}_n$ and ∂M_n, hence p_n and $p_n|\partial M_n$ are homotopy equivalences and i_n is a homotopy inverse of both p_n and $p_n|\partial M_n$. Thus, we have the homotopy equivalences

$$r_n : \overline{M}_n \to U_n \times \left[0, \frac{n}{n+1}\right] \quad \text{and} \quad r'_n = r_n|\partial M_n : \partial M_n \to U_n \times \left\{\frac{n}{n+1}\right\}$$

defined by $r_n(x) = (p_n(x), n/(n+1))$.

We shall construct homeomorphisms

$$\bar{f}_n : \overline{M}_n \to U_n \times \left[0, \frac{n}{n+1}\right], \qquad n \in \mathbb{N},$$

so that $\bar{f}_n \simeq r_n$,

$$\bar{f}_n(\partial M_n) = U_n \times \left\{\frac{n}{n+1}\right\}, \quad \text{i.e.,} \quad \bar{f}_n(M_n) = U_n \times \left[0, \frac{n}{n+1}\right),$$

and the following diagram commutes:

$$\begin{array}{ccc}
\overline{M}_n \times \left[0, \frac{n}{n+1}\right] & \xrightarrow{\subset} & \overline{M}_{n+1} \\
\Big\downarrow{\scriptstyle \bar{f}_n \times \mathrm{id}} & & \Big\downarrow{\scriptstyle \bar{f}_{n+1}} \\
U_n \times \left[0, \frac{n}{n+1}\right]^2 & \xrightarrow[\psi_n \times \mathrm{id}]{} & U_{n+1} \times \left[0, \frac{n+1}{n+2}\right]
\end{array}$$

Then $f_n = \bar{f}_n | M_n$ is the desired homeomorphism.

First, by Theorem 1, we have homeomorphisms $f : \overline{M}_1 \to U_1 \times [0, 1/2]$ and $f' : \partial M_1 \to U_1 \times \{1/2\}$ onto U_1 such that $f \simeq r_1$ and $f' \simeq r_1'$. Since $f' \simeq f | \partial M_1$, we can apply Theorem 2 to extend f' to a homeomorphism $\bar{f}_1 : \overline{M}_1 \to U_1 \times [0, 1/2]$ which is isotopic f, hence $\bar{f}_1 \simeq r_1$.

Now, assume that $\bar{f}_n$ has been obtained and consider the following sets:

$$\begin{aligned}
\overline{\partial} M_n &= \partial M_n \times \left[0, \frac{n}{n+1}\right] \cup \overline{M}_n \times \left\{\frac{n}{n+1}\right\}, \\
L_{n+1} &= \overline{M}_{n+1} \setminus \left(M_n \times \left[0, \frac{n}{n+1}\right)\right) \\
&= [n, n+1] \times U_{n+1} \times \left[0, \frac{n+1}{n+2}\right]^{n+1} \\
&\quad \cup \left(\bigcup_{i=1}^{n} [i-1, n] \times U_i\right) \times \left(\left[0, \frac{n+1}{n+2}\right]^{n+1} \setminus \left[0, \frac{n}{n+1}\right)^{n+1}\right), \\
B_n &= \psi_n\left(U_n \times \left\{\frac{n}{n+1}\right\}\right) \times \left[0, \frac{n}{n+1}\right] \\
&\quad \cup \psi_n\left(U_n \times \left[0, \frac{n}{n+1}\right]\right) \times \left\{\frac{n}{n+1}\right\}, \\
W_{n+1} &= \left(U_{n+1} \times \left[0, \frac{n+1}{n+2}\right]\right) \setminus \left(\psi_n\left(U_n \times \left[0, \frac{n}{n+1}\right)\right) \times \left[0, \frac{n}{n+1}\right)\right).
\end{aligned}$$

Then we have the following homeomorphism:

$$g_n = (\psi_n \times \mathrm{id})(\bar{f}_n \times \mathrm{id}) | \overline{\partial} M_n : \overline{\partial} M_n \to B_n.$$

Observe that L_{n+1} and W_{n+1} are $\ell_2(\tau)$-manifolds, $\overline{\partial} M_n$ and ∂M_{n+1} are disjoint Z-sets in L_{n+1}, and B_n and $U_{n+1} \times \{(n+1)/(n+2)\}$ are disjoint Z-

sets in W_{n+1}. Since $i_{n+1}(U_{n+1}) = U_{n+1} \times \{v_{n+1}\}$ and $U_{n+1} \times \{(n+1)/(n+2)\}$ are strong deformation retracts of L_{n+1} and W_{n+1} respectively, it follows that $r''_{n+1} = r_{n+1}|L_{n+1} : L_{n+1} \to W_{n+1}$ is a homotopy equivalence. By Theorem 1, we have homeomorphisms

$$g : L_{n+1} \to W_{n+1} \quad \text{and} \quad g' : \partial M_{n+1} \to U_{n+1} \times \left\{\frac{n+1}{n+2}\right\}$$

such that $g \simeq r''_{n+1}$ and $g' \simeq r'_{n+1} = r''_{n+1}|\partial M_{n+1}$. Then g' extends to a homeomorphism

$$g'' : \overline{\partial} M_n \cup \partial M_{n+1} \to B_n \cup U_{n+1} \times \left\{\frac{n+1}{n+2}\right\}$$

by setting $g''|\overline{\partial} M_n = g_n$.

Note that r_n is homotopic to the map

$$q_n : \overline{M}_n \to U_n \times \left[0, \frac{n}{n+1}\right]$$

defined by $q_n(x) = (p_n(x), 0)$ and $\psi_n q_n = p_n$. Then we have $\psi_n \overline{f}_n \simeq \psi_n r_n \simeq \psi_n q_n = p_n$. Let $c_n : \mathbf{I} \to \{n/(n+1)\}$ and $c_{n+1} : \mathbf{I} \to \{(n+1)/(n+2)\}$ be the constant maps. Since $r''_{n+1}|\overline{\partial} M_n = p_n \times c_{n+1}|\overline{\partial} M_n$, it follows that

$$g_n \simeq \psi_n \overline{f}_n \times c_n|\overline{\partial} M_n \simeq p_n \times c_n|\overline{\partial} M_n \simeq p_n \times c_{n+1}|\overline{\partial} M_n = r''_{n+1}|\overline{\partial} M_n,$$

where all homotopies are realized in W_{n+1} (the first two in B_n). Therefore,

$$g'' \simeq r''_{n+1}|\overline{\partial} M_n \cup \partial M_{n+1} \simeq g|\overline{\partial} M_n \cup \partial M_{n+1}.$$

Thus, we can apply Theorem 2 to extend g'' to a homeomorphism $\tilde{g} : L_{n+1} \to W_{n+1}$. By pasting $\tilde{g}$ with $(\psi_n \times \mathrm{id})(\overline{f}_n \times \mathrm{id})$, we can obtain the desired homeomorphism $\overline{f}_{n+1}$. Since $i_{n+1} p_{n+1} \simeq \mathrm{id}$ in $\overline{M}_{n+1}$, it follows that

$$\overline{f}_{n+1} \simeq \overline{f}_{n+1} i_{n+1} p_{n+1} = g' i_{n+1} p_{n+1} \simeq r_{n+1} i_{n+1} p_{n+1} \simeq r_{n+1}.$$

This completes the proof. ∎

Remark 2. For a closed tower $M_1 \subset M_2 \subset \cdots$ of $\ell_2(\tau)$-manifolds such that each M_i is a Z-set in M_{i+1}, $M = \bigcup_{n\in\mathbb{N}} [n-1, n] \times M_n$ is an $\ell_2(\tau)$-manifold. On the other hand, given a sequence $\Psi = (\psi_n)_{n\in\mathbb{N}}$ of collars $\psi_n : M_n \times [0,1) \to M_{n+1}$, the $\ell_2(\tau) \times \mathbb{R}^\infty$-manifold M_Ψ can be defined as in Remark 1. Similarly to Lemma 3, we can show $M \times \mathbb{R}^\infty \approx M_\Psi$. Since $M \times \mathbb{R}^\infty$ does not depend on Ψ, the topological type of M_Ψ is unique. Moreover, M_Ψ can be embedded in $\ell_2(\tau) \times \mathbb{R}^\infty$ as an open set.

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