BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 61, No. 2, 2013

NUMBER THEORY

# The Sylow $p$-Subgroups of Tame Kernels in Dihedral Extensions of Number Fields

by

Qianqian CUI and Haiyan ZHOU

*Presented by Andrzej SKOWROŃSKI*

**Summary.** Let $F/E$ be a Galois extension of number fields with Galois group $D_{2^n}$. In this paper, we give some expressions for the order of the Sylow $p$-subgroups of tame kernels of $F$ and some of its subfields containing $E$, where $p$ is an odd prime. As applications, we give some results about the order of the Sylow $p$-subgroups when $F/E$ is a Galois extension of number fields with Galois group $D_{16}$.

**1. Introduction.** Let $F$ be a number field, $\mathcal{O}_F$ the ring of integers in $F$, and $K_2(F)$ the Milnor $K$-group of $F$. The tame symbol on $F$ induces, for each finite prime ideal $\mathfrak{p}$, a homomorphism

$$\tau_{\mathfrak{p}} : K_2(F) \to k_{\mathfrak{p}}^*$$

defined by

$$\tau_{\mathfrak{p}}\{a, b\} \equiv (-1)^{\nu_{\mathfrak{p}}(a)\nu_{\mathfrak{p}}(b)} \frac{a^{\nu_{\mathfrak{p}}(b)}}{b^{\nu_{\mathfrak{p}}(a)}} \pmod{\mathfrak{p}},$$

where $\nu_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic valuation. The *tame kernel* of $F$ is the kernel of $\tau$, where

$$\tau = \bigoplus \tau_{\mathfrak{p}} : K_2(F) \to \bigoplus_{\mathfrak{p}\ \text{finite}} k_{\mathfrak{p}}^*.$$

In 1973, Quillen [6] proved that the $K$-group $K_2(\mathcal{O}_F)$ coincides with the tame kernel, and $K_2(\mathcal{O}_F)$ is finite.

There are many results describing the structure of the tame kernels of algebraic number fields and relating them to the class numbers of appropriate fields. The 2-primary part of the tame kernel $K_2(\mathcal{O}_F)$ for number fields

2010 *Mathematics Subject Classification*: 11R70, 12F10.
*Key words and phrases*: tame kernels, dihedral extensions of number fields.

DOI: 10.4064/ba61-2-4  

$F$ has been intensively studied (see [3], [6]–[8]). Furthermore, there are also some results concerning the $p$-primary part of the tame kernel when $p$ is odd (see [2], [4], [12]–[14]). Let $F/E$ be a Galois extension of number fields with Galois group $D_{2^n}$. The second author [12] obtained some results on tame kernels in the case $n = 3$, i.e., $\mathrm{Gal}(F/E) = D_8$.

In this paper, we prove some expressions for the order of the Sylow $p$-subgroups of tame kernels of $F$ and some of its subfields containing $E$ for any integer $n \geq 3$. As applications, in Section 3, we give some results about the order of the Sylow $p$-subgroups when $F/E$ is a Galois extension of number fields with Galois group $D_{16}$.

**2. Main results.** Throughout the paper we use the following notation:

- $D_{2^n}$ is the dihedral group of order $2^n$, i.e., $D_{2^n} = \langle \sigma, \tau \mid \sigma^{2^{n-1}} = 1, \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$.
- $E^m/E$ is a finite extension of number fields of degree $m$.
- $A(p)$ denotes the Sylow $p$-subgroup of a finite group $A$.
- $|A|$ denotes the order of a finite group $A$.
- $x =_p y$ means $v_p(x) = v_p(y)$, where $x, y \in \mathbb{Z}$.
- $C_m$ is a cyclic group of order $m$.
- $V_4$ is Klein's four group.

Now, we start with some well-known facts which will be the basis of this paper.

Let $F/E$ be a finite extension of number fields. In algebraic $K$-theory, a transfer $\mathrm{tr}_{F/E}$ is defined which is a group homomorphism

$$\mathrm{tr}_{F/E} : K_2(F) \to K_2(E).$$

Denote by $K_2(F/E)$ the kernel of the map $\mathrm{tr}_{F/E} : K_2(\mathcal{O}_F) \to K_2(\mathcal{O}_E)$. Obviously, the Sylow $p$-subgroup $K_2(F/E)(p)$ of $K_2(F/E)$ is the kernel of the map $\mathrm{tr}_{F/E} : K_2(\mathcal{O}_F)(p) \to K_2(\mathcal{O}_E)(p)$.

Lemma 1. *For every prime $p \nmid (F : E)$,*

$$K_2(\mathcal{O}_F)(p) \cong K_2(F/E)(p) \times K_2(\mathcal{O}_E)(p).$$

Lemma 2. *If $L$ is an intermediate field of $F/E$, then*

$$\mathrm{tr}_{F/E} = \mathrm{tr}_{L/E} \circ \mathrm{tr}_{F/L}.$$

Lemma 3. *If $F/E$ is a Galois extension with Galois group $G$, then for every prime $p \nmid (F : E)$, the homomorphism $j : K_2(\mathcal{O}_E)(p) \to K_2(\mathcal{O}_F)(p)$ induced by $E \subset F$ is injective, and the transfer $\mathrm{tr}_{F/E} : K_2(\mathcal{O}_F)(p) \to K_2(\mathcal{O}_E)(p)$ is surjective. Moreover, $j \circ \mathrm{tr}_{F/E} = N_{F/E}$, where $N_{F/E}(x) = \prod_{\sigma\in G} \sigma(x)$.*

Lemma 4 ([12, Theorem 1]). *Let $E^4/E$ be a Galois extension with Galois group $V_4 = \{1, a, b, ab\}$, $E^2_a$ the fixed field of $\langle a\rangle$, $E^2_b$ the fixed field of $\langle b\rangle$, and $E^2_{ab}$ the fixed field of $\langle ab\rangle$. Then for every odd prime $p$,*

$$K_2(E^4/E)(p) \cong K_2(E^2_a/E)(p) \times K_2(E^2_b/E)(p) \times K_2(E^2_{ab}/E)(p),$$

*and*

$$|K_2(\mathcal{O}_{E^4})|\,|K_2(\mathcal{O}_E)|^2 =_p |K_2(\mathcal{O}_{E^2_a})|\,|K_2(\mathcal{O}_{E^2_b})||K_2(\mathcal{O}_{E^2_{ab}})|.$$

Let $E^{2^n}/E$ be a Galois extension with Galois group $D_{2^n}$. In order to get the main theorem, we give the following basic information about the dihedral group $D_{2^n}$.

For every $\sigma^i\tau \in D_{2^n}$ ($0 \le i \le 2^{n-1}-1$, $i$ an integer), we have $(\sigma^i\tau)^2 = \sigma^i(\tau\sigma^i\tau) = \sigma^i\sigma^{-i} = 1$, i.e., $\sigma^i\tau$ is of order 2. Furthermore, $\langle\sigma^i\tau\rangle$ and $\langle\sigma^j\tau\rangle$ are conjugate subgroups iff $2 \mid i+j$. Therefore, the non-trivial subgroups of $D_{2^n}$ and the corresponding fixed fields are as follows:

- $2^{n-1}+1$ subgroups of order 2: $\langle\sigma^{2^{n-2}}\rangle$ and $\langle\sigma^i\tau\rangle$ ($0 \le i \le 2^{n-1}-1$, $i$ an integer). The corresponding fixed fields are respectively $E^{2^{n-1}}$ and $E_i^{2^{n-1}}$. Moreover, $\langle\sigma^{2i}\tau\rangle$ ($0 \le 2i \le 2^{n-1}-2$) are conjugate subgroups, and $\langle\sigma^{2i+1}\tau\rangle$ ($1 \le 2i+1 \le 2^{n-1}-1$) are conjugate subgroups.
- $2^{n-2}+1$ subgroups of order 4: $\langle\sigma^{2^{n-3}}\rangle$ and $\langle\sigma^{2^{n-2}}, \sigma^i\tau\rangle$ ($0 \le i \le 2^{n-2}-1$, $i$ an integer), where $\langle\sigma^{2^{n-3}}\rangle$ is a cyclic group of order 4, and every subgroup $\langle\sigma^{2^{n-2}}, \sigma^i\tau\rangle$ is isomorphic to $V_4$. The corresponding fixed fields are respectively $E^{2^{n-2}}$ and $E_i^{2^{n-2}}$.
- $2^{n-m}+1$ subgroups of order $2^m$ ($3 \le m \le n-1$): $\langle\sigma^{2^{n-m-1}}\rangle$ and $\langle\sigma^{2^{n-m}}, \sigma^i\tau\rangle$ ($0 \le i \le 2^{n-m}-1$, $i$ an integer), where $\langle\sigma^{2^{n-m-1}}\rangle$ is a cyclic group of order $2^m$, and every subgroup $\langle\sigma^{2^{n-m}}, \sigma^i\tau\rangle$ is isomorphic to $D_{2^m}$. The corresponding fixed fields are respectively $E^{2^{n-m}}$ and $E_i^{2^{n-m}}$.

Theorem 1. *Let $E^{2^n}/E$ be a Galois extension of number fields with Galois group $D_{2^n}$, $E^2$ the fixed field of $\langle\sigma\rangle$, and $E_0^{2^{n-1}}$ the fixed field of $\langle\tau\rangle$, $E_1^{2^{n-1}}$ the fixed field of $\langle\sigma\tau\rangle$. Then for every odd prime $p$,*

$$K_2(E^{2^n}/E^2)(p) \cong K_2(E_0^{2^{n-1}}/E)(p) \times K_2(E_1^{2^{n-1}}/E)(p), \tag{2.1}$$

*and*

$$|K_2(\mathcal{O}_{E^{2^n}})|\,|K_2(\mathcal{O}_E)|^2 =_p |K_2(\mathcal{O}_{E^2})|\,|K_2(\mathcal{O}_{E_0^{2^{n-1}}})|\,|K_2(\mathcal{O}_{E_1^{2^{n-1}}})|. \tag{2.2}$$

*Proof.* To prove (2.1), we will construct a map

$$\varphi : K_2(E^{2^n}/E^2)(p) \to K_2(E_0^{2^{n-1}}/E)(p) \times K_2(E_1^{2^{n-1}}/E)(p),$$

and prove that it is an isomorphism.

From $\mathrm{tr}_{E^{2^n}/E} = \mathrm{tr}_{E_i^{2^{n-1}}/E} \circ \mathrm{tr}_{E^{2^n}/E_i^{2^{n-1}}} = \mathrm{tr}_{E^2/E} \circ \mathrm{tr}_{E^{2^n}/E^2}$, we get $\mathrm{tr}_{E_i^{2^{n-1}}/E} \circ \mathrm{tr}_{E^{2^n}/E_i^{2^{n-1}}}(a) = \mathrm{tr}_{E^2/E} \circ \mathrm{tr}_{E^{2^n}/E^2}(a) = \mathrm{tr}_{E^2/E}(1) = 1$ for every $a \in K_2(E^{2^n}/E^2)(p)$, hence $\mathrm{tr}_{E^{2^n}/E_i^{2^{n-1}}}(a) \in K_2(E_i^{2^{n-1}}/E)(p)$, $i = 0, 1$.

Thus for every $a \in K_2(E^{2^n}/E^2)(p)$ we can define

$$\varphi(a) = (\mathrm{tr}_{E^{2^n}/E_0^{2^{n-1}}}(a), \mathrm{tr}_{E^{2^n}/E_1^{2^{n-1}}}(a)).$$

Obviously, $\varphi$ is a homomorphism.

If $\mathrm{tr}_{E^{2^n}/E_0^{2^{n-1}}}(a) = \mathrm{tr}_{E^{2^n}/E_1^{2^{n-1}}}(a) = 1$, then $a \cdot \tau(a) = a \cdot \sigma\tau(a) = 1$, hence $\sigma(a) = a$, so $j \circ \mathrm{tr}_{E^{2^n}/E^2}(a) = a \cdot \sigma(a) \cdot \sigma^2(a) \cdots \sigma^{2^{n-1}-1}(a) = a^{2^{n-1}} = 1$. This implies $a = 1$ since $a \in K_2(E^{2^n}/E^2)(p)$. So $\varphi$ is injective.

For every $b \in K_2(E_0^{2^{n-1}}/E)(p)$, by Lemma 3, there exists $c \in K_2(O_{E^{2^n}})(p)$ such that

$$b = j \circ \mathrm{tr}_{E^{2^n}/E_0^{2^{n-1}}}(c) = N_{E^{2^n}/E_0^{2^{n-1}}}(c) = c \cdot \tau(c);$$

then

$$\begin{aligned} N_{E^{2^n}/E}(c) &= j \circ \mathrm{tr}_{E^{2^n}/E}(c) \\ &= j \circ \mathrm{tr}_{E_0^{2^{n-1}}/E} \circ \mathrm{tr}_{E^{2^n}/E_0^{2^{n-1}}}(c) = j \circ \mathrm{tr}_{E_0^{2^{n-1}}/E}(b) = 1. \end{aligned}$$

Thus

$$j \circ \mathrm{tr}_{E^{2^n}/E^2}(b) = j \circ \mathrm{tr}_{E^{2^n}/E^2}(c \cdot \tau(c)) = N_{E^{2^n}/E}(c) = 1.$$

Hence $b \in K_2(E^{2^n}/E^2)(p)$, so $K_2(E_0^{2^{n-1}}/E)(p)$ can be considered as a subgroup of $K_2(E^{2^n}/E^2)$ $(p)$. Similarly, $K_2(E_1^{2^{n-1}}/E)(p)$ can also be considered as a subgroup of $K_2(E^{2^n}/E^2)(p)$.

If $d \in K_2(E_0^{2^{n-1}}/E)(p) \cap K_2(E_1^{2^{n-1}}/E)(p)$, it is obvious that $d$ is fixed by $\tau$ and by $\sigma\tau$ then it is fixed by $\sigma$. Since $d \in K_2(E^{2^n}/E^2)(p)$, we have $\mathrm{tr}_{E^{2^n}/E^2}(d) = d^{2^{n-1}} = 1$. So $d = 1$, i.e.,

$$K_2(E_0^{2^{n-1}}/E)(p) \cap K_2(E_1^{2^{n-1}}/E)(p) = 1.$$

Thus, we have proved (2.1). By (2.1), we have

$$|K_2(E^{2^n}/E^2)(p)| = |K_2(E_0^{2^{n-1}}/E)(p)|\,|K_2(E_1^{2^{n-1}}/E)(p)|. \tag{2.3}$$

By Lemma 1, we conclude that

$$\begin{gathered} |K_2(\mathcal{O}_{E^{2^n}})| =_p |K_2(E^{2^n}/E^2)|\,|K_2(\mathcal{O}_{E^2})|, \\ |K_2(\mathcal{O}_{E_i^{2^{n-1}}})| =_p |K_2(E_i^{2^{n-1}}/E)|\,|K_2(\mathcal{O}_E)|, \qquad i = 1, 2. \end{gathered}$$

Substituting this in (2.3) proves (2.2).

Theorem 2. *Let $E^{2^n}/E$ be a Galois extension of number fields with Galois group $D_{2^n}$, its subgroups and the corresponding fixed fields as stated* *above. Then for every odd prime $p$ and every $m \in \mathbb{Z}$, $0 \le m \le n-2$, we have*

$$(2.4)\qquad |K_2(\mathcal{O}_{E^{2^{n-m}}})|\,|K_2(\mathcal{O}_E)|^2 =_p |K_2(\mathcal{O}_{E^2})|\,|K_2(\mathcal{O}_{E_0^{2^{n-m-1}}})|\,|K_2(\mathcal{O}_{E_1^{2^{n-m-1}}})|.$$

*Proof.* By Theorem 1, we have proved (2.4) in the case $m = 0$. Next, we will prove it for $1 \le m \le n-2$.

Every subgroup $\langle \sigma^{2^{n-m-1}} \rangle$ is a normal subgroup of $D_{2^n}$, and the corresponding fixed field is $E^{2^{n-m}}$. Since $E^{2^n}/E$ is a Galois extension, by Galois theory $E^{2^{n-m}}/E$ is a Galois extension and $\mathrm{Gal}(E^{2^{n-m}}/E) \cong D_{2^n}/\langle \sigma^{2^{n-m-1}} \rangle$. Then

$$(2.5)\qquad \mathrm{Gal}(E^{2^{n-m}}/E) \cong D_{2^{n-m}}, \quad 1 \le m \le n-3,$$

$$(2.6)\qquad \mathrm{Gal}(E^4/E) \cong V_4.$$

By (2.5) and Theorem 1, we get (2.4) in the case $1 \le m \le n-3$. By (2.6) and Lemma 4, we get (2.4) in the case $m = n-2$. The proof is complete.

THEOREM 3. *Let $E^{2^n}/E$ be a Galois extension of number fields with Galois group $D_{2^n}$, its subgroups and the corresponding fixed fields as stated above. Then for every odd prime $p$ and every $m \in \mathbb{Z}$, $2 \le m \le n-1$, we have*

$$|K_2(O_{E^{2^n}})|\,|K_2(O_{E_0^{2^{n-m}}})|^2 =_p |K_2(O_{E^{2^{n-m+1}}})|\,|K_2(O_{E_0^{2^{n-1}}})|^2,$$

$$|K_2(O_{E^{2^n}})|\,|K_2(O_{E_1^{2^{n-m}}})|^2 =_p |K_2(O_{E^{2^{n-m+1}}})|\,|K_2(O_{E_1^{2^{n-1}}})|^2,$$

*and*

$$|K_2(\mathcal{O}_{E_i^{2^{n-m}}})| =_p \begin{cases} |K_2(\mathcal{O}_{E_0^{2^{n-m}}})|, & 0 \le i \le 2^{n-m}-1,\ i \text{ an even integer},\\ |K_2(\mathcal{O}_{E_1^{2^{n-m}}})|, & 0 \le i \le 2^{n-m}-1,\ i \text{ an odd integer}.\end{cases}$$

*Proof.* Since $E^{2^n}/E$ is a Galois extension, by Galois theory so is $E^{2^n}/E_i^{2^{n-m}}$. Moreover,

$$(2.7)\qquad \mathrm{Gal}(E^{2^n}/E_i^{2^{n-2}}) \cong V_4, \quad 0 \le i \le 2^{n-2}-1,$$

$$(2.8)\qquad \mathrm{Gal}(E^{2^n}/E_i^{2^{n-m}}) \cong D_{2^m}, \quad 3 \le m \le n-1,\ 0 \le i \le 2^{n-m}-1.$$

From (2.7) and Lemma 4, we get

$$(2.9)\qquad |K_2(\mathcal{O}_{E^{2^n}})|\,|K_2(\mathcal{O}_{E_i^{2^{n-2}}})|^2 =_p |K_2(\mathcal{O}_{E^{2^{n-1}}})|\,|K_2(\mathcal{O}_{E_i^{2^{n-1}}})|\,|K_2(\mathcal{O}_{E^{2^{n-1}}_{2^{n-1}+i}})|,$$

where $0 \le i \le 2^{n-2}-1$.

From (2.8) and Theorem 1, we get

$$(2.10)\quad |K_2(O_{E^{2^n}})|\,|K_2(O_{E_i^{2^{n-m}}})|^2 =_p |K_2(O_{E^{2^{n-m+1}}})|\,|K_2(O_{E_i^{2^{n-1}}})|\,|K_2(O_{E^{2^{n-1}}_{2^{n-m+1}+i}})|,$$

where $3 \le m \le n-1$, $0 \le i \le 2^{n-m}-1$.

Therefore, for $2 \le m \le n-1$ and $0 \le i \le 2^{n-m}-1$, we have

$$(2.11)\quad |K_2(\mathcal{O}_{E^{2^n}})|\,|K_2(\mathcal{O}_{E_i^{2^{n-m}}})|^2 =_p |K_2(\mathcal{O}_{E^{2^{n-m+1}}})|\,|K_2(\mathcal{O}_{E_i^{2^{n-1}}})|\,|K_2(\mathcal{O}_{E^{2^{n-1}}_{2^{n-m+1}+i}})|.$$

Since $\langle\tau\rangle, \langle\sigma^2\tau\rangle, \ldots, \langle\sigma^{2^{n-1}-2}\tau\rangle$ are conjugate subgroups, we conclude that $K_2(\mathcal{O}_{E_0^{2^{n-1}}})(p)$, $K_2(\mathcal{O}_{E_2^{2^{n-1}}})(p), \ldots$, $K_2(\mathcal{O}_{E^{2^{n-1}}_{2^{n-1}-2}})(p)$ are all isomorphic, so

$$(2.12)\quad |K_2(\mathcal{O}_{E_0^{2^{n-1}}})(p)| = |K_2(\mathcal{O}_{E_2^{2^{n-1}}})(p)| = \cdots = |K_2(\mathcal{O}_{E^{2^{n-1}}_{2^{n-1}-2}})(p)|.$$

Similarly,

$$(2.13)\quad |K_2(\mathcal{O}_{E_1^{2^{n-1}}})(p)| = |K_2(\mathcal{O}_{E_3^{2^{n-1}}})(p)| = \cdots = |K_2(\mathcal{O}_{E^{2^{n-1}}_{2^{n-1}-1}})(p)|.$$

Hence, when $i$ is an even integer, we have

$$(2.14)\quad |K_2(O_{E^{2^n}})|\,|K_2(O_{E_i^{2^{n-m}}})|^2 =_p |K_2(O_{E^{2^{n-m+1}}})|\,|K_2(O_{E_0^{2^{n-1}}})|^2.$$

When $i$ is an odd integer, we have

$$(2.15)\quad |K_2(O_{E^{2^n}})|\,|K_2(O_{E_i^{2^{n-m}}})|^2 =_p |K_2(O_{E^{2^{n-m+1}}})|\,|K_2(O_{E_1^{2^{n-1}}})|^2.$$

So the theorem is proved.

**3. Applications.** Let $E^{16}/E$ be a Galois extension of number fields with Galois group $D_{16} = \langle\sigma, \tau \mid \sigma^8 = 1,\ \tau^2 = 1,\ \tau\sigma\tau^{-1} = \sigma^{-1}\rangle$. Its non-trivial subgroups and the corresponding fixed fields are as follows:

- 9 subgroups of order 2: $\{1, \sigma^4\}$, $\{1, \sigma^i\tau\}$ $(0 \le i \le 7)$. The corresponding fixed fields are respectively $E^8$, $E_i^8$ $(0 \le i \le 7)$. Furthermore, $\{1, \tau\}$, $\{1, \sigma^2\tau\}$, $\{1, \sigma^4\tau\}$ and $\{1, \sigma^6\tau\}$ are conjugate subgroups, so $E_0^8$, $E_2^8$, $E_4^8$ and $E_6^8$ are isomorphic subfields. Similarly, $\{1, \sigma\tau\}$, $\{1, \sigma^3\tau\}$, $\{1, \sigma^5\tau\}$ and $\{1, \sigma^7\tau\}$ are conjugate subgroups, so $E_1^8$, $E_3^8$, $E_5^8$ and $E_7^8$ are isomorphic subfields.
- 5 subgroups of order 4: $\{1, \sigma^2, \sigma^4, \sigma^6\}$, $\{1, \sigma^4, \tau, \sigma^4\tau\}$, $\{1, \sigma^4, \sigma\tau, \sigma^5\tau\}$, $\{1, \sigma^4, \sigma^2\tau, \sigma^6\tau\}$ and $\{1, \sigma^4, \sigma^3\tau, \sigma^7\tau\}$. The corresponding fixed fields are respectively $E^4$, $E_0^4$, $E_1^4$, $E_2^4$ and $E_3^4$.
- 3 subgroups of order 8: $\{1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \sigma^6, \sigma^7\}$, $\{1, \sigma^2, \sigma^4, \sigma^6, \tau, \sigma^2\tau, \sigma^4\tau, \sigma^6\tau\}$ and $\{1, \sigma^2, \sigma^4, \sigma^6, \sigma\tau, \sigma^3\tau, \sigma^5\tau, \sigma^7\tau\}$. The corresponding fixed fields are respectively $E^2$, $E_0^2$ and $E_1^2$.

Proposition 1. *Let $E^{16}/E$ be a Galois extension of number fields with Galois group $D_{16}$, its subgroups and the corresponding fixed fields as stated above. Then for every odd prime $p$, we have*

$$|K_2(\mathcal{O}_{E^{16}})|\,|K_2(\mathcal{O}_E)|^2 =_p |K_2(\mathcal{O}_{E^2})|\,|K_2(\mathcal{O}_{E_0^8})|\,|K_2(\mathcal{O}_{E_1^8})|, \tag{3.1}$$

$$|K_2(\mathcal{O}_{E^8})|\,|K_2(\mathcal{O}_E)|^2 =_p |K_2(\mathcal{O}_{E^2})|\,|K_2(\mathcal{O}_{E_0^4})|\,|K_2(\mathcal{O}_{E_1^4})|, \tag{3.2}$$

$$|K_2(\mathcal{O}_{E^4})|\,|K_2(\mathcal{O}_E)|^2 =_p |K_2(\mathcal{O}_{E^2})|\,|K_2(\mathcal{O}_{E_0^2})|\,|K_2(\mathcal{O}_{E_1^2})|, \tag{3.3}$$

$$|K_2(\mathcal{O}_{E^{16}})|\,|K_2(\mathcal{O}_{E_0^4})|^2 =_p |K_2(\mathcal{O}_{E^8})|\,|K_2(\mathcal{O}_{E_0^8})|^2, \tag{3.4}$$

$$|K_2(\mathcal{O}_{E^{16}})|\,|K_2(\mathcal{O}_{E_1^4})|^2 =_p |K_2(\mathcal{O}_{E^8})|\,|K_2(\mathcal{O}_{E_1^8})|^2, \tag{3.5}$$

$$|K_2(\mathcal{O}_{E_0^4})| =_p |K_2(\mathcal{O}_{E_2^4})|, \tag{3.6}$$

$$|K_2(\mathcal{O}_{E_1^4})| =_p |K_2(\mathcal{O}_{E_3^4})|, \tag{3.7}$$

$$|K_2(\mathcal{O}_{E^{16}})|\,|K_2(\mathcal{O}_{E_0^2})|^2 =_p |K_2(\mathcal{O}_{E^4})|\,|K_2(\mathcal{O}_{E_0^8})|^2, \tag{3.8}$$

$$|K_2(\mathcal{O}_{E^{16}})|\,|K_2(\mathcal{O}_{E_1^2})|^2 =_p |K_2(\mathcal{O}_{E^4})|\,|K_2(\mathcal{O}_{E_1^8})|^2, \tag{3.9}$$

$$|K_2(\mathcal{O}_{E^8})|\,|K_2(\mathcal{O}_{E_0^2})|^2 =_p |K_2(\mathcal{O}_{E^4})|\,|K_2(\mathcal{O}_{E_0^4})|^2, \tag{3.10}$$

$$|K_2(\mathcal{O}_{E^8})|\,|K_2(\mathcal{O}_{E_1^2})|^2 =_p |K_2(\mathcal{O}_{E^4})|\,|K_2(\mathcal{O}_{E_1^4})|^2. \tag{3.11}$$

*Proof.* The formulae (3.1)–(3.9) follow at once from Theorems 2 and 3. By Galois theory, $E^8/E_0^2$ is a Galois extension with Galois group $V_4$; its three subextensions are $E^4/E_0^2$, $E_0^4/E_0^2$ and $E_2^4/E_0^2$. By Lemma 4, we get $|K_2(O_{E^8})|\,|K_2(O_{E_0^2})|^2 =_p |K_2(O_{E^4})|\,|K_2(O_{E_0^4})|\,|K_2(O_{E_2^4})|$. Hence, we get (3.10) by (3.6). Similarly, we get (3.11) from (3.7).

Example. Let $\mathbb{Q}^{16} = \mathbb{Q}(i, \sqrt[8]{2}\sqrt{2+\sqrt{2}})$. It is easy to verify that $\mathrm{Gal}(\mathbb{Q}^{16}/\mathbb{Q}) = D_{16}$, where

$$\sigma(i) = i, \quad \sigma\Big(\sqrt[8]{2}\,\sqrt{2+\sqrt{2}}\Big) = \sqrt[8]{2}\,\sqrt{2-\sqrt{2}}\,\zeta_8,$$
$$\tau(i) = -i, \quad \tau\Big(\sqrt[8]{2}\,\sqrt{2+\sqrt{2}}\Big) = \sqrt[8]{2}\,\sqrt{2+\sqrt{2}}.$$

Furthermore,

$$\mathbb{Q}^8 = \mathbb{Q}(i, \sqrt[4]{2}), \quad \mathbb{Q}_0^8 = \mathbb{Q}\Big(\sqrt[8]{2}\,\sqrt{2+\sqrt{2}}\Big),$$

$$\mathbb{Q}_1^8 = \mathbb{Q}\Big(\sqrt[8]{2}\,\sqrt{2+\sqrt{2}} + \sqrt[8]{2}\,\sqrt{2-\sqrt{2}}\,\zeta_8\Big), \quad \mathbb{Q}_2^8 = \mathbb{Q}\Big((1+i)\sqrt[8]{2}\,\sqrt{2+\sqrt{2}}\Big),$$

$$\mathbb{Q}_3^8 = \mathbb{Q}\Big(\sqrt[8]{2}\,\sqrt{2+\sqrt{2}} + \sqrt[8]{2}\,\sqrt{2-\sqrt{2}}\,\zeta_8^3\Big),$$

$$\mathbb{Q}_4^8 = \mathbb{Q}\Big(i\sqrt[8]{2}\,\sqrt{2+\sqrt{2}}\Big), \quad \mathbb{Q}_5^8 = \mathbb{Q}\Big(\sqrt[8]{2}\,\sqrt{2+\sqrt{2}} + \sqrt[8]{2}\,\sqrt{2-\sqrt{2}}\,\zeta_8^5\Big),$$

$\mathbb{Q}_6^8 = \mathbb{Q}\Big((1-i)\sqrt[8]{2}\sqrt{2+\sqrt{2}}\Big), \quad \mathbb{Q}_7^8 = \mathbb{Q}\Big(\sqrt[8]{2}\sqrt{2+\sqrt{2}} + \sqrt[8]{2}\sqrt{2-\sqrt{2}}\,\zeta_8^7\Big),$
$\mathbb{Q}^4 = \mathbb{Q}(i,\sqrt{2}), \quad \mathbb{Q}_0^4 = \mathbb{Q}(\sqrt[4]{2}), \quad \mathbb{Q}_1^4 = \mathbb{Q}((1+i)\sqrt[4]{2}), \quad \mathbb{Q}_2^4 = \mathbb{Q}(i\sqrt[4]{2}),$
$\mathbb{Q}_3^4 = \mathbb{Q}((1-i)\sqrt[4]{2}),$
$\mathbb{Q}^2 = \mathbb{Q}(\sqrt{-1}), \quad \mathbb{Q}_0^2 = \mathbb{Q}(\sqrt{2}), \quad \mathbb{Q}_1^2 = \mathbb{Q}(\sqrt{-2}).$

For every odd prime $p$, we know that $K_2(\mathcal{O}_{\mathbb{Q}^2})(p) = K_2(\mathcal{O}_{\mathbb{Q}_0^2})(p) = K_2(\mathcal{O}_{\mathbb{Q}_1^2})(p) = K_2(\mathcal{O}_{\mathbb{Q}^4})(p) = 1$. By Proposition 1, we have

$$|K_2(\mathcal{O}_{\mathbb{Q}_i^4})| =_p |K_2(\mathcal{O}_{\mathbb{Q}_j^4})|, \quad 0 \le i,j \le 3,$$
$$|K_2(\mathcal{O}_{\mathbb{Q}_i^8})| =_p |K_2(\mathcal{O}_{\mathbb{Q}_j^8})|, \quad 0 \le i,j \le 7,$$
$$|K_2(\mathcal{O}_{\mathbb{Q}^8})| =_p |K_2(\mathcal{O}_{\mathbb{Q}_0^4})|^2 =_p |K_2(\mathcal{O}_{\mathbb{Q}_1^4})|^2,$$
$$|K_2(\mathcal{O}_{\mathbb{Q}^{16}})| =_p |K_2(\mathcal{O}_{\mathbb{Q}_0^8})|^2 =_p |K_2(\mathcal{O}_{\mathbb{Q}_1^8})|^2.$$

**Acknowledgements.** The authors are grateful to the anonymous referees for their very careful reading of the paper and useful comments. The research work of H. Y. Zhou is supported by Post-Doctoral Funds of Jiangsu (1201065C).

## References

[1] J. Browkin, *On the p-rank of the tame kernel of algebraic number fields*, J. Reine Angew. Math. 432 (1992), 135–149.
[2] J. Browkin, *Tame kernels of cubic cyclic fields*, Math. Comp. 74 (2005), 967–999.
[3] J. Browkin, *Tame and wild kernels of quadratic imaginary number fields*, Math. Comp. 68 (1999), 291–305.
[4] J. Browkin and H. Gangl, *Tame kernels and second regulators of number fields and their subfields*, submitted to the volume dedicated to Professor Aderemi O. KuKu.
[5] F. Keune, *On the structure of $K_2$ of the ring of integers in a number field*, $K$-Theory 2 (1989), 625–645.
[6] H. R. Qin, *The 2-Sylow subgroup of $K_2(\mathcal{O}_F)$ for number fields $F$*, J. Algebra 284 (2005), 494–519.
[7] H. R. Qin, *The 2-Sylow subgroups of the tame kernel of imaginary quadratic fields*, Acta Arith. 69 (1995), 153–169.
[8] H. R. Qin, *The 4-rank of $K_2(O_F)$ for real quadratic fields*, Acta Arith. 72 (1995), 323–333.
[9] D. Quillen, *Finite generation of the groups $K_i$ of rings of algebraic integers*, in: Lecture Notes in Math. 341, Springer, 1973, 179–198.
[10] C. Soulé, *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. 268 (1984), 317–345.
[11] J. Tate, *Relation between $K_2$ and Galois cohomology*, Invent. Math. 36 (1976), 257–274.
[12] H. Y. Zhou, *Odd parts of tame kernels of dihedral extensions*, Acta Arith. 156 (2012), 341–349.

[13] H. Y. Zhou, *The tame kernel of multiquadratic number fields*, Comm. Algebra 37 (2009), 630–638.

[14] H. Y. Zhou, *Tame kernels of cubic cyclic fields*, Acta Arith. 124 (2006), 293–313.

Qianqian Cui, Haiyan Zhou (corresponding author)
School of Mathematical Sciences
Nanjing Normal University
Nanjing, 210023
P.R. China
E-mail: cuiqianqiande@126.com
haiyanxiaodong@gmail.com

*Received February 21, 2013* (7922)