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Generalized RBSDEs with Random Terminal Time and Applications to PDEs

by

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Summary. Generalized reflected backward stochastic differential equations have been considered so far only in the case of a deterministic interval. In this paper the existence and uniqueness of solution for generalized reflected backward stochastic differential equations in a convex domain with random terminal time is studied. Applications to the obstacle problem with Neumann boundary conditions for partial differential equations of elliptic type are given.

1. Introduction. Nonlinear backward stochastic differential equations, BSDEs for short, were introduced by Pardoux and Peng in 1990 [PP]. Since then, BSDEs have been studied extensively, due to their connections with different mathematical fields, mainly partial differential equations (PDE for short). Some of those results may be seen as generalizations of the celebrated Feynman–Kac formula. We should list here paper [P] that concerns the Cauchy problem for parabolic PDEs and elliptic equations with Dirichlet boundary conditions, and [PZ] that deals with parabolic equations with Neumann boundary conditions.

The notion of reflected backward stochastic differential equations (abbreviated as RBSDEs) in a general convex domain was introduced in [GP]. Similarly to BSDEs, they give probabilistic formulas for appropriate obstacle problems for PDEs. These connections for both parabolic and elliptic PDEs were considered in [J1, PR]. Recently some authors have been interested in connections between so called generalized RBSDEs (GRBSDEs for short) with deterministic terminal time and an obstacle problem of parabolic

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type with Neumann boundary conditions [J2, RX]. By generalized we mean equations with an additional component which is a stochastic integral with respect to an increasing process.

The aim of the present paper is to show the existence and uniqueness of solutions of GRBSDEs in general convex domains with random terminal time and connections with an obstacle problem of elliptic type with Neumann boundary conditions.

The paper is organized as follows. In Section 2 we give the definition of a solution of a GRBSDE with random terminal time and we formulate a main theorem of the paper. We consider an equation of the form

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\, ds + \int_{t\wedge\tau}^{\tau} \varphi(s, Y_s)\, d\Lambda_s - \int_{t\wedge\tau}^{\tau} Z_s\, dW_s + K_\tau - K_{t\wedge\tau}, \tag{1.1}$$

$t \in \mathbb{R}^+$, where τ is an almost surely finite stopping time, $W = (W_t)_{t\in\mathbb{R}^+}$ is an m-dimensional Wiener process and $\Lambda = (\Lambda_t)_{t\in\mathbb{R}^+}$ is a one-dimensional continuous and increasing process with $\Lambda_0 = 0$. Next, in Section 3 we show the connection between a GRBSDE with random terminal time and the obstacle problem of elliptic type with Neumann boundary conditions. Finally, in the last section we prove the main result of the paper, i.e. the theorem about existence and uniqueness of a solution of (1.1) (Theorem 2.2). We construct a sequence of GRBSDEs for which existence of solutions follows from the existence for GRBSDEs with deterministic terminal time (known from [J2]) and show that this sequence converges to a solution of (1.1). Moreover, we give some properties of the solution of (1.1).

Throughout the paper we will use the following notation. By $|\cdot|$ we mean the Euclidean norm in $\mathbb{R}^d$, while $\|x\|$ stands for $(\mathrm{trace}(x^*x))^{1/2}$, where x^* is the transposition of a matrix $x \in \mathbb{R}^{d\times m}$. For a process $K = (K^1, \ldots, K^d)$ we denote by $|K|_t = \sum_{i=1}^d |K^i|_t$ its variation on $[0,t]$, where $|K^i|_t$ is the total variation of K^i on $[0,t]$.

2. Generalized RBSDEs. Let $(\Omega, \mathcal{G}, \mathcal{P})$ be a complete probability space carrying a standard m-dimensional Wiener process $W = (W_t)_{t\in\mathbb{R}^+}$. Let $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be the usual augmentation of the filtration generated by W and assume that $\Lambda = (\Lambda_t)_{t\in\mathbb{R}^+}$ is an adapted, one-dimensional continuous and increasing process with $\Lambda_0 = 0$.

Let τ be an almost surely finite $\mathcal{F}$ stopping time and let ξ be an $\mathcal{F}_\tau$ measurable, square integrable random variable with values in $\bar{D}$, where D is a convex subset of $\mathbb{R}^d$. Suppose that functions $f : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \to \mathbb{R}^d$ and $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable and there exist constants $L, \kappa > 0$, $\beta < 0$, $\mu \in \mathbb{R}$ such that for any $t \in \mathbb{R}^+$, $y, y' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d\times m}$,

(A1) $|f(t, y, z) - f(t, y', z')| \le L(|y - y'| + \|z - z'\|)$,

(A2) $\langle y-y', f(t,y,z)-f(t,y',z)\rangle \le \mu|y-y'|^2$,
(A3) $|\varphi(t,y)-\varphi(t,y')| \le L|y-y'|$,
(A4) $\langle y-y', \varphi(t,y)-\varphi(t,y')\rangle \le \beta|y-y'|^2$,
(A5) $|\varphi(\cdot,\cdot,\cdot)| \le \kappa$,
(A6) for some real numbers λ and ν such that $\lambda > 2\mu + L^2$ and $\nu > \beta$,

$$E\Big(\int_0^\tau e^{\lambda t+\nu\Lambda_t}|f(t,0,0)|^2\,dt + \int_0^\tau e^{\lambda t+\nu\Lambda_t}|\varphi(t,0)|^2\,d\Lambda_t\Big) < \infty,$$

(A7) $Ee^{\lambda\tau+\nu\Lambda_\tau}(|\xi|^2+1) < \infty$ and

$$E\Big(\int_0^\tau e^{\lambda t+\nu\Lambda_t}|f(t,\xi_t,\zeta_t)|^2\,dt + \int_0^\tau e^{\lambda t+\nu\Lambda_t}|\varphi(t,\xi_t)|^2\,d\Lambda_t\Big) < \infty,$$

where $\xi_t = E(\xi|\mathcal{F}_t)$, ζ is an $\mathcal{F}$ progressively measurable $d\times m$-dimensional process with $E\int_0^\infty \|\zeta_t\|^2 dt < \infty$ and $\xi = E\xi + \int_0^\infty \zeta_t\,dW_t$,
(A8) there exists $q \ge 2$ such that for every $M > 0$,

$$E\int_0^M |f(t,0,0)|^{2q}\,dt < \infty.$$

Definition 2.1. A *solution* of the generalized reflected backward stochastic differential equation (GRBSDE) with random terminal time associated with data $(\tau,\xi,f,\varphi,\Lambda)$ is a triple $(Y,Z,K) = (Y_t,Z_t,K_t)_{t\in\mathbb{R}^+}$ of $\mathcal{F}$ progressively measurable processes in $\bar{D}\times\mathbb{R}^{d\times m}\times\mathbb{R}^d$ satisfying

$$\begin{aligned}(2.1)\qquad Y_t = \xi &+ \int_{t\wedge\tau}^\tau f(s,Y_s,Z_s)\,ds + \int_{t\wedge\tau}^\tau \varphi(s,Y_s)\,d\Lambda_s\\ &- \int_{t\wedge\tau}^\tau Z_s\,dW_s + K_\tau - K_{t\wedge\tau}, \qquad t\in\mathbb{R}^+,\end{aligned}$$

and such that

$$E\Big(\sup_{t\le\tau} e^{\lambda t}|Y_t|^2 + \int_0^\tau e^{\lambda t}\|Z_t\|^2\,dt + \int_0^\tau e^{\lambda t}|Y_t|^2\,d\Lambda_t\Big) < \infty,$$

where K is a continuous process with locally finite variation, $K_0 = 0$ and

$$(2.2)\qquad \int_0^\tau (Y_t - S_t)\,dK_t \le 0$$

for every $\mathcal{F}$ progressively measurable process $S = (S_t)_{t\in\mathbb{R}^+}$ with values in $\bar{D}$.
Moreover, on the set $\{t\ge\tau\}$ we have $Y_t = \xi$, $Z_t = 0$, $K_t = K_\tau$.

Theorem 2.2. *Let τ be an almost surely finite $\mathcal{F}$ stopping time and let assumptions* (A1)–(A8) *hold. Then there exists a unique solution of* (2.1).

We defer the proof of Theorem 2.2 to Section 4.

3. Partial differential equations. In [J2] and [RX] it was shown that a GRBSDE with deterministic terminal time gives a probabilistic formula for a viscosity solution to an obstacle problem for a parabolic PDE with Neumann boundary conditions. Here we will show that a GRBSDE with random terminal time gives a probabilistic formula for an obstacle problem for an elliptic PDE with Neumann boundary conditions.

In this section we will assume that $D=(a_1,b_1)\times\cdots\times(a_d,b_d)$. Let $\mathcal{O}$ and G be open connected bounded and smooth subsets of $\mathbb{R}^m$ such that $G\cap\mathcal{O}\neq\emptyset$ and $\partial\mathcal{O}\cap G\neq\emptyset$, $\partial G\cap\mathcal{O}\neq\emptyset$. Suppose that $\mathcal{O}=\{x;\,\phi(x)>0\}$, $\partial\mathcal{O}=\{x;\,\phi(x)=0\}$ for some $\phi\in C_b^2(\mathbb{R}^m)$ such that $|\nabla\phi(x)|=1$ for $x\in\partial\mathcal{O}$.

Let $b:\mathbb{R}^m\to\mathbb{R}^m$ and $\sigma:\mathbb{R}^m\to\mathbb{R}^{m\times m}$ be such that for some $L'>0$, and all $x,x'\in\mathbb{R}^m$,

(B1) $|b(x)-b(x')|+\|\sigma(x)-\sigma(x')\|\le L'|x-x'|$,

and let (X^x,A^x) be a solution of the SDE with reflection

$$X_t^x=x+\int_0^t b(X_s^x)\,ds+\int_0^t\sigma(X_s^x)\,dW_s+A_t^x,\qquad t\in\mathbb{R}^+,$$

where $P(X^x\in\bar{\mathcal{O}})=1$, A^x is a process with locally finite variation $|A^x|$, which increases only if $X_t^x\in\partial\mathcal{O}$; $A_0^x=0$, $X_0=x\in\mathcal{O}\cap G$ ([LS]). Define $\tau^x=\inf\{t\ge 0;\,X_t^x\notin G\}$.

Assume that

(B2) functions $f:\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^{d\times m}\to\mathbb{R}^d$, $\varphi:\mathbb{R}^m\times\mathbb{R}^d\to\mathbb{R}^d$ and $g:\partial G\cap\bar{\mathcal{O}}\to\bar{D}$ are continuous and there exist constants $\kappa,p\ge 0$, $L>0$, $\mu\in\mathbb{R}$ and $\beta<0$ such that $\mu+L^2<0$ and for any $x\in\mathbb{R}^m$, $y,y'\in\mathbb{R}^d$, $z,z'\in\mathbb{R}^{d\times m}$,

$$\begin{gathered}|g(x)|\le\kappa(1+|x|^p),\\ \langle y-y',f(x,y,z)-f(x,y',z)\rangle\le\mu|y-y'|^2,\\ |f(x,y,z)-f(x,y',z')|\le L(|y-y'|+\|z-z'\|),\\ |f(x,y,0)|\le\kappa(1+|y|),\qquad|\varphi(x,y)|\le\kappa\\ |\varphi(x,y)-\varphi(x,y')|\le L|y-y'|,\\ \langle y-y',\varphi(x,y)-\varphi(x,y')\rangle\le\beta|y-y'|^2,\\ E\int_0^{\tau^x}|f(X_t^x,\xi_t,\zeta_t)|^2\,dt<\infty,\end{gathered}$$

where $\xi=g(X_{\tau^x}^x)$, $\xi_t=E(g(X_{\tau^x}^x)|\mathcal{F}_t)$, $\xi=E\xi+\int_0^\infty\zeta_t\,dW_t$.

(B3) $E\tau^x<\infty$.

Let (Y^x,Z^x,K^x) be a solution of the GRBSDE with data $(\tau^x,g(X_{\tau^x}^x),F,\Phi,|A^x|)$, where $F(t,\omega,y,z)=f(X_t^x(\omega),y,z)$, $\Phi(t,\omega,y)=\varphi(X_t^x(\omega),y)$,

$t \in \mathbb{R}^+$, $\omega \in \Omega$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{d\times m}$, i.e.

$$Y^x_{t\wedge\tau^x} = g(X^x_{\tau^x}) + \int_{t\wedge\tau^x}^{\tau^x} f(X^x_\theta, Y^x_\theta, Z^x_\theta)\,d\theta + \int_{t\wedge\tau^x}^{\tau^x} \varphi(X^x_\theta, Y^x_\theta)\,d|A^x|_\theta - \int_{t\wedge\tau^x}^{\tau^x} Z^x_\theta\,dW_\theta + K^x_{\tau^x} - K^x_{t\wedge\tau^x}, \qquad t \in \mathbb{R}^+.$$

In this section we assume that $\mu + L^2 < 0$. Then assumption (A7) reduces to $E(|X^x_{\tau^x}|^{2p} + |A^x|_{\tau^x}) < \infty$. But note that from [PZ, Proposition 3.2] one can deduce in particular that for all $x \in \bar{\mathcal{O}}$ and for any stopping time T such that $E|T| < \infty$, we have $E|A^x|_T < \infty$.

We consider the following problem for a PDE with Neumann boundary conditions: find $u : \bar{\mathcal{O}} \cap \bar{G} \to \bar{D}$ such that

(3.1)
$$\begin{cases} \min\big(u_i(x) - a_i, \max\big(u_i(x) - b_i, -Lu_i(x) - f_i(x, u(x), (\nabla u_i\sigma)(x))\big)\big) = 0, \\ \hfill x \in \mathcal{O} \cap G, \\ \min\left(u_i(x) - a_i, \max\left(u_i(x) - b_i, -\dfrac{\partial u_i}{\partial n}(x) - \varphi_i(x, u(x))\right)\right) = 0, \\ \hfill x \in \partial\mathcal{O} \cap G, \\ u(x) = g(x), \qquad x \in \partial G \cap \bar{\mathcal{O}}, \end{cases}$$

for $i = 1, \ldots, d$, where

$$Lu_i(x) = \frac{1}{2} \sum_{1\le j,k\le m} \frac{\partial^2 u_i}{\partial x_j \partial x_k}(x)(\sigma\sigma^*)_{jk}(x) + \sum_{1\le j\le m} \frac{\partial u_i}{\partial x_j}(x) b_j(x).$$

Definition 3.1 ([CIL]). (i) $u \in C(\bar{\mathcal{O}} \cap \bar{G}; \bar{D})$ is called a *viscosity subsolution* of (3.1) if $u_i(x) \le g_i(x)$, $1 \le i \le d$, $x \in \partial G \cap \bar{\mathcal{O}}$, and moreover for any $\psi \in C^2(\mathbb{R}^m)$, whenever $x \in \bar{\mathcal{O}} \cap G$ is a local maximum of $u_i - \psi$, then

$$\text{(3.2)} \qquad \min\big(u_i(x) - a_i, \max\big(u_i(x) - b_i, -L\psi(x) - f_i(x, u(x), (\nabla\psi\sigma)(x))\big)\big) \le 0$$

if $x \in \mathcal{O} \cap G$, and

$$\min\left(u_i(x) - a_i, \max\left(u_i(x) - b_i, \min\left(-\frac{\partial\psi}{\partial n}(x) - \varphi_i(x, u(x)), -L\psi(x) - f_i(x, u(x), (\nabla\psi\sigma)(x))\right)\right)\right) \le 0, \qquad x \in \partial\mathcal{O} \cap G.$$

(ii) $u \in C(\bar{\mathcal{O}} \cap \bar{G}; \bar{D})$ is called a *viscosity supersolution* of (3.1) if $u_i(x) \ge g_i(x)$, $1 \le i \le d$, $x \in \partial G \cap \bar{\mathcal{O}}$, and moreover for any $\psi \in C^2(\mathbb{R}^m)$, whenever

$x \in \bar{\mathcal{O}} \cap G$ is a local minimum of $u_i - \psi$, then

$$\min\big(u_i(x) - a_i, \max\big(u_i(x) - b_i, -L\psi(x) - f_i(x, u(x), (\nabla\psi\sigma)(x))\big)\big) \geq 0$$

if $x \in \mathcal{O} \cap G$, and

$$\min\Bigg(u_i(x) - a_i, \max\Bigg(u_i(x) - b_i, \max\Bigg(-\frac{\partial\psi}{\partial n}(x) - \varphi_i(x, u(x)), -L\psi(x) - f_i(x, u(x), (\nabla\psi\sigma)(x))\Bigg)\Bigg)\Bigg) \geq 0, \quad x \in \partial\mathcal{O} \cap G.$$

(iii) $u \in C(\bar{\mathcal{O}} \cap \bar{G}; \bar{D})$ is called a *viscosity solution* of (3.1) if it is both a viscosity subsolution and supersolution.

Theorem 3.2. *Assume* (B1)–(B3). *The function u defined as $u(x) = Y_0^x$ for $x \in \bar{\mathcal{O}} \cap \bar{G}$ is a continuous viscosity solution of* (3.1).

Proof. Note that the continuity of u follows by the same arguments as in the proof of Proposition 4.1 in [PZ] upon using the proof of Proposition 4.1 in [P].

Of course, if $x \in \partial G \cap \bar{\mathcal{O}}$, then $\tau^x = 0$ and $u(x) = Y_0^x = g(X_0) = g(x)$. We will only show that u is a subsolution of (3.1); the proof that it is a supersolution is similar. Take any $1 \leq i \leq d$, $\psi \in C^2(\mathbb{R}^m)$ and let $x \in \bar{\mathcal{O}} \cap G$ be a local maximum of $u_i - \psi$. Without loss of generality we may assume that $u_i(x) = \psi(x)$.

First, consider the case when $x \in \mathcal{O} \cap G$. Note that if $u_i(x) = a_i$ or $u_i(x) = b_i$, the inequality (3.2) is obvious. Hence we can assume that $u_i(x) = \psi(x) \in (a_i, b_i)$ and we have to show

$$-L\psi(x) - f_i(x, u(x), (\nabla\psi\sigma)(x)) \leq 0.$$

Suppose for contradiction that $L\psi(x) + f_i(x, u(x), (\nabla\psi\sigma)(x)) < 0$. From continuity of ψ and f we can choose $\alpha > 0$ such that whenever $|y - x| \leq \alpha$ then $u_i(y) \leq \psi(y)$, $\psi(y) \in (a_i, b_i)$ and

$$L\psi(y) + f_i(y, u(y), (\nabla\psi\sigma)(y)) < 0.$$

Define $\upsilon = \inf\{t > 0;\ |X_t^x - x| \geq \alpha\} \wedge \tau_x \wedge \alpha$ and let

$$(\bar{Y}_t, \bar{Z}_t, \bar{K}_t) = ((Y_{t\wedge\upsilon}^x)^i, \mathbf{1}_{[0,\upsilon]}(t)(Z_t^x)^i, (K_{t\wedge\upsilon}^x)^i), \quad t \in [0, \alpha].$$

For $t \in [0, \alpha]$, the triple $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t)$ satisfies the one-dimensional RBSDE

$$\bar{Y}_t = u_i(X_\upsilon^x) + \int_t^\alpha \mathbf{1}_{[0,\upsilon]}(\theta) f_i(X_\theta^x, u(X_\theta^x), \bar{Z}_\theta)\, d\theta - \int_t^\alpha \bar{Z}_\theta\, dW_\theta + \bar{K}_\alpha - \bar{K}_t.$$

On the other hand, by the Itô formula, the pair

$$(\hat{Y}_t, \hat{Z}_t) = (\psi(X_{t\wedge\upsilon}^x), \mathbf{1}_{[0,\upsilon]}(t)(\nabla\psi\sigma)(X_t^x)), \quad t \in [0, \alpha].$$

is a solution of the BSDE

$$\hat{Y}_t = \psi(X^x_\upsilon) - \int_t^\alpha \mathbf{1}_{[0,\upsilon]}(\theta) L\psi(X^x_\theta)\, d\theta - \int_t^\alpha \hat{Z}_\theta\, dW_\theta, \qquad t \in [0,\alpha].$$

Since $\hat{Y}_t \in (a_i, b_i)$ for $t \in [0,\alpha]$, the triple $(\hat{Y}, \hat{Z}, 0)$ is a solution of the above RBSDE. Therefore by the assumption that $u_i \le \psi$ and the choice of α and υ, and with the help of Lemma 4.2 below, we deduce that $\bar{Y}_0 < \hat{Y}_0$, i.e. $u_i(x) < \psi(x)$, which is a contradiction.

Now we consider the case $x \in \partial\mathcal{O} \cap G$. As above we can assume that $u_i(x) \in (a_i, b_i)$. Suppose towards a contradiction that

$$\max\Big(\frac{\partial\psi}{\partial n}(x) + \varphi_i(x, u(x)), L\psi(x) + f_i(x, u(x), (\nabla\psi\sigma)(x))\Big) < 0.$$

Let $\alpha > 0$ be such that whenever $|y - x| \le \alpha$ then $u_i(y) \le \psi(y)$, $\psi(y) \in (a_i, b_i)$ and

$$\max\Big(\frac{\partial\psi}{\partial n}(y) + \varphi_i(y, u(y)), L\psi(y) + f_i(y, u(y), (\nabla\psi\sigma)(y))\Big) < 0.$$

Define $\upsilon = \inf\{t \ge 0;\ |X^x_t - x| \ge \alpha\} \wedge \alpha \wedge \tau_x$ and let

$$(\bar{Y}_t, \bar{Z}_t, \bar{K}_t) = ((Y^x_{t\wedge\upsilon})^i, \mathbf{1}_{[0,\upsilon]}(t)(Z^x_t)^i, (K^x_{t\wedge\upsilon})^i), \qquad t \in [0,\alpha].$$

The triple $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t)$ satisfies the one-dimensional GRBSDE

$$\begin{aligned}\bar{Y}_t = u_i(X^x_\upsilon) &+ \int_t^\alpha \mathbf{1}_{[0,\upsilon]}(\theta) f_i(X^x_\theta, u(X^x_\theta), \bar{Z}_\theta)\, d\theta \\ &+ \int_t^\alpha \mathbf{1}_{[0,\upsilon]}(\theta)\varphi_i(X^x_\theta, u(X^x_\theta))\, d|A^x|_\theta - \int_t^\alpha \bar{Z}_\theta\, dW_\theta + \bar{K}_\alpha - \bar{K}_t.\end{aligned}$$

On the other hand, by the Itô formula, the pair

$$(\hat{Y}_t, \hat{Z}_t) = (\psi(X^x_{t\wedge\upsilon}), \mathbf{1}_{[0,\upsilon]}(t)(\nabla\psi\sigma)(X^x_t)), \qquad t \in [0,\alpha],$$

satisfies the generalized BSDE

$$\begin{aligned}\hat{Y}_t = \psi(X^x_\upsilon) &- \int_t^\alpha \mathbf{1}_{[0,\upsilon]}(\theta) L\psi(X^x_\theta)\, d\theta - \int_t^\alpha \mathbf{1}_{[0,\upsilon]}(\theta)\frac{\partial\psi}{\partial n}(X^x_\theta)\, d|A^x|_\theta \\ &- \int_t^\alpha \hat{Z}_\theta\, dW_\theta, \qquad t \in [0,\alpha],\end{aligned}$$

and $\hat{Y}_t \in (a_i, b_i)$ for $t \in [0,\alpha]$. From $u_i \le \psi$ and the choice of α and υ, with the help of Lemma 4.2, we get $\bar{Y}_0 < \hat{Y}_0$, i.e. $u_i(x) < \psi(x)$, which is a contradiction. ■

4. Proof of the main theorem. Before proving Theorem 2.2 we will show some useful facts.

PROPOSITION 4.1. *Let* (Y, Z, K) *be a solution of* (2.1). *Suppose that conditions* (A1)–(A4), (A6), (A7) *are satisfied. Then for any* $a \in D$ *there exists a constant* $C > 0$ *such that*

$$\text{(a) } E\Big(\sup_{t\le\tau} e^{\lambda t}|Y_t - a|^2 + \int_0^\tau e^{\lambda t}|Y_t - a|^2\, d\Gamma_t + \int_0^\tau e^{\lambda t}\|Z_t\|^2\, dt + \int_0^\tau e^{\lambda t}\, d|K|_t\Big) \le CE\Big(e^{\lambda\tau}|\xi - a|^2 + \int_0^\tau e^{\lambda t}|f(t,a,0)|^2\, dt + \int_0^\tau e^{\lambda t}|\varphi(t,a)|^2\, d\Lambda_t\Big),$$

$$\text{(b) } E\Big(\sup_{t\le\tau} e^{\lambda t+\nu\Lambda_t}|Y_t - a|^2 + \int_0^\tau e^{\lambda t+\nu\Lambda_t}|Y_t - a|^2\, d\Gamma_t + \int_0^\tau e^{\lambda t+\nu\Lambda_t}\|Z_t\|^2\, dt\Big) \le CE\Big(e^{\lambda\tau+\nu\Lambda_\tau}|\xi - a|^2 + \int_0^\tau e^{\lambda t+\nu\Lambda_t}(|f(t,a,0)|^2\, dt + |\varphi(t,a)|^2\, d\Lambda_t)\Big),$$

where $\Gamma_t = \Lambda_t + t$.

Proof. (a) From the Itô formula, for $t \in \mathbb{R}^+$,

$$(4.1)\qquad e^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau} - a|^2 + \int_{t\wedge\tau}^\tau e^{\lambda s}(\lambda|Y_s - a|^2 + \|Z_s\|^2)\, ds = e^{\lambda\tau}|\xi - a|^2 + 2\int_{t\wedge\tau}^\tau e^{\lambda s}(Y_s - a)f(s, Y_s, Z_s)\, ds + 2\int_{t\wedge\tau}^\tau e^{\lambda s}(Y_s - a)\varphi(s, Y_s)\, d\Lambda_s + 2\int_{t\wedge\tau}^\tau e^{\lambda s}(Y_s - a)\, dK_s - 2\int_{t\wedge\tau}^\tau e^{\lambda s}(Y_s - a)Z_s\, dW_s.$$

Using (A1)–(A4) we obtain

$$\begin{aligned} 2\langle y - a, f(t,y,z)\rangle &\le 2\mu|y - a|^2 + 2L|y - a|\,\|z\| + 2|y - a|\,|f(t,a,0)| \\ &\le (2\mu + L^2/\varepsilon + \eta)|y - a|^2 + \varepsilon\|z\|^2 + \frac{1}{\eta}|f(t,a,0)|^2, \\ 2\langle y - a, \varphi(t,y)\rangle &\le 2\beta|y - a|^2 + 2|y - a|\,|\varphi(t,a)| \\ &\le \beta|y - a|^2 + \frac{1}{|\beta|}|\varphi(t,a)|^2. \end{aligned}$$

Let ε and η be such that $\tilde{\varepsilon} = 1 - \varepsilon > 0$ and $\tilde{\lambda} = \lambda - (2\mu + L^2/\varepsilon + \eta) > 0$. With this notation, (4.1) has the form

$$
\begin{aligned}
(4.2)\qquad & e^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau} - a|^2 + \int_{t\wedge\tau}^{\tau} e^{\lambda s}(\tilde{\lambda}|Y_s - a|^2 + \tilde{\varepsilon}\|Z_s\|^2)\,ds \\
&\qquad + |\beta| \int_{t\wedge\tau}^{\tau} e^{\lambda s}|Y_s - a|^2\,d\varLambda_s \\
&\le e^{\lambda\tau}|\xi - a|^2 + \frac{1}{\eta}\int_{t\wedge\tau}^{\tau} e^{\lambda s}|f(s,a,0)|^2\,ds + \frac{1}{|\beta|}\int_{t\wedge\tau}^{\tau} e^{\lambda s}|\varphi(s,a)|^2\,d\varLambda_s \\
&\quad + 2\int_{t\wedge\tau}^{\tau} e^{\lambda s}(Y_s - a)\,dK_s - 2\int_{t\wedge\tau}^{\tau} e^{\lambda s}(Y_s - a)Z_s\,dW_s.
\end{aligned}
$$

Using (2.2) and integrating we get

$$
\begin{aligned}
(4.3)\qquad & E\Big(e^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau} - a|^2 + \int_{t\wedge\tau}^{\tau} e^{\lambda s}|Y_s - a|^2\,d\varGamma_s + \int_{t\wedge\tau}^{\tau} e^{\lambda s}\|Z_s\|^2\,ds\Big) \\
&\le CE\Big(e^{\lambda\tau}|\xi - a|^2 + \int_{t\wedge\tau}^{\tau} e^{\lambda s}|f(s,a,0)|^2\,ds + \int_{t\wedge\tau}^{\tau} e^{\lambda s}|\varphi(s,a)|^2\,d\varLambda_s\Big).
\end{aligned}
$$

In order to get an estimate on $E\sup_{t\le\tau} e^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau} - a|^2$ first note that from the Burkholder–Davis–Gundy and Schwarz inequalities we have

$$
E\sup_{t\le\tau}\Big|\int_{t\wedge\tau}^{\tau} e^{\lambda s}(Y_s - a)Z_s\,dW_s\Big| \le \frac{1}{4}E\sup_{t\le\tau} e^{\lambda t}|Y_t - a|^2 + CE\int_0^{\tau} e^{\lambda t}\|Z_t\|^2\,dt.
$$

Therefore using (2.2) and taking supremum in (4.2) we obtain

$$
\begin{aligned}
E\sup_{t\le\tau} e^{\lambda t}|Y_t - a|^2 &\le CE\Big(e^{\lambda\tau}|\xi - a|^2 + \int_0^{\tau} e^{\lambda s}|f(s,a,0)|^2\,ds \\
&\quad + \int_0^{\tau} e^{\lambda s}|\varphi(s,a)|^2\,d\varLambda_s\Big) + \frac{1}{2}E\sup_{t\le\tau} e^{\lambda t}|Y_t - a|^2 + CE\int_0^{\tau} e^{\lambda s}\|Z_s\|^2\,ds.
\end{aligned}
$$

Hence by (4.3),

$$
\begin{aligned}
E\sup_{t\le\tau} e^{\lambda t}|Y_t - a|^2 &\le CE\Big(e^{\lambda\tau}|\xi - a|^2 + \int_0^{\tau} e^{\lambda s}|f(s,a,0)|^2\,ds \\
&\qquad + \int_0^{\tau} e^{\lambda s}|\varphi(s,a)|^2\,d\varLambda_s\Big).
\end{aligned}
$$

It is known that for any convex set D, $\langle h - a, \mathbf{n}_h\rangle \le -\mathrm{dist}(a, \partial D)$, $h \in \partial D$, $a \in D \setminus \partial D$, where $\mathbf{n}_h$ is the normal inward vector at $h \in \partial D$ (see e.g. [M]).

Therefore,

$$\int_0^\tau e^{\lambda s}(Y_s - a)\,dK_s = \int_t^T e^{\lambda s}(Y_s - a)\mathbf{n}_{Y_s}\,d|K|_s \le -\mathrm{dist}(a, \partial D)\int_0^\tau e^{\lambda s}\,d|K|_s$$

and by (4.2),

$$E\int_0^\tau e^{\lambda t}\,d|K|_t \le CE\Big(e^{\lambda\tau}|\xi - a|^2 + \int_0^\tau e^{\lambda t}|f(t,a,0)|^2\,dt + \int_0^\tau e^{\lambda t}|\varphi(t,a)|^2\,d\Lambda_t\Big).$$

(b) We use integration by parts for $e^{\lambda(t\wedge\tau)+\nu\Lambda_{t\wedge\tau}}|Y_{t\wedge\tau} - a|^2$. Similarly to (a), we have

$$\begin{aligned}
Ee^{\lambda(t\wedge\tau)+\nu\Lambda_{t\wedge\tau}}|Y_{t\wedge\tau} - a|^2 &+ E\int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}(\lambda|Y_s - a|^2 + \|Z_s\|^2)\,ds \\
&\qquad + \nu E\int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}|Y_s - a|^2\,d\Lambda_s \\
&\le Ee^{\lambda\tau+\nu\Lambda_\tau}|\xi - a|^2 + (2\mu + L^2/\varepsilon + \eta)E\int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}|Y_s - a|^2\,ds \\
&\quad + \frac{1}{\eta}E\int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}|f(s,a,0)|^2\,ds + \varepsilon E\int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}\|Z_s\|^2\,ds \\
&\quad + \beta E\int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}|Y_s - a|^2\,d\Lambda_s + \frac{1}{|\beta|}E\int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}|\varphi(s,a)|^2\,d\Lambda_s.
\end{aligned}$$

Let $\varepsilon < 1$ and η be such that $2\mu + L^2/\varepsilon + \eta < \lambda$. Since $\nu > \beta$,

$$\begin{aligned}
E\Big(e^{\lambda(t\wedge\tau)+\nu\Lambda_{t\wedge\tau}}|Y_{t\wedge\tau} - a|^2 &+ \int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}(|Y_s - a|^2(ds + d\Lambda_s) + \|Z_s\|^2 ds)\Big) \\
&\le CE\Big(e^{\lambda\tau+\nu\Lambda_\tau}|\xi - a|^2 + \int_{t\wedge\tau}^{\tau} e^{\lambda s+\nu\Lambda_s}(|f(s,a,0)|^2\,ds + |\varphi(s,a)|^2\,d\Lambda_s)\Big).
\end{aligned}$$

Again arguing as in (a) we complete the proof. ∎

REMARK. The above estimates for $\sup_{t\le\tau} e^{\lambda t}|Y_t - a|^2$, $\int_0^\tau e^{\lambda t}\|Z_t\|^2\,dt$ and $\int_0^\tau e^{\lambda t}|Y_t - a|^2\,d\Gamma_t$ remain true for $a \in \bar{D}$.

LEMMA 4.2. *Suppose that $d = 1$. Let f, f' and φ, φ' satisfy* (A1)–(A4). *Let (Y,Z,K), (Y',Z',K') be solutions of the GRBSDEs with random terminal time with data (τ,ξ,f,φ), (τ,ξ',f',φ') respectively, where ξ, ξ' satisfy* (A7). *If $\xi \le \xi'$, $f(t,y,z) \le f'(t,y,z)$, $\varphi(t,y) \le \varphi'(t,y)$, $\mathcal{P}$-a.s., then $Y_t \le Y'_t$, $t \in \mathbb{R}^+$, $\mathcal{P}$-a.s. Additionally, if $P(\xi < \xi') > 0$, $f(t,y,z) < f'(t,y,z)$ or $\varphi(t,y) < \varphi'(t,y)$, then $Y_0 < Y'_0$.*

Proof. From the Itô formula

$$
\begin{aligned}
&e^{\lambda t}|(Y_{t\wedge\tau}-Y'_{t\wedge\tau})^+|^2+\int_{t\wedge\tau}^{\tau} e^{\lambda s}(\lambda|(Y_{s\wedge\tau}-Y'_{s\wedge\tau})^+|^2+\mathbf{1}_{\{Y_s>Y'_s\}}|Z_s-Z'_s|^2)\,ds\\
&\quad= e^{\lambda\tau}|(\xi-\xi')^+|^2+2\int_{t\wedge\tau}^{\tau} e^{\lambda s}(Y_s-Y'_s)^+(f(s,Y_s,Z_s)-f'(s,Y'_s,Z'_s))\,ds\\
&\qquad+2\int_{t\wedge\tau}^{\tau} e^{\lambda s}(Y_s-Y'_s)^+(\varphi(s,Y_s)-\varphi'(s,Y'_s))\,d\Lambda_s\\
&\qquad+2\int_{t\wedge\tau}^{\tau} e^{\lambda s}(Y_s-Y'_s)^+d(K_s-K'_s)-2\int_{t\wedge\tau}^{\tau} e^{\lambda s}(Y_s-Y'_s)^+(Z_s-Z'_s)\,dW_s\\
&\quad\le(2\mu+L^2/\varepsilon)\int_{t\wedge\tau}^{\tau} e^{\lambda s}|(Y_s-Y'_s)^+|^2\,ds+\varepsilon\int_{t\wedge\tau}^{\tau} e^{\lambda s}\mathbf{1}_{\{Y_s>Y'_s\}}|Z_s-Z'_s|^2\,ds\\
&\qquad+2\beta\int_{t\wedge\tau}^{\tau}\mathbf{1}_{\{Y_s>Y'_s\}}|Y_s-Y'_s|^2\,d\Lambda_s-2\int_{t\wedge\tau}^{\tau}(Y_s-Y'_s)^+(Z_s-Z'_s)\,dW_s.
\end{aligned}
$$

Since $(\int_0^t(Y_s-Y'_s)^+(Z_s-Z'_s)\,dW_s)_{t\in\mathbb{R}^+}$ is a martingale and $\beta<0$, choosing $\varepsilon<1$ such that $2\mu+L^2/\varepsilon<\lambda$ we complete the proof. ∎

Proof of Theorem 2.2. First we will show uniqueness for (2.1). Suppose that (Y,Z,K) and (Y',Z',K') are two solutions of (2.1). By the Itô formula and arguing as in the proof of Lemma 4.2 we find that

$$
\begin{aligned}
&e^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau}-Y'_{t\wedge\tau}|^2+\int_{t\wedge\tau}^{\tau} e^{\lambda s}(\lambda|Y_s-Y'_s|^2+\|Z_s-Z'_s\|^2)\,ds\\
&\qquad\le(2\mu+L^2/\varepsilon)\int_{t\wedge\tau}^{\tau} e^{\lambda s}|Y_s-Y'_s|^2\,ds+\varepsilon\int_{t\wedge\tau}^{\tau} e^{\lambda s}\|Z_s-Z'_s\|^2\,ds\\
&\qquad\quad+2\beta\int_{t\wedge\tau}^{\tau} e^{\lambda s}|Y_s-Y'_s|^2\,d\Lambda_s-2\int_{t\wedge\tau}^{\tau} e^{\lambda s}(Y_s-Y'_s)(Z_s-Z'_s)\,dW_s.
\end{aligned}
$$

Choosing $\varepsilon<1$ such that $2\mu+L^2/\varepsilon<\lambda$ we obtain

$$
E\Big(e^{\lambda(t\wedge\tau)}|Y_{t\wedge\tau}-Y'_{t\wedge\tau}|^2+\int_{t\wedge\tau}^{\tau} e^{\lambda s}\|Z_s-Z'_s\|^2\,ds+\int_{t\wedge\tau}^{\tau} e^{\lambda s}|Y_s-Y'_s|^2\,d\Lambda_s\Big)=0,
$$

which shows the uniqueness.

The proof of existence for (2.1) is divided into two steps. In the first step we will construct a sequence (Y^M,Z^M,K^M), $M\in\mathbb{N}$, of solutions of some GRBSDEs and in the second step we will show that this sequence converges to the solution of (2.1).

STEP 1. For each natural M we will construct a solution (Y^M, Z^M, K^M) of a GRBSDE of the form

$$(4.4)\qquad Y_t^M = \xi + \int_{t\wedge\tau}^{M\wedge\tau} f(s, Y_s^M, Z_s^M)\, ds + \int_{t\wedge\tau}^{M\wedge\tau} \varphi(s, Y_s^M)\, d\Lambda_s - \int_{t\wedge\tau}^{\tau} Z_s^M\, dW_s + K_{M\wedge\tau}^M - K_{t\wedge\tau}^M, \quad t \in \mathbb{R}^+,$$

in the following way. Let $\xi_M = E(\xi|\mathcal{F}_M)$ and let Λ^τ be a stopped process for the stopping time τ, i.e. $\Lambda_t^\tau = \Lambda_{t\wedge\tau}$. For $t \in [0, M]$ consider

$$Y_t^M = \xi_M + \int_t^M \mathbf{1}_{[0,\tau]}(s) f(s, Y_s^M, Z_s^M)\, ds + \int_t^M \varphi(s, Y_s^M)\, d\Lambda_s^\tau - \int_t^M Z_s^M\, dW_s + K_M^M - K_t^M.$$

The terminal value ξ_M is an $\mathcal{F}_M$ measurable random variable, the function $\mathbf{1}_{[0,\tau(\omega)]}(t) f(t, \omega, y, z)$ is Lipschitz continuous with respect to y and z, and by (A3), $\varphi(t, \omega, y)$ is Lipschitz continuous with respect to y. Therefore by [J2] there exists a unique solution (Y^M, Z^M, K^M) of (4.4) on $[0, M]$. Note that on the set $\{t \geq \tau\}$ we have $\xi_M = \xi$ and

$$Y_t^M = \xi + 0 - \int_t^M Z_s^M\, dW_s + K_M^M - K_t^M.$$

Since $\xi \in \bar{D}$, by uniqueness of solution of BSDEs (see [P]) it follows that $K_M^M = K_t^M = K_\tau^M$. Therefore $Y_t^M = \xi - \int_t^M Z_s^M\, dW_s$, and in particular $Y_\tau^M = E(\xi|\mathcal{F}_\tau) = \xi$. On the other hand, by the Itô formula

$$E\Big(|Y_\tau^M|^2 + \int_\tau^M \|Z_s^M\|^2\, ds\Big) = E|\xi|^2.$$

As a consequence, on the set $\{t \geq \tau\}$, $Y_t^M = \xi$ and $Z_t^M = 0$.

For $t > M$ we put $Y_t^M = \xi_t$, $Z_t^M = \zeta_t$ and $K_t^M = K_M^M$. Note that these processes satisfy

$$Y_t^M = \xi - \int_t^\tau Z_s^M\, dW_s,$$

and on the set $\{t \geq \tau\}$ we have $Y_t^M = \xi_t = \xi$, $Z_t^M = 0$.

By Proposition 4.1(a) the processes defined above satisfy

$$E\Big(\sup_{t\leq\tau} e^{\lambda t}|Y_t^M - a|^2 + \int_0^\tau e^{\lambda t}(|Y_t^M - a|^2\, d\Gamma_t + \|Z_t^M\|^2\, dt + d|K^M|_t)\Big) \leq CE\Big(e^{\lambda\tau}|\xi - a|^2 + \int_0^\tau e^{\lambda t}|f(t, a, 0)|^2\, dt + \int_0^\tau e^{\lambda t}|\varphi(t, a)|^2\, d\Lambda_t\Big).$$

Since (Y_t^M, Z_t^M, K_t^M), $t \in [0, M]$, is a unique solution of (4.4), we have $E\sup_{t\le M} e^{\lambda t}|K_t^M|^2 < \infty$ and $\int_0^M (Y_t^M - S_t)\,dK_t^M \le 0$ for any $\mathcal{F}$ progressively measurable process $S = (S_t)_{t\in\mathbb{R}^+}$ with values in $\bar{D}$ (see [J2, Definition 2.1]). By the equality $K_t^M = K_\tau^M$ on the set $\{t \ge \tau\}$ we also have

$$\int_0^\tau (Y_t^M - S_t)\,dK_t^M \le 0.$$

STEP 2. We will show that the sequence (Y^M, Z^M, K^M), $M \in \mathbb{N}$, defined above is a Cauchy sequence for the norm

$$\|(Y^M, Z^M, K^M)\|_\Lambda^2 = E\Big(\sup_{t\le\tau} e^{\lambda t}|Y_t^M|^2 + \int_0^\tau e^{\lambda t}(|Y_t^M|^2\,d\Lambda_t + \|Z_t^M\|^2\,dt) + \sup_{t\le\tau} e^{\lambda t}|K_t^M|^2\Big).$$

Take $N, M \in \mathbb{N}$, $N < M$. For $t \in [N, M]$ we have $Y_t^N = \xi - \int_{t\wedge\tau}^\tau Z_s^N\,dW_s$ and

$$Y_t^M = \xi + \int_{t\wedge\tau}^{M\wedge\tau} f(s, Y_s^M, Z_s^M)\,ds + \int_{t\wedge\tau}^{M\wedge\tau} \varphi(s, Y_s^M)\,d\Lambda_s - \int_{t\wedge\tau}^{\tau} Z_s^M\,dW_s + K_{M\wedge\tau}^M - K_{t\wedge\tau}^M.$$

In particular, $Y_M^N = \xi - \int_{M\wedge\tau}^\tau Z_s^N\,dW_s$ and $Y_M^M = \xi - \int_{M\wedge\tau}^\tau Z_s^M\,dW_s$. By the uniqueness of solution of BSDE on the set $\{M\wedge\tau \le t \le \tau\}$ it follows that $Y_M^N = E(\xi|\mathcal{F}_M) = Y_M^M$ and $Z_t^N = Z_t^M$ (see [P]). Therefore

$$Y_t^M - Y_t^N = \int_{t\wedge\tau}^{M\wedge\tau} f(s, Y_s^M, Z_s^M)\,ds + \int_{t\wedge\tau}^{M\wedge\tau} \varphi(s, Y_s^M)\,d\Lambda_s - \int_{t\wedge\tau}^{M\wedge\tau} (Z_s^M - Z_s^N)\,dW_s + K_{M\wedge\tau}^M - K_{t\wedge\tau}^M.$$

And by the Itô formula for $t \in [N, M]$,

$$\begin{aligned}
&e^{\lambda t}|Y_t^M - Y_t^N|^2 + \int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(\lambda|Y_s^M - Y_s^N|^2 + \|Z_s^M - Z_s^N\|^2)\,ds \\
&\quad = 2\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(Y_s^M - Y_s^N) f(s, Y_s^M, Z_s^M)\,ds \\
&\qquad + 2\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(Y_s^M - Y_s^N)\varphi(s, Y_s^M)\,d\Lambda_s + 2\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(Y_s^M - Y_s^N)\,dK_s^M \\
&\qquad - 2\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(Y_s^M - Y_s^N)(Z_s^M - Z_s^N)\,dW_s.
\end{aligned}$$

Arguing as in the proof of Proposition 4.1 (see (4.2)) we have

$$
\begin{aligned}
(4.5)\qquad e^{\lambda t}|Y_t^M - Y_t^N|^2 &+ \int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(\tilde{\lambda}|Y_s^M - Y_s^N|^2 + \tilde{\varepsilon}\|Z_s^M - Z_s^N\|^2)\,ds \\
&+ |\beta| \int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}|Y_s^M - Y_s^N|^2\,d\Lambda_s \\
\le \frac{1}{\eta}\int_{t\wedge\tau}^{M\wedge\tau} & e^{\lambda s}|f(s, Y_s^N, Z_s^N)|^2\,ds + \frac{1}{|\beta|}\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}|\varphi(s, Y_s^N)|^2\,d\Lambda_s \\
+ 2\int_{t\wedge\tau}^{M\wedge\tau} & e^{\lambda s}(Y_s^M - Y_s^N)\,dK_s^M - 2\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(Y_s^M - Y_s^N)(Z_s^M - Z_s^N)\,dW_s.
\end{aligned}
$$

Since on the set $\{s \ge N\}$ we have $Y_s^N = \xi_s$, $Z_s^N = \zeta_s$, after integrating the above inequality and by (2.2),

$$
\begin{aligned}
Ee^{\lambda t}|Y_t^M - Y_t^N|^2 &+ E\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(|Y_s^M - Y_s^N|^2\,d\Gamma_s + \|Z_s^M - Z_s^N\|^2\,ds) \\
&\le CE\Big(\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}|f(s,\xi_s,\zeta_s)|^2\,ds + \int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}|\varphi(s,\xi_s)|^2\,d\Lambda_s\Big).
\end{aligned}
$$

Note that by the Burkholder–Davis–Gundy and Schwarz inequalities,

$$
\begin{aligned}
E\sup_{N\le t\le M}&\Big|\int_{t\wedge\tau}^{M\wedge\tau} e^{\lambda s}(Y_s^M - Y_s^N)(Z_s^M - Z_s^N)\,dW_s\Big| \\
&\le \frac{1}{4}E\sup_{N\le t\le M} e^{\lambda t\wedge\tau}|Y_{t\wedge\tau}^M - Y_{t\wedge\tau}^N|^2 + CE\int_{N\wedge\tau}^{M\wedge\tau} e^{\lambda t}\|Z_t^M - Z_t^N\|^2\,dt.
\end{aligned}
$$

Hence by (4.5),

$$
\begin{aligned}
E\sup_{N\le t\le M} e^{\lambda t}|Y_t^M - Y_t^N|^2 &+ E\int_{N\wedge\tau}^{M\wedge\tau} e^{\lambda s}(|Y_s^M - Y_s^N|^2\,d\Gamma_s + \|Z_s^M - Z_s^N\|^2\,ds) \\
&\le CE\Big(\int_{N\wedge\tau}^{M\wedge\tau} e^{\lambda s}|f(s,\xi_s,\zeta_s)|^2\,ds + \int_{N\wedge\tau}^{M\wedge\tau} e^{\lambda s}|\varphi(s,\xi_s)|^2\,d\Lambda_s\Big).
\end{aligned}
$$

It is clear that the right hand side of the above inequality tends to zero as $N \to \infty$. By (4.4) we also get

$$
\lim_{N\to\infty} E\sup_{N\le t\le M} e^{\lambda t}|K_t^M - K_t^N|^2 = \lim_{N\to\infty} E\sup_{N\le t\le M} e^{\lambda t}|K_t^M|^2 = 0.
$$

Suppose now that $t \le N < M$. Since

$$Y_t^M = Y_N^M + \int_{t\wedge\tau}^{N\wedge\tau} f(s, Y_s^M, Z_s^M)\, ds + \int_{t\wedge\tau}^{N\wedge\tau} \varphi(s, Y_s^M)\, d\Lambda_s - \int_{t\wedge\tau}^{N\wedge\tau} Z_s^M\, dW_s + K_N^M - K_t^M,$$

we have

$$\begin{aligned} Y_t^M - Y_t^N = Y_N^M - Y_N^N &+ \int_{t\wedge\tau}^{N\wedge\tau} (f(s, Y_s^M, Z_s^M) - f(s, Y_s^N, Z_s^N))\, ds \\ &+ \int_{t\wedge\tau}^{N\wedge\tau} (\varphi(s, Y_s^M) - \varphi(s, Y_s^N))\, d\Lambda_s \\ &- \int_{t\wedge\tau}^{N\wedge\tau} (Z_s^M - Z_s^N)\, dW_s + K_N^M - K_t^M - (K_N^N - K_t^N). \end{aligned}$$

Arguing as in the proof of uniqueness we obtain

$$\begin{aligned} E\Big(e^{\lambda(t\wedge\tau)}|Y_t^M - Y_t^N|^2 + \int_0^{N\wedge\tau} e^{\lambda s}\|Z_s^M - Z_s^N\|^2\, ds\Big) &\le E e^{\lambda(N\wedge\tau)}|Y_N^M - Y_N^N|^2 \\ &\le CE\Big(\int_{N\wedge\tau}^{\tau} e^{\lambda s}|f(s, \xi_s, \zeta_s)|^2\, ds + \int_{N\wedge\tau}^{\tau} e^{\lambda s}|\varphi(s, \xi_s)|^2\, d\Lambda_s\Big), \end{aligned}$$

where the second inequality comes from the previous part of the proof (for $t \in [N, M]$). It is not difficult to see that $E\int_{N\wedge\tau}^{\tau} e^{\lambda s}(|f(s, \xi_s, \zeta_s)|^2\, ds + |\varphi(s, \xi_s)|^2\, d\Lambda_s)$ converges to zero as $N \to \infty$. Similarly,

$$\begin{aligned} &E \sup_{t\le N} e^{\lambda t}|Y_t^M - Y_t^N|^2 \\ &\quad \le CE\Big(\int_{N\wedge\tau}^{\tau} e^{\lambda s}|f(s, \xi_s, \zeta_s)|^2\, ds + \int_{N\wedge\tau}^{\tau} e^{\lambda s}|\varphi(s, \xi_s)|^2\, d\Lambda_s\Big), \end{aligned}$$

where the right hand side tends to zero as $N \to \infty$. Hence

$$\lim_{N\to\infty} E \sup_{t\le N} e^{\lambda t}|K_t^M - K_t^N|^2 = 0.$$

To finish the proof consider the case $t > M > N$. Since $Y_t^N = Y_t^M = \xi_t$, $K_t^N = K_N^N$ and $K_t^M = K_M^M$, by uniqueness of solution we have $Z_t^N = Z_t^M$.

Hence, for any $t \in \mathbb{R}^+$, the sequence (Y^M, Z^M, K^M) defined by (4.4) is Cauchy for the norm $\|(Y^M, Z^M, K^M)\|_\Lambda$, and its limit (Y, Z, K) solves (2.1). (It satisfies estimates from Proposition 4.1.) Moreover, $\int_0^\tau (Y_t - S_t)\, dK_t \le 0$ for any $\mathcal{F}$ progressively measurable process $S = (S_t)_{t\in\mathbb{R}^+}$. ∎

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