

A Determinantal Proof of the Product Formula for the Multivariate Transfinite Diameter

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Summary. We give an elementary proof of the product formula for the multivariate transfinite diameter using multivariate Leja sequences and an identity on vandermondians.

1. Introduction and statement of the result. Let \mathbb{N}_d^n denote the set of n -indices of length at most d endowed with the graded lexicographic order (\prec). The cardinality of \mathbb{N}_d^n , denoted by N_d^n , is equal to $\binom{n+d}{n}$. A *vandermondian* (of order d) is the determinant of an $N_d^n \times N_d^n$ matrix of the form (z_α^β) where $z_\alpha \in \mathbb{C}^n$, $[\cdot]^\beta$ is the usual monomial and the rows and columns are ordered according to \prec . Such a determinant is denoted by $\text{VDM}(\mathbf{z})$ where $\mathbf{z} := (z_\alpha : \alpha \in \mathbb{N}_d^n)$. It is a polynomial of degree

$$\ell_d^n := n \binom{n+d}{n+1}$$

in the $(N_d^n)^n$ coordinates of the z_α 's. The d th *diameter* $D_d(K)$ of a compact subset K of \mathbb{C}^n is defined by

$$(1) \quad D_d(K) = \sup\{|\text{VDM}(\mathbf{z})|^{1/\ell_d^n} : \mathbf{z} \in K^{N_d^n}\},$$

and a collection \mathbf{z} for which the supremum is achieved in (1) is called a *Fekete system* (of order d) for K . Now, the *transfinite diameter* $D(K)$ is the limit of $D_d(K)$ as d goes to ∞ . That such a limit exists is by no means obvious (when $n > 1$). It is a beautiful result of V. Zaharjuta [9] who not only proved the convergence of $(D_d(K))$ but also related its limit to complex polynomial approximation.

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Here, for the sake of brevity, we shall content ourselves with recalling one useful relation. Given $\alpha \in \mathbb{N}^n$ we define

$$T(\alpha, K) := \inf\{\|\tau\|_K : \tau \in \mathbb{UP}_\alpha\}$$

where \mathbb{UP}_α is the set of polynomials with \prec -leading monomial z^α , i.e., $\tau(z) = z^\alpha + \sum_{\beta \prec \alpha} c_\beta z^\beta$ and $\|\cdot\|_K$ is the sup-norm on K . A polynomial for which the infimum is achieved is called an α -Chebyshev polynomial. Zaharjuta showed that

$$(2) \quad D(K) = \lim_{d \rightarrow \infty} \left(\prod_{|\alpha|=d} T(\alpha, K) \right)^{1/(dh_d^n)},$$

where

$$h_d^n := \binom{n+d-1}{d} = |\mathbb{N}_d^n \setminus \mathbb{N}_{d-1}^n|.$$

Just like its (better known) univariate analogue, the multivariate transfinite diameter is intimately connected with (pluri)potential theory. Most of what is known so far about these relations can be found in [4, 5, 2] and in the references of those papers. Our note rather deals with the elementary properties of the transfinite diameter and gives a (mainly) algebraic proof of the following formula.

THEOREM 1. *For $i = 1, 2$, let K_i be a compact subset of \mathbb{C}^{n_i} , $n = n_1 + n_2$ and $K := K_1 \times K_2 \subset \mathbb{C}^n$. Then*

$$(3) \quad D(K) = (D(K_1))^{n_1/n} \cdot (D(K_2))^{n_2/n}.$$

This result was first proved in [4, pp. 286–292] by computing transfinite diameters with the help of orthogonal polynomials with respect to a positive measure satisfying the Bernstein–Markov inequality. That proof requires two non-immediate facts. First, that the map $K \mapsto D(K)$ is continuous under non-increasing sequences (of compact sets) and, second, that, for every K , there exists a measure satisfying the Bernstein–Markov inequality with support as close as we like to K .

However, seeing the relatively simple formula (3), we may suspect there does exist a proof that works with the determinants involved in the definition (1) of the d th diameters. Actually such a proof was found a long time ago by Schiffer and Siciak [8] but it worked only in the case $n_i = 1$. The proof presented here is much in the same spirit as theirs. The idea is to take a Fekete system for K_1 , another for K_2 and to “combine” them to form something like a Fekete system for $K_1 \times K_2$. We shall amend this very rough idea and elaborate on it in the next section, and our proof of (a slightly more informative version of) Theorem 1 will easily follow in the last section.

2. Tools. (a) We shall not be able to work with Fekete systems but rather with similar extremal collections of points, called (*block*) *Leja sequences*. If E is a compact subset of \mathbb{C}^m , points of such a sequence are indexed by m -indices and are inductively constructed as follows. We choose any $a_0 \in E$ and if $\mathcal{L}_d := (a_\alpha : |\alpha| \leq d)$ then we define $\mathcal{L}_{d+1} = (a_\alpha : |\alpha| \leq d + 1)$ by requiring

$$(4) \quad \text{vdm}(\mathcal{L}_{d+1}) = \sup\{|\text{vdm}(\mathbf{z})| : \mathbf{z} \in E^{N_{d+1}^m}, z_\alpha = a_\alpha \text{ for } |\alpha| \leq d\}.$$

Note that it is not excluded that $\text{vdm}(\mathcal{L}_d) = 0$ for every d . A very similar sequence, call it a (*point*) *Leja sequence*, was introduced independently by Jędrzejowski [7] and Bloom *et al.* [3]. In both cases, the authors constructed it by adding one point at each step (so that they needed N_d^m steps to construct N_d^m points) whereas we add a whole *block* of h_d^m points at step d (so that we need $d + 1$ steps to construct N_d^m points). We could work as well with their (point) Leja sequences, but since the notion of block is essential in our discussion, as will soon be apparent, we prefer to stick to the more natural setting. The point is that, as long as we are concerned with transfinite diameters, (block) Leja sequences behave like sequences of Fekete systems.

THEOREM 2. *We have*

$$D(E) = \lim_{d \rightarrow \infty} |\text{vdm}(\mathcal{L}_d)|^{1/\ell_d^m}.$$

Proof. We only sketch it for it is a mere adaptation of the proof given in [7] or in [3, pp. 461–462] for a (point) Leja sequence. First, since, by definition, the Fekete system is the best possible choice, we have

$$D(E) \geq \limsup_{d \rightarrow \infty} |\text{vdm}(\mathcal{L}_d)|^{1/\ell_d^m}.$$

Hence we just need to prove that the \liminf satisfies the reverse inequality. We may also assume that E is *unisolvent* for $\mathbb{P}(\mathbb{C}^m)$ (that is, no non-zero polynomial vanishes identically on E) for otherwise the conclusion is easy (see the argument in [7, p. 69]). We shall prove by induction that

$$(5) \quad |\text{vdm}(\mathcal{L}_d)| \geq \prod_{|\alpha| \leq d} T(\alpha, E) \quad (d \geq 0),$$

and a (careful) use of (2) will then yield the required \liminf inequality (see [7, 3] for details regarding this last step).

The inequality (5) is obvious for $d = 0$. We assume it holds true for d and proceed to prove it for $d + 1$. Let $\mathbf{z} = (z_\alpha)$ as in (4). For every β of length $d + 1$, we put

$$M_\beta := \det(z_u^v : u \preceq \beta, v \preceq \beta).$$

Since, the \prec -first N_d^m entries of \mathbf{z} are fixed (equal to the a_α 's), M_β is a (continuous) function of the variables z_u for $d < |u|, u \preceq \beta$. Let γ be the \prec -smallest m -index of length $d + 1$. Expanding M_γ with respect to the last

column, we see that $M_\gamma = \text{VDM}(\mathcal{L}_d)\tau_\gamma(z_\gamma)$ where $\tau_\gamma \in \text{UP}_\gamma$ and the coefficients of τ_γ depend only on the a_α 's. Such a formula is valid however only because $\text{VDM}(\mathcal{L}_d) \neq 0$ and this follows from the induction hypothesis. Indeed, since E is unisolvent, $T(\alpha, E) > 0$ for every α , which, by (5), ensures $\text{VDM}(\mathcal{L}_d) \neq 0$. Now, using again the unisolvence of E , we may choose (and fix) $z_\gamma \in E$ so that $|\tau_\gamma(z_\gamma)| = \|\tau_\gamma\|_E > 0$ (which implies $M_\gamma \neq 0$). Let now θ be the \prec -successor of γ . Expanding M_θ with respect to its last column we find $M_\theta = M_\gamma\tau_\theta(z_\theta)$ where $\tau_\theta \in \text{UP}_\theta$ and the coefficients of τ_θ depend only on the a_α 's and on the already fixed z_γ . Again, this is possible because $M_\gamma \neq 0$. Using the unisolvence of E , we may next find (and fix) $z_\theta \in E$ such that $|\tau_\theta(z_\theta)| = \|\tau_\theta\|_E > 0$ (which implies $M_\theta \neq 0$). At this point, we have

$$|M_\theta| = |\text{VDM}(\mathcal{L}_d)| \|\tau_\gamma\|_E \|\tau_\theta\|_E \geq |\text{VDM}(\mathcal{L}_d)| T(\gamma, E) T(\theta, E).$$

Continuing in this way, after h_{d+1}^m inductive steps, for every α of length $d+1$, we construct a polynomial $\tau_\alpha \in \text{UP}_\alpha$ and $z_\alpha \in E$ with $\|\tau_\alpha\|_E = |\tau_\alpha(z_\alpha)| > 0$ such that, for the corresponding system \mathbf{z} , we have

$$\text{VDM}(\mathbf{z}) = \text{VDM}(\mathcal{L}_d) \prod_{|\alpha|=d+1} \tau_\alpha(z_\alpha),$$

hence $|\text{VDM}(\mathbf{z})| \geq |\text{VDM}(\mathcal{L}_d)| \prod_{|\alpha|=d+1} T(\alpha, E)$. By construction of \mathcal{L}_{d+1} , we therefore have $|\text{VDM}(\mathcal{L}_{d+1})| \geq |\text{VDM}(\mathcal{L}_d)| \prod_{|\alpha|=d+1} T(\alpha, E)$. Now, using again the induction hypothesis we arrive at $\text{VDM}(\mathcal{L}_{d+1}) \geq \prod_{|\alpha|\leq d+1} T(\alpha, E)$, which is (5) for $d+1$. ■

The interested reader should consult the paper [1] for an application of (point) Leja sequences to the reconstruction of the pluricomplex Green function.

(b) Following our idea, we want to *combine* Leja sequences for K_1 and K_2 to get something like a Leja sequence for $K_1 \times K_2$. We first say what we mean by *combining* sequences. We shall use a (natural) process introduced in [6] which generalizes an old idea of Biermann. Given, for $i = 1, 2$, a collection $X_i = (x_{i,\alpha} : |\alpha| \leq d)$ of $N_d^{n_i}$ points in \mathbb{C}^{n_i} , the *intertwining* of X_1 and X_2 is the collection of $N_d^{n_1+n_2}$ points in $\mathbb{C}^{n_1+n_2}$ defined by $X_1 \oplus X_2 := ((x_{1,\alpha}, x_{2,\beta}) : |\alpha| + |\beta| \leq d)$. The following theorem is proved in [6, Sec. 5] by using multivariate Lagrange interpolation theory. It is this result which enables us to show that the intertwining of Leja sequences for the factor sets K_i behaves like a Leja sequence for the product set $K = K_1 \times K_2$. It was found and used by Schiffer and Siciak in the case $n_i = 1$ for which the intertwining of the X_i reduces to the aforementioned construction of Biermann.

THEOREM 3. Let $X_i, i = 1, 2$, and $X_1 \oplus X_2$ as above. Then

$$(6) \quad \text{VDM}(X_1 \oplus X_2) = \prod_{j=1}^d (\text{VDM}(X_1^j))^{h_{d-j}^{n_2}} \prod_{j=1}^d (\text{VDM}(X_2^j))^{h_{d-j}^{n_1}},$$

where X_i^j is the j th block of X_i , that is, $X_i^j := (x_{i,\alpha} : |\alpha| \leq j)$.

It is very important for us that for every $j, 1 \leq j \leq d$, the j th block of the d th term $\mathcal{L}_{i,d}$ of the Leja sequence for K_i is just the j th term of this same sequence, i.e., $\mathcal{L}_{i,d}^j = \mathcal{L}_{i,j}$. It has been constructed precisely in order that this property be satisfied. The same would no longer be true with Fekete systems and this is the very reason why we need to use Leja sequences here. This being said, we immediately deduce the following corollary to Theorem 3.

COROLLARY 4. For $i = 1, 2$, let $(\mathcal{L}_{i,d})$ be a Leja sequence for K_i . Then

$$(7) \quad \text{VDM}(\mathcal{L}_{1,d} \oplus \mathcal{L}_{2,d}) = \prod_{j=1}^d (\text{VDM}(\mathcal{L}_{1,j}))^{h_{d-j}^{n_2}} \prod_{j=1}^d (\text{VDM}(\mathcal{L}_{2,j}))^{h_{d-j}^{n_1}}.$$

(c) We finally need some information to be able to calculate with the fairly complicated exponents involved in the various formulae.

LEMMA 5. Let $d, n_i, i = 1, 2$, be positive integers and $n = n_1 + n_2$. Then

$$\sum_{k=1}^d k h_k^{n_1} h_{d-k}^{n_2} = (n_1/n) d h_d^n.$$

Proof. This is Lemma 3 in [4]. It is proved using a standard generating power series technique. ■

LEMMA 6. Let $d, n_i, i = 1, 2$, be positive integers and $n = n_1 + n_2$. Then

$$\sum_{k=1}^d \ell_k^{n_1} h_{d-k}^{n_2} = (n_1/n) \ell_d^n.$$

Proof. Lemma 5 (applied with $n_1 + 1$) yields

$$\sum_{k=1}^d k h_k^{n_1+1} h_{d-k}^{n_2} = \frac{n_1 + 1}{n + 1} d h_d^{n+1}$$

and the claim follows by observing that, for every $m > 0$,

$$m h_m^{n_1+1} = \frac{n_1 + 1}{n_1} \ell_m^{n_1}. \quad \blacksquare$$

3. Abridged proof of Theorem 1. As a by-product of our proof, we shall see that the intertwining of Leja sequences for K_1 and K_2 permits one to recover the transfinite diameter of $K = K_1 \times K_2$. Note that it is very

unlikely that this intertwining sequence itself is a Leja sequence for K . We continue to denote by $\mathcal{L}_{i,d}$ a (block) Leja sequence for K_i .

STEP 1. We prove $(D(K))^n \geq (D(K_1))^{n_1} \cdot (D(K_2))^{n_2}$. Since $D_d(K) \geq |\text{VDM}(\mathcal{L}_{1,d} \oplus \mathcal{L}_{2,d})|^{1/\ell_d^n}$, the inequality follows from the next lemma.

LEMMA 7. *We have*

$$\lim_{d \rightarrow \infty} |\text{VDM}(\mathcal{L}_{1,d} \oplus \mathcal{L}_{2,d})|^{1/\ell_d^n} = (D(K_1))^{n_1/n} \cdot (D(K_2))^{n_2/n}.$$

Proof. The details are a bit tedious but standard. We shall assume $D(K_i) > 0$. The proof is even simpler if one of the $D(K_i)$ is 0. Fix $\varepsilon > 0$. In view of Theorem 2 for d large enough, say $d \geq d_0$, we have

$$(8) \quad e^{-\varepsilon} D(K_i) \leq |\text{VDM}(\mathcal{L}_{i,d})|^{1/\ell_d^{n_i}} \leq e^\varepsilon D(K_i).$$

Choose positive $r < R$ (depending on ε) such that

$$(9) \quad rD(K_i) \leq |\text{VDM}(\mathcal{L}_{i,d})|^{1/\ell_d^{n_i}} \leq RD(K_i) \quad \text{for } d < d_0.$$

(Here we need $\text{VDM}(\mathcal{L}_{i,d}) \neq 0$. This follows from (5) which holds true since $D(K_i) > 0$ implies that K_i is unisolvent for $\mathbb{P}_d(\mathbb{C}^n)$.) Working with $i = 1$ and writing

$$\left(\prod_{j=1}^d (\text{VDM}(\mathcal{L}_{1,j}))^{h_{d-j}^{n_2}} \right)^{1/\ell_d^n} = \prod_{j=1}^d \left(|\text{VDM}(\mathcal{L}_{1,j})|^{1/\ell_j^{n_1}} \right)^{\ell_j^{n_1} h_{d-j}^{n_2} / \ell_d^n},$$

we readily infer, on distinguishing the cases $d \geq d_0$ and $d < d_0$ and using (8), (9) and Lemma 6, that

$$(10) \quad r^{s_d} e^{-\varepsilon n_1/n - s_d} (D(K_1))^{n_1/n} \leq \left(\prod_{j=1}^d (\text{VDM}(\mathcal{L}_{1,j}))^{h_{d-j}^{n_2}} \right)^{1/\ell_d^n} \leq R^{s_d} e^{\varepsilon n_1/n - s_d} (D(K_1))^{n_1/n}$$

where $s_d := (1/\ell_d^n) \sum_{j=1}^{d_0-1} \ell_j^{n_1} h_{d-j}^{n_2}$ tends to 0 as $d \rightarrow \infty$. We finish the proof by using the same estimate for K_2 together with Corollary 4 and letting d tend to ∞ and then ε to 0. ■

STEP 2. We now prove $(D(K))^n \leq (D(K_1))^{n_1} \cdot (D(K_2))^{n_2}$. Here, the Leja sequences are of no use but, thanks to (2), elementary considerations on the Chebyshev polynomials will lead to the inequality. Since the graded lexicographic order is compatible with addition, the product of an α^1 -Chebyshev polynomial for K_1 and an α^2 -Chebyshev polynomial for K_2 belongs to $\mathbb{UP}_{(\alpha^1, \alpha^2)}$ so that $T((\alpha^1, \alpha^2), K) \leq T(\alpha^1, K_1) \cdot T(\alpha^2, K_2)$. Hence, writing every $\alpha \in \mathbb{N}_d^n$ as $\alpha = (\alpha^1, \alpha^2)$ with α^1 (resp. α^2) formed by the first n_1 (resp.

last n_2) coordinates of α , we have

$$(11) \quad \prod_{|\alpha|=d} T(\alpha, K) \leq \left(\prod_{j=0}^d \prod_{|\alpha^1|=j} (T(\alpha_1, K_1))^{h_d^{n_2-j}} \right) \cdot \left(\prod_{j=0}^d \prod_{|\alpha^2|=j} (T(\alpha_2, K_2))^{h_d^{n_1-j}} \right).$$

Let $C_i > D(K_i)$ for $i = 1, 2$. In view of (2),

$$\prod_{|\alpha^i|=d} T(\alpha^i, K_i) \leq C_i^{dh_d^{n_i}} \quad \text{for } d \text{ large enough, say } d \geq d_0,$$

and, for some $R > 0$ (depending on the C_i 's),

$$\prod_{|\alpha^i|=d} T(\alpha^i, K_i) \leq (RC_i)^{dh_d^{n_i}} \quad \text{for } d < d_0$$

(both estimates for $i = 1, 2$). Inserting these estimates in (11) (distinguishing the cases $d \geq d_0$ and $d \leq d_0$) and using Lemma 5, we find

$$(12) \quad \left(\prod_{|\alpha|=d} T(\alpha, K) \right)^{1/(dh_d^n)} \leq R^{s_{1,d}+s_{2,d}} \cdot C_1^{n_1/n} \cdot C_2^{n_2/n},$$

where

$$s_{i,d} = \frac{1}{dh_d^n} \sum_{j=0}^{d_0-1} j h_j^{n_i} h_{d-j}^{n-n_i} \rightarrow 0 \quad \text{as } d \rightarrow \infty.$$

Now, letting $d \rightarrow \infty$ and using once more (2), we arrive at $D(K) \leq C_1^{n_1/n} \cdot C_2^{n_2/n}$. The required inequality follows since C_i can be taken arbitrarily close to $D(K_i)$. This completes the proof of Theorem 1.

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